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EXISTENCE AND MULTIPLICITY OF POSITIVE SOLUTIONS FOR A THIRD-ORDER TWO-POINT BOUNDARY VALUE PROBLEM

Sergey Smirnov

University of Latvia, Riga, LATVIA

ABSTRACT. We study the existence and multiplicity of positive solutions for a third-order two-point boundary value problem by applying Krasnosel'skii's fixed point theorem. To illustrate the applicability of the obtained results, we consider some examples.

1. Introduction

We study a two-point boundary value problem consisting of the nonlinear third-order differential equation

$$x''' = a(t)f(x), \qquad 0 < t < 1, \tag{1}$$

and the boundary conditions

$$x(0) = 0, \qquad x(1) = 0, \qquad x'(1) = 0. \tag{2}$$

We assume throughout that $f\colon [0,\infty) \to [0,\infty)$ is continuous, $a\colon [0,1] \to [0,\infty)$ is continuous and does not vanish identically on any subinterval of $[0,1]$.

The purpose of the paper is to give results on the existence and multiplicity of positive solutions to (1), (2) by applying Krasnosel'skii's fixed point theorem. By a positive solution of (1), (2) we understand $C^3[0,1]$ function which is positive on $0 < t < 1$ and satisfies differential equation (1) for $0 < t < 1$ and boundary conditions (2). However, note that if $f(0) = 0$, then boundary value problem (1), (2) always has the trivial solution. Since $f(x)$ is not defined for $x < 0$,

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every solution of (1), (2) is nonnegative. We will show in the sequel that every nonnegative nontrivial solution of (1), (2) is positive.

In fact, our main results state that for each given positive integer n, we can indicate f so that problem (1), (2) has at least n positive solutions. To obtain these results, we first rewrite problem (1), (2) as an equivalent integral equation by constructing the corresponding Green's function. Then, we apply Krasnosel'skii's fixed point theorem in a cone [11, 16].

Much research has been done on two-point boundary value problems for third-order differential equations in the last decades. As recent contributions, we cite the papers of Cabada and Dimitrov [4], Gritsans and Sadyrbaev [10], Kelevedjiev and Todorov [15]. Krasnosel'skii's fixed point theorem in a cone has been widely used to study the existence and multiplicity of positive solutions of second and higher-order boundary value problems. One of the early works on the subject was a paper by Guo and Lakshmikantham [12]. Further research in this direction was carried out by Erbe and Wang [5], Henderson and Thompson [13], Anderson and Davis [2], Baxley and Haywood [3], Graef, Qian and Yang [6], Graef and Yang [7–9], Zhao, Wang and Ge [20]. For survey of known results and additional references we refer the reader to the monograph by Agarwal, O'Regan and Wong [1]. A great contribution to the development of the subject was made by Webb and Infante [14, 17–19]. The author would like to highlight the paper by Graef and Yang [7], which motivated the present investigation. We prove the existence of multiple positive solutions for our main problem in the same manner as in [7], but we use another splitting of the cone.

Since our main tool in this paper is Krasnosel'skii's fixed point theorem, let us state this theorem for the reader's convenience.

Theorem 1.1 (Krasnosel'skii, [16]). *Let E be a Banach space and $K \subset E$ be a cone in E. Assume Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$, $T : K \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right) \to K$ is a completely continuous operator such that:*

$$(A) \qquad \|Tx\| \le \|x\|,\ \forall x \in K \cap \partial\Omega_1 \quad \text{and} \quad \|Tx\| \ge \|x\|,\ \forall x \in K \cap \partial\Omega_2, \quad \text{or}$$

$$(B) \qquad \|Tx\| \ge \|x\|,\ \forall x \in K \cap \partial\Omega_1 \quad \text{and} \quad \|Tx\| \le \|x\|,\ \forall x \in K \cap \partial\Omega_2.$$

Then T has a fixed point in $K \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right)$.

The paper contains four sections besides the Introduction. In Section 2, we rewrite the main problem as an equivalent integral equation, by constructing the corresponding Green's function. Also, we give some inequalities for the Green's function here. In Section 3, we consider the existence of at least one positive solution for the problem. Section 4 is devoted to the existence of multiple positive solutions for the problem. Finally, we provide some examples to illustrate our results in Section 5.

