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# FINITE VOLUME SCHEMES FOR THE AFFINE MORPHOLOGICAL SCALE SPACE (AMSS) MODEL

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ABSTRACT. Finite volume (FV) numerical schemes for the approximation of Affine Morphological Scale Space (AMSS) model are proposed. For the scheme parameter $\theta$, $0 \leq \theta \leq 1$ the numerical schemes of Crank-Nicolson type were derived. The explicit ($\theta = 0$), semi-implicit, fully-implicit ($\theta = 1$) and Crank-Nicolson ($\theta = 0.5$) schemes were studied. Stability estimates for explicit and implicit schemes were derived. On several numerical experiments the properties and comparison of the numerical schemes are presented.

## 1. Introduction

In the substantial paper [1], Alvarez, Guichard, Lions and Morel gave axiomatically all image multi-scale theories and gave explicit formulas for the partial differential equations generated by scale spaces (see also [7]). They had proved that all causal, local, isometric and contrast invariant scale spaces are given by curvature evolution equations of the type

$$u_t = g\big(\mathrm{curv}(u), t\big)|\nabla u|. \tag{1}$$

Among them two particular motions became very useful for planar shape deformation and recognition. The first one is represented by mean curvature flows

$$u_t = \mathrm{curv}(u)|\nabla u|. \tag{2}$$


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The second one is devoted to affine curvature evolutions

$$u_t = \big(\mathrm{curv}(u)\big)^{\frac{1}{3}} |\nabla u|. \tag{3}$$

There are many interesting papers concerning these two models, either from theoretical (existence and properties of the solutions) or from numerical (numerical solution and its convergence in some sense) point of views. In this introduction, we focus only on the numerical solution of the AMSS model. We want to mention the paper [7], where the numerical scheme based on the finite difference method is presented. Another approach can be seen in [3], where the exact solution for the model is presented and the accuracy of the numerical scheme can be computed. The paper is a continuation of our previous paper [2], where basic numerical scheme based on the finite volume methodology is presented, namely the semi-implicit finite volume scheme and its iterative improvement.

In this paper, we want to describe four numerical schemes, namely, explicit, semi-implicit, fully-implicit and Crank-Nicolson schemes. They are all based on the finite volume method as it was used in [5] for the MCM model. We focused our study on the properties of numerical schemes concerning experimental order of convergence, CPU times and theoretical stability results for some of them.

The paper is organised as follows. In Section 2, we present the AMSS model and its discretization. In Section 3, we summarise four numerical schemes for AMSS model and in Section 4, we present stability estimation for explicit and implicit schemes. Numerical experiments concerning the properties of proposed numerical schemes are in Section 5.

## 2. AMSS model and its finite volume discretization

The model can be written in the following way

$$u_t - |\nabla u| \left( \mathrm{div} \left( \frac{\nabla u}{|\nabla u|} \right) \right)^{\frac{1}{3}} = 0, \quad \text{a.e. } (t,x) \in (0,T) \times \Omega \tag{4}$$

with the initial condition

$$u(0,x) = u_0(x), \quad \text{a.e. } x \in \Omega, \tag{5}$$

and the boundary condition

$$u(t,x) = 0, \quad \text{a.e. } (t,x) \in (0,T) \times \partial\Omega, \tag{6}$$

here $\Omega \subset \mathbb{R}^d$, with $d \in \mathbb{N}$, and $\partial\Omega$ is its boundary. As the model is highly nonlinear and degenerate, we use the so-called Evans-Spruck [6] regularization to avoid zero values in denominator. Moreover, we use additional regularization as in [5]. Thus our regularized problem is of the form

$$u_t - f(|\nabla u|)\left(\operatorname{div}\left(\frac{\nabla u}{f(|\nabla u|)}\right)\right)^{\frac{1}{3}} = 0, \quad \text{a.e. } (t,x) \in (0,T)\times\Omega, \tag{7}$$

where

$$f(s) = \min\{\sqrt{s^2+a}, b\} \text{ for some parameters } a > 0, b > 0. \tag{8}$$

In order to describe the schemes, we now introduce some notations for the space discretisation as in [5].

**Definition 2.1** (Space discretisation)**.** Let $\Omega$ be a polyhedral open bounded connected subset of $\mathbb{R}^d$, with $d \in \mathbb{N}$, and $\partial\Omega = \overline{\Omega}\backslash\Omega$ its boundary. A discretisation of $\Omega$, denoted by $\mathcal{D}$, is defined as the triplet $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$, where:

1) $\mathcal{M}$ is a finite family of nonempty connected open disjoint subsets of $\Omega$ (the "control volumes") such that $\overline{\Omega} = \cup_{p\in\mathcal{M}}\overline{p}$. For any $p \in \mathcal{M}$, let $\partial p = \overline{p}\backslash p$ be the boundary of $p$; let $|p| > 0$ denote the measure of $p$ and let $h_p$ denote the diameter of $p$ and $h_{\mathcal{D}}$ denote the maximum value of $(h_p)_{m\in\mathcal{M}}$.
2) $\mathcal{E}$ is a finite family of disjoint subsets of $\overline{\Omega}$ (the "edges" of the mesh), such that, for all $\sigma \in \mathcal{E}$, $\sigma$ is a nonempty open subset of a hyperplane of $\mathbb{R}^d$, whose $(d-1)$-dimensional measure $|\sigma|$ is strictly positive. We also assume that, for all $p \in \mathcal{M}$, there exists a subset $\mathcal{E}_p$ of $\mathcal{E}$ such that $\partial p = \cup_{\sigma\in\mathcal{E}_p}\overline{\sigma}$. For any $\sigma \in \mathcal{E}$, we denote by $\mathcal{M}_\sigma = \{p \in \mathcal{M}, \sigma \in \mathcal{E}_p\}$. We then assume that, for all $\sigma \in \mathcal{E}$, either $\mathcal{M}_\sigma$ has exactly one element and then $\sigma \subset \partial\Omega$ (the set of these interfaces, called boundary interfaces, is denoted by $\mathcal{E}_{\text{ext}}$) or $\mathcal{M}_\sigma$ has exactly two elements (the set of these interfaces, called interior interfaces, is denoted by $\mathcal{E}_{\text{int}}$). For all $\sigma \in \mathcal{E}$, we denote by $x_\sigma$ the barycentre of $\sigma$. For all $p \in \mathcal{M}$ and $\sigma \in \mathcal{E}_p$, we denote by $\mathbf{n}_{p,\sigma}$ the unit vector normal to $\sigma$ outward to $p$.
3) $\mathcal{P}$ is a family of points of $\Omega$ indexed by $\mathcal{M}$, denoted by $\mathcal{P} = (x_p)_{p\in\mathcal{M}}$, such that for all $p \in \mathcal{M}$, $x_p \in p$ and $p$ is assumed to be $x_p$-star-shaped, which means that for all $x \in p$, the inclusion $[x_p, x] \subset p$ holds. Denoting by $d_{p\sigma}$ the Euclidean distance between $x_p$ and the hyperplane including $\sigma$, one assumes that $d_{p\sigma} > 0$. Then we denote by $D_{p,\sigma}$ the cone with vertex $x_p$ and basis $\sigma$.
4) We make an important following assumption

$$d_{p\sigma}\mathbf{n}_{p,\sigma} = x_\sigma - x_p, \ \ \forall p \in \mathcal{M}, \ \ \forall \sigma \in \mathcal{E}_p. \tag{9}$$

We denote

$$\theta_{\mathcal{D}} = \min_{p\in\mathcal{M}} \min_{\sigma\in\mathcal{E}_p} \frac{d_{p\sigma}}{h_p}. \tag{10}$$

**Definition 2.2** (Space-time discretisation)**.** Let $\Omega$ be a polyhedral open bounded connected subset of $\mathbb{R}^d$, with $d \in \mathbb{N}$ and let $T > 0$ be given. We say that $(\mathcal{D},\tau)$ is a space-time discretisation of $(0,T)\times\Omega$ if $\mathcal{D}$ is a space discretisation of $\Omega$ in the sense of Definition 2.1 and if there exists $N_T\in\mathbb{N}$ with $T=(N_T+1)\tau$.

Let $(\mathcal{D},\tau)$ be a space-time discretisation of $\Omega\times(0,T)$. We define the set $H_{\mathcal{D}}\subset\mathbb{R}^{\mathcal{M}}\times\mathbb{R}^{\mathcal{E}}$ such that $u_\sigma=0$ for all $\sigma\in\mathcal{E}_{\rm ext}$. We define the following functions on $H_{\mathcal{D}}$

$$N_p(u)^2=\frac{1}{|p|}\sum_{\sigma\in\mathcal{E}_p}\frac{|\sigma|}{d_{p\sigma}}(u_\sigma-u_p)^2,\ \ \forall p\in\mathcal{M},\ \ \forall u\in H_{\mathcal{D}}. \tag{11}$$

Let us recall that

$$\|u\|_{1,\mathcal{D}}^2=\sum_{p\in\mathcal{M}}|p|N_p(u)^2 \tag{12}$$

defines a norm on $H_{\mathcal{D}}$ (see [5] and references therein). We then define the set $\mathbb{N}_0=\{n=0,\dots,N_T\}$ and $H_{\mathcal{D},\tau}$ of all $u=(u^{n+1})_{n\in\mathbb{N}_0}$ such that $u^{n+1}\in H_{\mathcal{D}}$ for all $n\in\mathbb{N}_0$, and we set

$$\|u\|_{1,\mathcal{D},\tau}^2=\sum_{n=0}^{N_T}\tau\|u^{n+1}\|_{1,\mathcal{D}}^2,\ \ \forall u\in H_{\mathcal{D},\tau}. \tag{13}$$

Finally, on $\Omega\subset\mathbb{R}^d$ and time interval $(0,T)$ we define $G_{\mathcal{D},\tau}$ by

$$\begin{aligned}&G_{\mathcal{D},\tau}(x,t)=d\frac{u_\sigma^{n+1}-u_p^{n+1}}{d_{p\sigma}}\mathbf{n}_{p\sigma},\\&\text{for a.e. } x\in D_{p\sigma},\quad \text{for a.e. } t\in\big(n\tau,(n+1)\tau\big),\ \ \forall p\in\mathcal{M},\ \ \forall\sigma\in\mathcal{E}_p,\ \ \forall n\in\mathbb{N}_0.\end{aligned} \tag{14}$$

The idea to obtain the numerical scheme is based on the finite volume methodology. We use the approach for approximation as in [2]. First, we rearrange and then we integrate the equation (7) for an arbitrary $p\in\mathcal{M}$ with the boundary $\partial p$ and a unit outward normal $\mathbf{n_p}$. We obtain

$$\int_p\left(\frac{u_t}{f(|\nabla u|)}\right)^3\mathrm{d}x-\int_{\partial p}\frac{\partial u}{\partial\mathbf{n_p}}\frac{1}{f(|\nabla u|)}\,\mathrm{d}s\ =\ 0. \tag{15}$$

We denote by $\delta_t u_p^n$ the approximation of time derivative on the finite volume $p$ and time interval $\big(n\tau,(n+1)\tau\big)$, which is usually given by

$$\delta_t u_p^n=\frac{u_p^{n+1}-u_p^n}{\tau}.$$

Using the standard finite volume methodology [4], we can derive an explicit scheme in the following way

$$\begin{aligned}&\left(\frac{\delta_t u_p^n}{f(N_p(u^n))}\right)^3|p|-\frac{1}{f(N_p(u^n))}\sum_{\sigma\in\mathcal{E}_p}\frac{|\sigma|}{d_{p\sigma}}(u_\sigma^n-u_p^n)=0,\\&\forall p\in\mathcal{M},\ \ \forall n\in\mathbb{N}.\end{aligned} \tag{16}$$

Moreover, the following relation is given for the interior edges [5]

$$\frac{u_\sigma^n - u_p^n}{f(N_p(u^n))\, d_{p\sigma}} + \frac{u_\sigma^n - u_q^n}{f(N_q(u^n))\, d_{q\sigma}} = 0, \ \ \forall \sigma \in \mathcal{E}_{\text{int}} \quad \text{with} \quad \mathcal{M}_\sigma = \{p, q\}, \ \ \forall n \in \mathbb{N}. \tag{17}$$

Involving the above relation, we obtain the following scheme

$$\begin{aligned} &\left(\delta_t u_p^n\right)^3 \frac{1}{f(N_p(u^n))^3} |p| - \sum_{\sigma \in \mathcal{E}_p} \frac{(u_q^n - u_p^n)|\sigma|}{f(N_p(u^n))\, d_{p\sigma} + f(N_q(u^n))\, d_{q\sigma}} = 0, \\ &\forall p \in \mathcal{M}, \ \ \forall n \in \mathbb{N}. \end{aligned} \tag{18}$$