2. Construction and estimation of the Green's function

Our first goal is to rewrite problem (1), (2) as an equivalent integral equation. Therefore, let us consider the linear equation

$$x''' = h(t), \ 0 < t < 1, \tag{3}$$

together with boundary conditions (2).

Proposition 2.1. *If $h : [0,1] \to \mathbb{R}$ is a continuous function, then boundary value problem* (3), (2) *has a unique solution*

$$\begin{aligned} x(t) = &\int_0^t \left[\frac{1}{2}\, s^2(1-t)^2\right] h(s)ds \\ &+ \int_t^1 \left[\frac{1}{2}\,(1-s)\,t\,\big((s-t)+(1-t)s\big)\right] h(s)ds, \end{aligned}$$

that we can rewrite as

$$x(t) = \int_0^1 G(t,s)h(s)ds,$$

where

$$G(t,s) = \frac{1}{2}\begin{cases} s^2(1-t)^2, & 0 \le s \le t \le 1, \\ (1-s)\,t\,\big((s-t)+(1-t)s\big), & 0 \le t \le s \le 1. \end{cases} \tag{4}$$

P r o o f. To prove the proposition we use the variation of parameters formula

$$x(t) = c_1 + c_2 t + c_3 t^2 + \frac{1}{2}\int_0^t (s-t)^2 h(s)ds.$$

In view of boundary conditions (2), we get $c_1 = 0$ and

$$c_2 = -\int_0^1 (s-1)^2 h(s)ds - \int_0^1 (s-1)h(s)ds,$$

$$c_3 = \frac{1}{2}\int_0^1 (s-1)^2 h(s)ds + \int_0^1 (s-1)h(s)ds.$$

Thus, we have

$$x(t) = \left(t^2 - t\right) \int_0^1 (s-1)h(s)ds + \left(\frac{1}{2}\,t^2 - t\right) \int_0^1 (s-1)^2 h(s)ds$$
$$+ \frac{1}{2}\int_0^t (s-t)^2 h(s)ds$$
$$= \int_0^t \left[\frac{1}{2}\,s^2(1-t)^2\right] h(s)ds + \int_t^1 \left[\frac{1}{2}\,(1-s)\,t\,\big((s-t)+(1-t)s\big)\right] h(s)ds.$$

The uniqueness follows from the fact, that the homogeneous problem $x''' = 0$, (2) has only the trivial solution. ☐

Hence boundary value problem (1), (2) is equivalent to the integral equation

$$x(t) = \int_0^1 G(t,s)a(s)f\big(x(s)\big)\,ds, \quad 0 \le t \le 1, \tag{5}$$

in the sense that x is a solution of (1), (2) if and only if it is a solution of (5). Here $G(t,s)$ denotes the Green's function for the problem $x''' = 0$, (2), and is explicitly given by (4).

In order to prove the existence of positive solutions, we also need some inequalities for the Green's function $G(t,s)$.

Proposition 2.2. *For all $(t,s) \in [0,1] \times [0,1]$, we have*

$$0 \le G(t,s) \le \frac{s^2(1-s)}{2(1+s)}. \tag{6}$$

If $(t,s) \in (0,1) \times (0,1)$, then

$$G(t,s) > 0. \tag{7}$$

P r o o f. The first part of inequality (6) and inequality (7) is obvious. Let us find the maximum of $G(t,s)$ for each s with respect to t.

For $0 \le s \le t \le 1$, the maximum occurs at

$$t = s \quad \text{and is equal to} \quad \frac{s^2(1-s)^2}{2}.$$

If $0 \le t \le s \le 1$, the maximum occurs at

$$t = \frac{s}{1+s} \quad \text{and is equal to} \quad \frac{s^2(1-s)}{2(1+s)}.$$

Since for all $(t,s) \in [0,1] \times [0,1]$,

$$\frac{s^2(1-s)^2}{2} = \frac{s^2(1-s)(1-s)(1+s)}{2(1+s)} = \frac{s^2(1-s)(1-s^2)}{2(1+s)} \le \frac{s^2(1-s)}{2(1+s)},$$

we get the proof. ☐

Note that inequality (6) is sharp.

Proposition 2.3. *For all* $(t,s) \in [1/14, 1/2] \times [0,1]$, *we have*

$$G(t,s) \geq \frac{1}{4} \cdot \frac{s^2(1-s)}{2(1+s)}. \tag{8}$$

P r o o f. Consider

$$\frac{G(t,s)}{\frac{s^2(1-s)}{2(1+s)}} = \begin{cases} \dfrac{(1-t)^2(1+s)}{(1-s)}, & 0 \leq s \leq t \leq 1, \\[2ex] \dfrac{t(1+s)\big((s-t)+(1-t)s\big)}{s^2}, & 0 \leq t \leq s \leq 1. \end{cases}$$

Further, for $\Lambda_1 = \{(t,s) : 1/14 \leq t \leq 1/2,\ 0 \leq s \leq 1,\ s \leq t\}$, we have

$$\min_{\Lambda_1} \frac{G(t,s)}{\frac{s^2(1-s)}{2(1+s)}} = \min_{\Lambda_1} \frac{(1-t)^2(1+s)}{(1-s)} = \frac{1}{4}.$$

If $\Lambda_2 = \{(t,s) : 1/14 \leq t \leq 1/2,\ 0 \leq s \leq 1,\ t \leq s\}$, then

$$\min_{\Lambda_2} \frac{G(t,s)}{\frac{s^2(1-s)}{2(1+s)}} = \min_{\Lambda_2} \frac{t(1+s)\big((s-t)+(1-t)s\big)}{s^2} = \frac{13}{49}.$$

Therefore,

$$\frac{G(t,s)}{\frac{s^2(1-s)}{2(1+s)}} \geq \frac{1}{4} \quad \text{for} \quad \frac{1}{14} \leq t \leq \frac{1}{2}, \quad 0 \leq s \leq 1. \qquad \square$$

Proposition 2.4. *Every nonnegative nontrivial solution* $x(t)$ *of* (1), (2) *is positive.*

P r o o f. Suppose that there exists $t_0 \in (0,1)$ such that $x(t_0) = 0$. Since boundary value problem (1), (2) is equivalent to integral equation (5) we get

$$x(t_0) = \int_0^1 G(t_0,s)a(s)f\big(x(s)\big)\,ds = 0.$$

Since $G(t_0,s)a(s)f\big(x(s)\big) \geq 0$ for all $s \in [0,1]$, then

$$G(t_0,s)a(s)f\big(x(s)\big) = 0 \quad \text{for all} \quad s \in [0,1].$$

Since $G(t_0,s) > 0$ for all $s \in (0,1)$ we get that $x'''(s) = a(s)f\big(x(s)\big) = 0$ for all $s \in (0,1)$. Therefore, $x(s)$ is a polynomial of degree at most two. Since $x(s)$ satisfies boundary conditions (2) it follows that $x(s) = 0$ for all $s \in [0,1]$. We get the contradiction. $\square$

3. Existence of positive solutions

In this section, we study the existence of positive solutions for boundary value problem (1), (2). Throughout, we shall use the following notations:

$$f_0 = \lim_{x\to 0+} \frac{f(x)}{x}, \qquad f_\infty = \lim_{x\to +\infty} \frac{f(x)}{x},$$

$$I_1 = \left(\int\limits_0^1 \frac{s^2(1-s)}{2(1+s)}\, a(s)ds \right)^{-1} < \left(\max_{0\le t\le 1} \int\limits_{1/14}^{1/2} G(t,s)\, a(s)ds \right)^{-1} = I_2.$$

For our constructions, consider the Banach space $C[0,1]$ with the norm

$$\|x\| = \max_{0\le t\le 1} |x(t)|, \ x \in C[0,1].$$

Define a cone K in $C[0,1]$ by

$$K = \left\{ x \in C[0,1] \;\middle|\; x(t) \ge 0, \ \min_{\frac{1}{14}\le t\le \frac{1}{2}} x(t) \ge \frac{1}{4}\|x\| \right\},$$

and an integral operator $T : K \to C[0,1]$ by

$$(Tx)(t) = \int\limits_0^1 G(t,s)a(s)f\big(x(s)\big)ds, \quad 0 \le t \le 1.$$

It is easy to see that boundary value problem (1), (2) has a solution x if and only if x is a fixed point of the operator T. Also, it is well-known that $T : K \to C[0,1]$ is a completely continuous operator.