In the expression above, we can notice that the second term is in fact the approximation of curvature on the finite volume $p$. So we can pose

$$K_p^n = \frac{1}{|p|} \sum_{\sigma \in \mathcal{E}_p} \frac{|\sigma|(u_q^n - u_p^n)}{d_{p\sigma} f(N_p(u^n)) + d_{q\sigma} f(N_q(u^n))} \tag{19}$$

and then we obtain

$$\begin{aligned} &\left(\delta_t u_p^n\right)^3 \frac{1}{f(N_p(u^n))^3} |p| - K_p^n |p| = 0, \\ &\forall p \in \mathcal{M}, \ \ \forall n \in \mathbb{N}. \end{aligned} \tag{20}$$

For approximation of the time derivative, from (20) we can immediately obtain

$$\delta_t u_p^n = f\big(N_p(u^n)\big)\big(K_p^n\big)^{\frac{1}{3}}. \tag{21}$$

Now, if we approximate

$$\begin{aligned} \left(\delta_t u_p^n\right)^3 &\approx \frac{u_p^{n+1} - u_p^n}{\tau} \Big(f\big(N_p(u^n)\big)\big(K_p^n\big)^{\frac{1}{3}}\Big)^2 \\ &= \frac{u_p^{n+1} - u_p^n}{\tau} f\big(N_p(u^n)\big)^2 \big(K_p^n\big)^{\frac{2}{3}} \end{aligned} \tag{22}$$

from (20) we have

$$\frac{u_p^{n+1} - u_p^n}{\tau} \frac{(f(N_p(u^n))(K_p^n)^{\frac{1}{3}})^2}{f(N_p(u^n))^3} |p| - K_p^n |p| = 0,$$

and finally, we obtain after straightforward rearrangement

$$\begin{aligned} &\frac{u_p^{n+1} - u_p^n}{\tau} |p| - f\big(N_p(u^n)\big)\big(K_p^n\big)^{\frac{1}{3}} |p| = 0, \\ &\forall p \in \mathcal{M}, \ \ \forall n \in \mathbb{N}_0. \end{aligned} \tag{23}$$

Similarly, we can do approach for the fully-implicit scheme and we obtain

$$\begin{aligned} &\big(\delta_t u_p^n\big)^3 \frac{1}{f(N_p(u^{n+1}))^3}|p| - K_p^{n+1}|p| = 0, \\ &\forall p \in \mathcal{M}, \quad \forall n \in \mathbb{N}_0. \end{aligned} \tag{24}$$

We use again the similar approach for approximation the time derivative.

The nonlinear term for approximation of the time derivative can be obtained immediately from (24)

$$\delta_t u_p^n = f\big(N_p(u^{n+1})\big)\big(K_p^{n+1}\big)^{\frac{1}{3}}, \tag{25}$$

and similarly we can approximate

$$\begin{aligned} \big(\delta_t u_p^n\big)^3 &= \frac{u_p^{n+1} - u_p^n}{\tau} \Big(f\big(N_p(u^{n+1})\big)\big(K_p^{n+1}\big)^{\frac{1}{3}}\Big)^2 \\ &= \frac{u_p^{n+1} - u_p^n}{\tau} \Big(f\big(N_p(u^{n+1})\big)\Big)^2 \big(K_p^{n+1}\big)^{\frac{2}{3}} \end{aligned} \tag{26}$$

from (24) we finally obtain

$$\begin{aligned} &\frac{u_p^{n+1} - u_p^n}{\tau}|p| - f\big(N_p(u^{n+1})\big)\big(K_p^{n+1}\big)^{\frac{1}{3}}|p| = 0, \\ &\forall p \in \mathcal{M}, \quad \forall n \in \mathbb{N}_0. \end{aligned} \tag{27}$$

Now, for some $\theta \in \langle 0, 1\rangle$ we can write the combination of both schemes of Crank--Nicolson type. First we define:

$$u_p^0 = \frac{1}{|p|} \int_p u_0(x)\,\mathrm{d}x, \quad \forall p \in \mathcal{M}, \tag{28}$$

$$u_\sigma^0 = \frac{1}{|\sigma|} \int_\sigma u_0(s)\,\mathrm{d}s, \quad \forall \sigma \in \mathcal{E}, \tag{29}$$

the boundary condition is fulfilled thanks to

$$u_\sigma^n = 0, \quad \forall \sigma \in \mathcal{E}_{\mathrm{ext}}, \quad \forall n \in \mathbb{N}_0 \tag{30}$$

and

$$\begin{aligned} &\frac{(u_p^{n+1} - u_p^n)}{\tau} - \theta f\big(N_p(u^{n+1})\big)\big(K_p^{n+1}\big)^{\frac{1}{3}} - (1-\theta) f\big(N_p(u^n)\big)\big(K_p^n\big)^{\frac{1}{3}} = 0, \\ &\forall p \in \mathcal{M}, \quad \forall n \in \mathbb{N}_0. \end{aligned} \tag{31}$$

Now, considering a family of values $(u_p^n)_{p\in\mathcal{M}, n\in\mathbb{N}_0}$, given by (28), (29), (30) and (31), we define [5] as the approximate solution $u_{\mathcal{D},\tau}$ in $\Omega \times \mathbb{R}_+$ by

$$\begin{aligned} &u_{\mathcal{D},\tau}(x, 0) = u_p^0, \quad u_{\mathcal{D},\tau}(x, t) = u_p^{n+1} \\ &\text{for a.e. } x \in p, \quad \forall t \in \big(n\tau, (n+1)\tau\big), \quad \forall p \in \mathcal{M}, \quad \forall n \in \mathbb{N}_0. \end{aligned} \tag{32}$$

## 3. Summary of the numerical schemes

Now, we present four schemes that will be used in our computations.