Proposition 3.1. *$T(K) \subset K$.*

P r o o f. From inequality (6), it follows that for $x \in K$, $(Tx)(t) \ge 0$ on $[0,1]$. Also, for $x \in K$, we have from (6) that

$$(Tx)(t) = \int\limits_0^1 G(t,s)a(s)f\big(x(s)\big)ds \le \int\limits_0^1 \frac{s^2(1-s)}{2(1+s)}\, a(s)f\big(x(s)\big)ds,$$

so that

$$\|Tx\| \le \int\limits_0^1 \frac{s^2(1-s)}{2(1+s)}\, a(s)f\big(x(s)\big)ds. \tag{9}$$

And next, if $x \in K$, we have by (8) and (9) ,

$$\min_{\frac{1}{14}\le t\le\frac{1}{2}} (Tx)(t) = \min_{\frac{1}{14}\le t\le\frac{1}{2}} \int\limits_0^1 G(t,s)a(s)f\big(x(s)\big)ds$$

$$\ge \frac{1}{4}\int\limits_0^1 \frac{s^2(1-s)}{2(1+s)}\, a(s)f\big(x(s)\big)ds \ge \frac{1}{4}\|Tx\|. \qquad \square$$

Theorem 3.2. *If $f_0 < I_1$ and $f_\infty > 4\,I_2$ (particularly $f_0 = 0$ and $f_\infty = \infty$-superlinear case), then boundary value problem* (1), (2) *has at least one positive solution.*

P r o o f. Since $f_0 < I_1$, we may choose $r_1 > 0$ such that $f(x) \le I_1 x$ for $0 < x \le r_1$. Thus, if $x \in K$ and $\|x\| = r_1$, then

$$\begin{aligned}(Tx)(t) &= \int_0^1 G(t,s)a(s)f\big(x(s)\big)ds \le \int_0^1 \frac{s^2(1-s)}{2(1+s)}\,a(s)f\big(x(s)\big)ds\\ &\le I_1\|x\| \int_0^1 \frac{s^2(1-s)}{2(1+s)}\,a(s)ds = \|x\|, \qquad 0 \le t \le 1.\end{aligned}$$

Now, if we let $\Omega_1 = \{x \in C[0,1] \mid \|x\| < r_1\}$, then $\|Tx\| \le \|x\|$ for $x \in K \cap \partial\Omega_1$.

On the other hand, since $f_\infty > 4\,I_2$, there exists $\widehat{r}_2 > 0$ such that $f(x) \ge 4\,I_2 x$ for $x \ge \widehat{r}_2$. Let $r_2 = \max\{2\,r_1, 4\,\widehat{r}_2\}$ and $\Omega_2 = \{x \in C[0,1] \mid \|x\| < r_2\}$. Then $x \in K$ and $\|x\| = r_2$ implies

$$\min_{\frac{1}{14} \le t \le \frac{1}{2}} x(t) \ge \frac{1}{4}\,\|x\| = \frac{1}{4}\,r_2 \ge \widehat{r}_2.$$

Thus, if $s \in [1/14, 1/2]$, then $x(s) \in [\widehat{r}_2, r_2]$, and we get

$$\begin{aligned}\|Tx\| &= \max_{0 \le t \le 1} \int_0^1 G(t,s)a(s)f\big(x(s)\big)ds \ge \max_{0 \le t \le 1} \int_{1/14}^{1/2} G(t,s)a(s)f\big(x(s)\big)ds\\ &\ge 4\,I_2 \max_{0 \le t \le 1} \int_{1/14}^{1/2} G(t,s)a(s)x(s)ds\\ &\ge 4\,I_2\,\frac{1}{4}\,\|x\| \max_{0 \le t \le 1} \int_{1/14}^{1/2} G(t,s)a(s)ds = \|x\|.\end{aligned}$$

Hence, $\|Tx\| \ge \|x\|$ for $x \in K \cap \partial\Omega_2$.