### 3.1. Explicit scheme

Numerical scheme defined by (28), (29), (30) and (31) gives for $\theta = 0$, *explicit* numerical scheme, namely

$$u_p^{n+1} = u_p^n + \tau f\big(N_p(u^n)\big)\big(K_p^n\big)^{\frac{1}{3}}, \quad \forall p \in \mathcal{M}, \quad \forall n \in \mathbb{N}_0. \tag{33}$$

### 3.2. Fully-implicit scheme

Numerical scheme (28), (29), (30) and (31) gives for $\theta = 1$, *fully-implicit* numerical scheme, namely

$$u_p^{n+1} - \tau f\big(N_p(u^{n+1})\big)\big(K_p^{n+1}\big)^{\frac{1}{3}} = u_p^n, \quad \forall p \in \mathcal{M}, \quad \forall n \in \mathbb{N}_0. \tag{34}$$

This nonlinear algebraic system (34) can be computed in an iterative way. Let

$$u_p^{n+1,0} = u_p^n, \quad f\big(N_p(u^{n+1,0})\big) = f\big(N_p(u^n)\big) \quad \text{and} \quad K_p^{n+1,0} = K_p^n$$

and other iterations by

$$\begin{aligned} &u_p^{n+1,k+1} - \tau \frac{f\Big(N_p\Big(u^{n+1,k}\Big)\Big)}{\max\Big\{a_1, (K_p^{n+1,k})^{\frac{2}{3}}\Big\}} \tilde{K}_p^{n+1,k+1} = u_p^n, \\ &\forall p \in \mathcal{M}, \quad \forall n \in \mathbb{N}_0, \end{aligned} \tag{35}$$

where

$$\tilde{K}_p^{n+1,k+1} = \frac{1}{|p|} \sum_{\sigma \in \mathcal{E}_p} \frac{|\sigma|\big(u_q^{n+1,k+1} - u_p^{n+1,k+1}\big)}{d_{p\sigma} f\big(N_p(u^{n+1,k})\big) + d_{q\sigma} f\big(N_q(u^{n+1,k})\big)} \tag{36}$$

and $a_1 > 0$ is the prescribed small, fixed parameter.

There are several possibilities to stop the iterations, e.g., if two following iterations differ less than the prescribed tolerance or if the residuum of linear algebraic system with the unknowns $u_p^{n+1,k}$ and coefficients with the same iteration $k$ is smaller than the prescribed tolerance.

### 3.3. Semi-implicit scheme

The numerical scheme (28), (29), (30) and (31) gives for $\theta = 1$ *semi-implicit* numerical scheme, with modification that nonlinear terms are computed in previous time step, that means it is a special case of fully-implicit scheme without iterations

$$\begin{aligned} &u_p^{n+1} - \tau \frac{f(N_p(u^n))}{\max\big\{a_1, (K_p^n)^{\frac{2}{3}}\big\}} \bar{K}_p^{n+1} = u_p^n, \\ &\forall p \in \mathcal{M}, \quad \forall n \in \mathbb{N}_0, \end{aligned} \tag{37}$$

where

$$\bar{K}_p^{n+1} = \frac{1}{|p|} \sum_{\sigma \in \mathcal{E}_p} \frac{|\sigma|(u_q^{n+1} - u_p^{n+1})}{d_{p\sigma} f(N_p(u^n)) + d_{q\sigma} f(N_q(u^n))} \tag{38}$$

and $a_1 > 0$ is the prescribed small, fixed parameter.

### 3.4. Crank-Nicolson scheme

In this case, we use the numerical scheme (28),(29), (30) and (31) for $\theta = \frac{1}{2}$ and we give the *Crank-Nicolson* numerical scheme, namely

$$u_p^{n+1} - \frac{\tau}{2} f\big(N_p(u^{n+1})\big)\big(K_p^{n+1}\big)^{\frac{1}{3}} = u_p^n + \frac{\tau}{2} f\big(N_p(u^n)\big)\big(K_p^n\big)^{\frac{1}{3}},$$

$$\forall p \in \mathcal{M}, \quad \forall n \in \mathbb{N}_0. \tag{39}$$

Now using the similar approach as for fully-implicit scheme, we can derive the iteration scheme as follows: $u_p^{n+1,0} = u_p^n$, $f\big(N_p(u^{n+1,0})\big) = f(N_p(u^n))$ and $K_p^{n+1,0} = K_p^n$ and other iterations by

$$u_p^{n+1,k+1} - \frac{\tau}{2} \frac{f(N_p(u^{n+1,k}))}{\max\big\{a_1, (K_p^{n+1,k})^{\frac{2}{3}}\big\}} \tilde{K}_p^{n+1,k+1} = u_p^n + \frac{\tau}{2} f\big(N_p(u^n)\big)\big(K_p^n\big)^{\frac{1}{3}}$$

$$\forall p \in \mathcal{M}, \quad \forall n \in \mathbb{N}_0. \tag{40}$$

## 4. Stability estimates

We want to prove stability estimations for the numerical solution obtained by proposed numerical schemes. We state the assumptions called (A) for the data in the following:

1) $\Omega$ is a finite connected open subset of $\mathbb{R}^d$ with boundary $\partial\Omega$ defined by a finite union of subsets of hyperplanes of $\mathbb{R}^d$,
2) $u_0 \in H_0^1(\Omega)$,
3) $f$ is defined by (8).

### 4.1. Semi-implicit scheme

We are dealing with the semi-implicit scheme, it means that as in (37) we remind that

$$u_p^{n+1} - \tau \frac{f(N_p(u^n))}{\max\{a_1, (K_p^n)^{\frac{2}{3}}\}} \bar{K}_p^{n+1} = u_p^n, \tag{41}$$

$$\forall p \in \mathcal{M}, \quad \forall n \in \mathbb{N}_0.$$

Now let us state the $L^\infty$ stability of the semi-implicit scheme. The result is similar as in [5].

**Lemma 4.1** ($L^\infty$ stability of the semi-implicit scheme). *Under assumption (A), let $(\tau,\mathcal{D})$ be a space-time discretisation of $(0,T)\times\Omega$ in the sense of Definition 2.2. We denote by*

$$|u_0|_{\mathcal{D},\infty} = \max_{p\in\mathcal{M}} |u_p^0|, \tag{42}$$

*(Note that, if $u_0 \in L^\infty(\Omega)$ and then $|u_0|_{\mathcal{D},\infty} \le \|u_0\|_{L^\infty(\Omega)}$.)*

*Let $(u_p^n)_{p\in\mathcal{M},n\in\mathbb{N}}$ be a solution of (28), (29) and (30), (37). Then it holds*

$$|u_p^n| \le |u_0|_{\mathcal{D},\infty} \quad \forall p\in\mathcal{M}, \quad \forall n = 0,\dots,N_T.$$

Proof. Suppose that for the fixed time step $n+1$ the maximum of all $u_p^{n+1}$ is achieved at the finite volume $p$. We immediately know that

$$\frac{f(N_p(u^n))}{\max\{a_1,(K_p^n)^{\frac{2}{3}}\}} > 0$$

and from the definition of $\bar{K}_p^{n+1}$ and the fact that maximum of all $u_p^{n+1}$ is achieved at the finite volume $p$, we have

$$\bar{K}_p^{n+1} \le 0$$

which brings that the second term in the semi-implicit scheme is nonnegative and it leads to

$$u_p^{n+1} \le u_p^n. \tag{43}$$

Then, we recursively get the estimate (43), similarly reasoning for the minimum values. □

**Remark.** The same conclusion with similar arguments is for the fully-implicit scheme.

### 4.2. Explicit scheme

We remind that in this case we have

$$u_p^{n+1} = u_p^n + \tau f\big(N_p(u^n)\big)(K_p^n)^{\frac{1}{3}}, \quad \forall p\in\mathcal{M}, \quad \forall n\in\mathbb{N}_0. \tag{44}$$

**Lemma 4.2** ($L^\infty$ stability of the explicit scheme). *Under assumption (A), let $(\tau,\mathcal{D})$ be a space-time discretisation of $(0,T)\times\Omega$ in the sense of Definition 2.2.*

*Let $(u_p^n)_{p\in\mathcal{M},n\in\mathbb{N}}$ be a solution of (28), (29) and (30), (34). Let*

$$a = C_1|p| \tag{45}$$

*hold for parameter $a$ defined in (8) for some real positive constant $C_1$.*

*Let* $C_2 = C_1 b^{-\frac{2}{3}} \sum_{\sigma\in\mathcal{E}_p} \frac{|\sigma|}{d_{p\sigma}}$. *Suppose*

$$\tau < C_2|p|. \tag{46}$$

*Then for* $C = b^{\frac{2}{3}} C_1^{\frac{1}{3}}$ *it holds that*

$$|u_p^n| \le |u_0|_{\mathcal{D},\infty} + CT, \quad \forall p \in \mathcal{M}, \quad \forall n = 0, \dots, N_T.$$

P r o o f. From (8) and (11) we express

$$f\big(N_p(u^n)\big) = \min\left\{\sqrt{\frac{1}{|p|}\sum_{\sigma\in\mathcal{E}_p}\frac{|\sigma|}{d_{p\sigma}}(u_\sigma - u_p)^2 + a}, b\right\}, \qquad \forall p \in \mathcal{M}, \quad \forall u \in H_{\mathcal{D}}. \tag{47}$$

and from (19) and (17) we can derive

$$K_p^n = \frac{1}{|p|}\sum_{\sigma\in\mathcal{E}_p}\frac{|\sigma|}{d_{p\sigma}}\frac{(u_\sigma^n - u_p^n)}{f(N_p(u^n))}. \tag{48}$$

We rearrange the second term of right-hand side of (44) and we obtain

$$u_p^{n+1} = u_p^n + \frac{\tau}{|p|^{\frac{1}{3}}}\left(f\big(N_p(u^n)\big)^2 \sum_{\sigma\in\mathcal{E}_p}\frac{|\sigma|}{d_{p\sigma}}\big(u_\sigma^n - u_p^n\big)\right)^{\frac{1}{3}}. \tag{49}$$

Suppose first

$$\left|\sum_{\sigma\in\mathcal{E}_p}\frac{|\sigma|}{d_{p\sigma}}(u_\sigma^n - u_p^n)\right| \le a.$$

We can estimate from (49) using the properties of $f$

$$|u_p^{n+1}| \le |u_p^n| + \frac{\tau b^{\frac{2}{3}} a^{\frac{1}{3}}}{|p|^{\frac{1}{3}}}. \tag{50}$$

Using (45) we obtain

$$\big|u_p^{n+1}\big| \le \big|u_p^n\big| + C\tau \le \max_{p\in\mathcal{M}}\big|u_p^n\big| + C\tau, \qquad \forall p \in \mathcal{M}, \quad \forall n \in \mathbb{N}_0. \tag{51}$$

Suppose now

$$\left|\sum_{\sigma\in\mathcal{E}_p}\frac{|\sigma|}{d_{p\sigma}}(u_\sigma^n - u_p^n)\right| > a.$$

We rearrange (49)

$$u_p^{n+1} = u_p^n + \frac{\tau}{|p|^{\frac{1}{3}}} \left( \frac{f(N_p(u^n))}{\sum_{\sigma\in\mathcal{E}_p} \frac{|\sigma|}{d_{p\sigma}}(u_\sigma^n - u_p^n)} \right)^{\frac{2}{3}} \sum_{\sigma\in\mathcal{E}_p} \frac{|\sigma|}{d_{p\sigma}}(u_\sigma^n - u_p^n)$$

$$= u_p^n \left( 1 - \frac{\tau}{|p|^{\frac{1}{3}}} \left( \frac{f(N_p(u^n))}{\sum_{\sigma\in\mathcal{E}_p} \frac{|\sigma|}{d_{p\sigma}}(u_\sigma^n - u_p^n)} \right)^{\frac{2}{3}} \sum_{\sigma\in\mathcal{E}_p} \frac{|\sigma|}{d_{p\sigma}} \right) \tag{52}$$

$$+ \frac{\tau}{|p|^{\frac{1}{3}}} \left( \frac{f(N_p(u^n))}{\sum_{\sigma\in\mathcal{E}_p} \frac{|\sigma|}{d_{p\sigma}}(u_\sigma^n - u_p^n)} \right)^{\frac{2}{3}} \sum_{\sigma\in\mathcal{E}_p} \frac{|\sigma|}{d_{p\sigma}} u_\sigma^n.$$

Now for stability estimate the coefficient by the term $u_p^n$ must be positive, so the stability condition results

$$\frac{\tau}{|p|^{\frac{1}{3}}} \left( \frac{f(N_p(u^n))}{\sum_{\sigma\in\mathcal{E}_p} \frac{|\sigma|}{d_{p\sigma}}(u_\sigma^n - u_p^n)} \right)^{\frac{2}{3}} \sum_{\sigma\in\mathcal{E}_p} \frac{|\sigma|}{d_{p\sigma}} \le \frac{\tau}{|p|^{\frac{1}{3}}} \left(\frac{b}{a}\right)^{\frac{2}{3}} \sum_{\sigma\in\mathcal{E}_p} \frac{|\sigma|}{d_{p\sigma}} < 1$$

and we obtain

$$\tau < |p|^{\frac{1}{3}} \left(\frac{a}{b}\right)^{\frac{2}{3}} \sum_{\sigma\in\mathcal{E}_p} \frac{|\sigma|}{d_{p\sigma}} = C_1 |p| b^{-\frac{2}{3}} \sum_{\sigma\in\mathcal{E}_p} \frac{|\sigma|}{d_{p\sigma}} = C_2 |p|. \tag{53}$$