Therefore, from Theorem 1.1, it follows that T has a fixed point in $K \cap \big(\overline{\Omega}_2 \setminus \Omega_1\big)$. Since the set $K \cap \big(\overline{\Omega}_2 \setminus \Omega_1\big)$ does not contain zero element (trivial solution), it follows that this fixed point yields a positive solution of (1), (2). $\square$

Theorem 3.3. *If $f_0 > 4\,I_2$ and $f_\infty < I_1$ (particularly $f_0 = \infty$ and $f_\infty = 0$-sublinear case), then boundary value problem* (1), (2) *has at least one positive solution.*

Proof. Since $f_0 > 4\,I_2$, we may choose $r_1 > 0$ such that $f(x) \geq 4\,I_2 x$ for $0 < x \leq r_1$. Then, for $x \in K$ and $\|x\| = r_1$, we have

$$\begin{aligned}\|Tx\| &= \max_{0\leq t\leq 1}\int_0^1 G(t,s)a(s)f\big(x(s)\big)ds \geq \max_{0\leq t\leq 1}\int_{1/14}^{1/2} G(t,s)a(s)f\big(x(s)\big)ds\\ &\geq 4\,I_2\max_{0\leq t\leq 1}\int_{1/14}^{1/2} G(t,s)a(s)x(s)ds\\ &\geq 4\,I_2\,\frac{1}{4}\,\|x\|\max_{0\leq t\leq 1}\int_{1/14}^{1/2} G(t,s)a(s)ds = \|x\|.\end{aligned}$$

Thus, we may let $\Omega_1 = \{x \in C[0,1] \bigm| \|x\| < r_1\}$ so that $\|Tx\| \geq \|x\|$ for $x \in K \cap \partial\Omega_1$.

Now, since $f_\infty < I_1$, there exists $\widehat{r}_2 > 0$ so that $f(x) \leq I_1 x$ for $x \geq \widehat{r}_2$.

Case (a): Suppose f is bounded, say $f(x) \leq N$ for all $x \in (0,\infty)$. In this case choose $r_2 = \max\big\{2\,r_1,\ N\,I_1^{-1}\big\}$ so that for $x \in K$ with $\|x\| = r_2$ we have

$$(Tx)(t) = \int_0^1 G(t,s)a(s)f\big(x(s)\big)ds \leq N\int_0^1 \frac{s^2(1-s)}{2(1+s)}\,a(s)ds \leq r_2$$

and therefore, we get $\|Tx\| \leq \|x\|$.

Case (b): If f is unbounded, we can choose $r_2 > \max\{2\,r_1,\ \widehat{r}_2\}$ such that $f(x) \leq f(r_2)$ for $0 < x \leq r_2$. Then for $x \in K$ and $\|x\| = r_2$ we have

$$\begin{aligned}(Tx)(t) &= \int_0^1 G(t,s)a(s)f\big(x(s)\big)ds \leq \int_0^1 \frac{s^2(1-s)}{2(1+s)}\,a(s)f\big(x(s)\big)ds\\ &\leq \int_0^1 \frac{s^2(1-s)}{2(1+s)}\,a(s)f(r_2)ds \leq I_1\,r_2\int_0^1 \frac{s^2(1-s)}{2(1+s)}\,a(s)ds = r_2\\ &= \|x\|, \qquad 0 \leq t \leq 1.\end{aligned}$$

Therefore, in both cases, we may put $\Omega_2 = \{x \in C[0,1] \bigm| \|x\| < r_2\}$ and for $x \in K\cap\partial\Omega_2$ we have $\|Tx\| \leq \|x\|$. Thus, from Theorem 1.1, it follows that T has a fixed point in $K \cap \big(\overline{\Omega}_2 \setminus \Omega_1\big)$. Since the set $K \cap \big(\overline{\Omega}_2 \setminus \Omega_1\big)$ does not contain zero element (trivial solution), it follows that this fixed point yields a positive solution of (1), (2). $\square$

Theorem 3.4. *If there exist constants* $0 < r_1 < r_2$ *such that* $f(x) \le I_1 r_1$ *for* $x \in [0, r_1]$ *and* $f(x) \ge I_2 r_2$ *for* $x \in [r_2/4, r_2]$, *then boundary value problem* (1), (2) *has at least one positive solution* $x(t)$ *such that* $r_1 \le \|x\| \le r_2$.