Using the condition (45) and (46) for this case we obtain similarly as in [4]

$$|u_p^{n+1}| \le \max_{p\in\mathcal{M}} |u_p^n|, \quad \forall p \in \mathcal{M}, \quad \forall n \in \mathbb{N}_0. \tag{54}$$

Collecting both cases we can conclude from (51) and (54) that

$$\forall p \in \mathcal{M}, \quad \forall n \in \mathbb{N}_T,$$

$$|u_p^{n+1}| \le \max_{p\in\mathcal{M}} |u_p^n| + C\tau \le \max_{p\in\mathcal{M}} \|u_p^0|, + CT, \tag{55}$$

which concludes the proof. □

## 5. Numerical experiments

We want to present the results obtained by proposed numerical schemes and therefore, we use the example where the exact solution of (4) is well-known. This example was used in [3] and [2]. The exact solution is of the form

$$u(x, y, t) = \max\Big\{ 1 - (x^2 + y^2)^{\frac{2}{3}} - \frac{4}{3}t, 0 \Big\}^2. \tag{56}$$

Initial condition is obtained from the exact solution for $t = 0$. We use the homogeneous Dirichlet boundary condition. Our domain $\Omega$ in this case is a square $\Omega = [-2, 2] \times [-2, 2]$. The time interval is $I = [0, 0.4]$.

One can see the shape of the exact solution and cuts of the solution for $y = 0$ for time $T = 0; 0.2; 0.4$ in Figure 1.

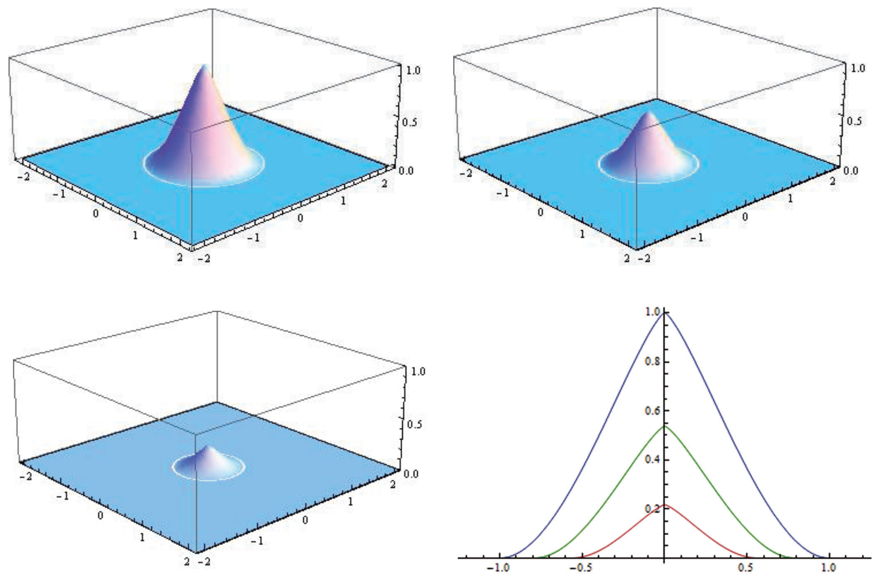

FIGURE 1. Initial condition (top left) and the exact solution at time T=0.2 (top right), the exact solution at time T=0.4 (bottom left) and the shape of cuts ($y = 0$) for initial condition (blue) and the exact solution at time T=0.2 (green), T=0.4 (red).

The results are computed for all numerical schemes presented at the end of previous section:

- explicit (EX),
- semi-implicit (SI),
- fully-implicit (FI),
- Crank-Nicolson (CN).

There are several parameters and relations of parameters and also several choices of the methods that can be used in proposed numerical schemes. We did many experiments for all of them and here we present the best results with parameters were the criterion of error versus CPU time we took in mind.

For all methods we use the parameter of the scheme $b = 10^8$ and $a = 10^{-12}$. For this choice the explicit method is valid for $C_1 \leq 10^{-11}$ for all used meshes. We did it in this way to have the same regularization parameters for all methods. There is also possibility to select the parameter $a = C_1|p|$ for all methods, but in this case results obtained for coarser meshes are worse. We use the uniform mesh and $h = |\sigma| > 0$ is the measure of the edge for the finite volume. For solving the linear algebraic system, we used SOR method with parameter $\alpha = 1.6$ and the stopping criterion was denoted by TOL. For fully-implicit and Crank-Nicolson schemes we used nonlinear iterations, too and the stopping criterion was denoted by TOLN.

In the tables below we present the errors obtained by the numerical schemes and the experimental order of convergence (EOC). The considered error is

$$E_2 = \|u_{\mathcal{D},\tau} - \bar{u}\|_{L^2(\Omega\times(0,T))}, \tag{57}$$

where numerical function is defined in (32) and by $\bar{u}$ we denote the value of exact solution given by (56), that means

$$\bar{u} = u(x_p, t_n) \quad \text{for } x \in p \quad \text{and} \quad t \in < t_{n-1}, t_n >.$$

Further by $N$ we denote the number of finite volumes along one side of the domain $\Omega$.

Parameters and notations for schemes are:

- For explicit method due to the conditional stability of the method we use the relation $\tau = \frac{h^2}{4}$ and denote this time step as $\tau_{\mathrm{EX}}$ and error from (57) is denoted by $E_2^{\mathrm{EX}}$.
- For semi-implicit scheme we use $\tau = h^2$ and denote this time step as $\tau_{\mathrm{SI}}$ and error from (57) is denoted by $E_2^{\mathrm{SI}}$. For this case the tolerance for linear solver SOR is $\mathrm{TOL} = 0.1h^2$ and average number of SOR iterations is about 12–15.
- For fully-implicit schemes we use $\tau = h^2$ and denote this time step as $\tau_{\mathrm{FI}}$ and error from (57) is denoted by $E_2^{\mathrm{FI}}$. For this case the tolerance for linear solver SOR is $\mathrm{TOL} = h^2$ and average number of iterations is about 5. For nonlinear iterations we use the tolerance $\mathrm{TOLN} = 0.1h^2$ and the average number of iterations for all computations is about 4.
- For the Crank-Nicolson scheme we use $\tau = \frac{h}{4}$ and denote this time step as $\tau_{\mathrm{CN}}$ and error from (57) is denoted by $E_2^{\mathrm{CN}}$. For this case the tolerance for linear solver SOR is $\mathrm{TOL} = h^2$ and average number of iterations is about 15. For nonlinear iterations we use the tolerance $\mathrm{TOLN} = 4h^2$ and the average number of iterations for all computations is about 9.

The results obtained for explicit scheme and semi-implicit scheme can be found in Table 1.

TABLE 1. Error reports and EOCs for explicit and semi-implicit schemes.

| $N$ | $\tau_{\mathrm{EX}}$ | $E_2^{\mathrm{EX}}$ | EOC | $\tau_{\mathrm{SI}}$ | $E_2^{\mathrm{SI}}$ | EOC |
|---|---|---|---|---|---|---|
| 20 | 1.0e-02 | 8.23e-03 | - | 4.0e-02 | 3.350e-02 | - |
| 40 | 2.50e-03 | 2.43e-03 | 1.762 | 1.0e-02 | 1.41e-02 | 1.307 |
| 80 | 6.25e-04 | 7.85-04 | 1.629 | 2.5e-03 | 5.40e-03 | 1.390 |
| 160 | 1.56e-04 | 3.29e-04 | 1.257 | 6.25e-04 | 1.95e-03 | 1.474 |
| 320 | 3.92e-05 | 1.64e-04 | 1.006 | 1.56e-04 | 6.89e-04 | 1.498 |

The results obtained for fully-implicit iterative scheme and C-N scheme are in Table 2.

TABLE 2. Error reports and EOCs for fully-implicit and Crank-Nicolson schemes.

| $N$ | $\tau_{\mathrm{FI}}$ | $E_2^{\mathrm{FI}}$ | EOC | $\tau_{\mathrm{CN}}$ | $E_2^{\mathrm{CN}}$ | EOC |
|---|---|---|---|---|---|---|
| 20 | 4.0e-02 | 1.50e-02 | - | 5.0e-02 | 2.49e-02 | - |
| 40 | 1.0e-02 | 4.45e-03 | 1.749 | 2.5e-02 | 1.25e-02 | 0.987 |
| 80 | 2.5e-03 | 1.42e-03 | 1.650 | 1.25-02 | 3.35e-03 | 1.906 |
| 160 | 6.25e-04 | 5.08e-04 | 1.482 | 6.25e-03 | 7.48e-04 | 2.161 |
| 320 | 1.56e-04 | 2.05e-04 | 1.306 | 3.125e-03 | 1.91e-04 | 1.972 |

From tables above, one can see that best results from EOC point of view can be achieved using the Crank-Nicolson scheme. In this case the EOC is nearly of second order. For semi-implicit and fully-implicit scheme we obtain EOC about 1.5 and for Explicit scheme the EOC decreases down to first order. One can realize that we do not know the accurate stability condition for this nonlinear equation although we know from previous section the estimation of stability condition. We try of course the relation $\tau = h^2$. Explicit scheme does not converge in this case. It is better if for the relation $\tau = Ch^2$ is the constant $C$ smaller but then the number of time steps is increasing and it cost much CPU time.

That is why we choose the relation $\tau = h^2/4$. And notice that for the finest scheme we can obtain the best $L_2$ error. We can obtain comparable results for $N = 160$ and $N = 320$ also for all iterative schemes that mean FI and CN for this case if we modify the tolerance TOLN, but it is worse for another $N$ and even more it is worse for comparison of error versus CPU time.

Numerical results obtained using the Crank-Nicolson scheme for $T = 0; 0.2; 0.4$ can be seen in Figure 2 for $N = 40$.

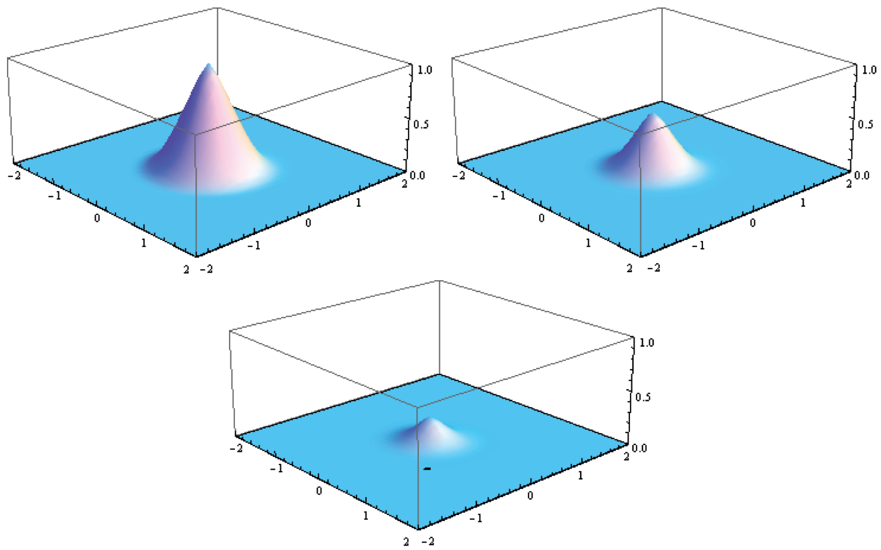


FIGURE 2. Initial condition (top left) and numerical solution at time T=0.2 (top right), numerical solution at time T=0.4 (bottom) for the Crank--Nicolson scheme for $N = 40$.

Next, we compare numerical schemes for error and CPU time point of view for this example. One can see the results in Log Log scale in Figure 3 for all presented schemes and in Figure 4 we can see the zoom focused to the smaller error. Explicit scheme for this comparison is the best to obtain error up to certain accuracy. For smallest error it is not true due to many time steps which must be computed.

Finally, we compare only three unconditionally stable schemes, i.e., semi-implicit scheme, fully-implicit scheme and the Crank-Nicolson scheme.

In Figure 5, there are results for these three schemes presented, and again the zoom on smaller errors is in Figure 6. From these Figures, it is obvious that C-N scheme achieves the best results, namely for smaller errors.

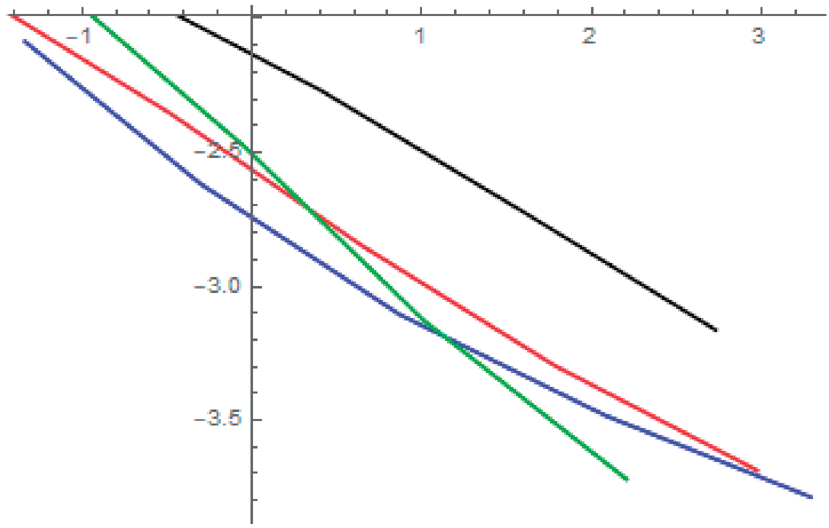


Figure 3. Comparison of error versus CPU time for explicit (blue), semi--implicit (black), fully-implicit (red) and C-N scheme (green) in Log Log scale.

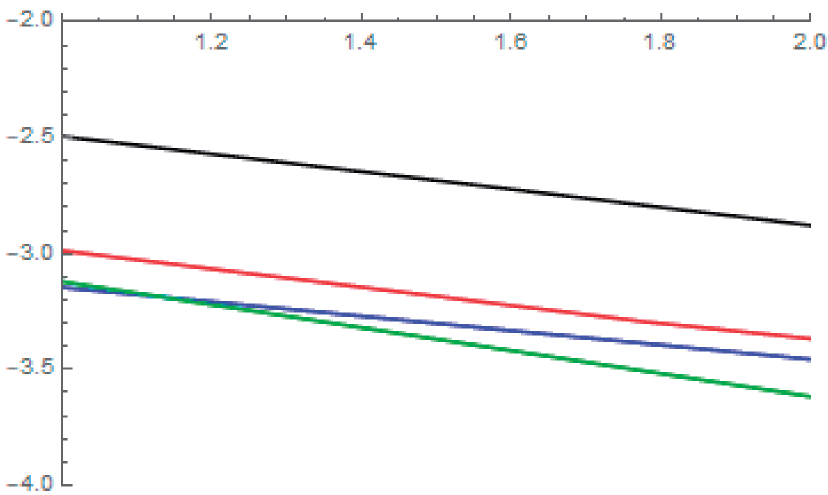


Figure 4. Comparison of error versus CPU time for explicit (blue), semi--implicit (black), fully-implicit (red) and C-N scheme (green) in Log Log scale.

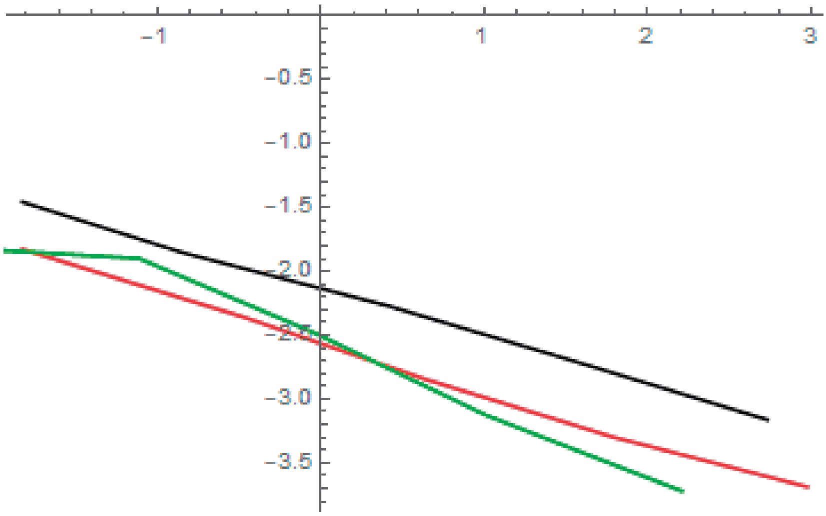


FIGURE 5. Comparison of error versus CPU time for semi-implicit (black), fully-implicit scheme (red) and C-N scheme (green) in Log Log scale.

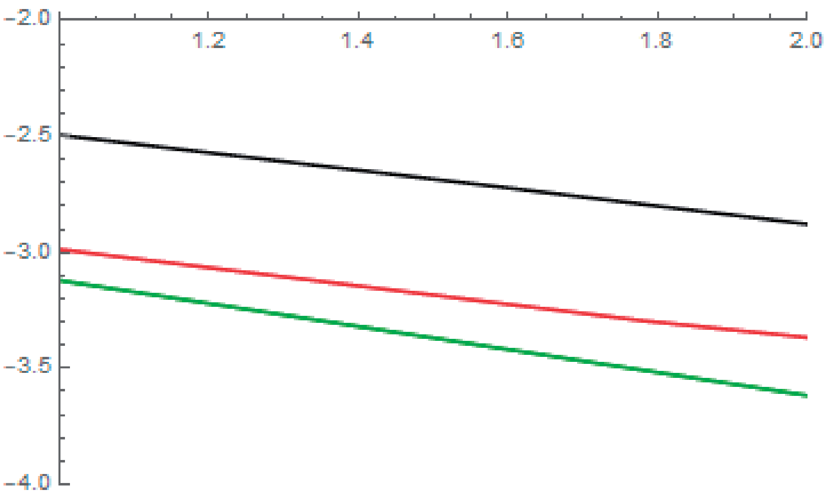


FIGURE 6. Comparison of error versus CPU time for semi-implicit (black), fully-implicit scheme (red) and C-N scheme (green) in Log Log scale.

## 6. Conclusion

We studied AMSS model from numerical point of view or more precisely its regularization form. We have presented numerical schemes based on the finite volume methodology of Crank-Nicolson type. Further, we focused especially on schemes, namely explicit, semi-implicit, fully-implicit and Crank-Nicolson schemes. For explicit and implicit schemes we have proved stability for the numerical solution. The stability of Crank-Nicolson scheme for AMSS model is an open question yet. On several experiments we have performed the errors and experimental order of convergence for the examples where the exact solution is well-known. Graphs where we compare the proposed numerical methods from error versus CPU time point of view are also included. For future study we want to show the numerical Affine Invariance Property of proposed scheme and prove the convergence results.

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