Proof. If $x \in K$ and $\|x\| = r_1$, then

$$(Tx)(t) = \int_0^1 G(t,s)a(s)f\big(x(s)\big)ds \le \int_0^1 \frac{s^2(1-s)}{2(1+s)}\,a(s)f\big(x(s)\big)ds$$

$$\le I_1 r_1 \int_0^1 \frac{s^2(1-s)}{2(1+s)}\,a(s)ds = r_1 = \|x\|, \qquad 0 \le t \le 1.$$

If we let $\Omega_1 = \{x \in C[0,1] \bigm| \|x\| < r_1\}$, then $\|Tx\| \le \|x\|$ for $x \in K \cap \partial\Omega_1$.

If $x \in K$ with $\|x\| = r_2$, then for every $s \in [1/14, 1/2]$ we have

$$\min_{\frac{1}{14} \le s \le \frac{1}{2}} x(s) \ge \frac{1}{4}\|x\| = \frac{1}{4} r_2 \quad \text{and} \quad x(s) \in [r_2/4, r_2].$$

Therefore,

$$\|Tx\| = \max_{0 \le t \le 1} \int_0^1 G(t,s)a(s)f\big(x(s)\big)ds \ge \max_{0 \le t \le 1} \int_{1/14}^{1/2} G(t,s)a(s)f\big(x(s)\big)ds$$

$$\ge I_2 r_2 \max_{0 \le t \le 1} \int_{1/14}^{1/2} G(t,s)a(s)ds = r_2 = \|x\|.$$

Thus, we may let $\Omega_2 = \{x \in C[0,1] \bigm| \|x\| < r_2\}$ so that $\|Tx\| \ge \|x\|$ for $x \in K \cap \partial\Omega_2$.

Therefore, from Theorem 1.1, it follows that T has a fixed point in $K \cap \big(\overline{\Omega}_2 \setminus \Omega_1\big)$. Since the set $K \cap \big(\overline{\Omega}_2 \setminus \Omega_1\big)$ does not contain zero element (trivial solution), it follows that this fixed point yields a positive solution of (1), (2). $\square$

4. Existence of multiple positive solutions

The following two propositions will be used in the proof of our main result of this section.

Proposition 4.1. *Suppose that there exists* $r > 0$ *such that* $f(x) \le I_1 \frac{4}{5} r$ *for* $x \in [0, r]$. *If* $x \in K$ *with* $\|x\| = \rho$, *where* $\frac{4}{5} r \le \rho \le r$, *then* $\|Tx\| \le \rho$.

P r o o f. If $x \in K$ with $\|x\| = \rho$, then for $t \in [0,1]$ we have

$$(Tx)(t) = \int_0^1 G(t,s)a(s)f\big(x(s)\big)ds \le \int_0^1 \frac{s^2(1-s)}{2(1+s)}\, a(s)f\big(x(s)\big)ds$$

$$\le I_1\, \frac{4}{5}\, r \int_0^1 \frac{s^2(1-s)}{2(1+s)} a(s)ds = \frac{4}{5}\, r \le \rho, \quad \text{or} \quad \|Tx\| \le \rho. \qquad \square$$

Proposition 4.2. *Suppose that there exists $r > 0$ such that $f(x) \ge I_2\, r$ for $x \in [r/5, r]$. If $x \in K$ with $\|x\| = \rho$, where $\frac{4}{5}\, r \le \rho \le r$, then $\|Tx\| \ge \rho$.*

P r o o f. If $x \in K$ with $\|x\| = \rho$, where $\frac{4}{5}\, r \le \rho \le r$, then for every $s \in [1/14, 1/2]$ we have

$$\min_{\frac{1}{14} \le s \le \frac{1}{2}} x(s) \ge \frac{1}{4}\, \|x\| = \frac{1}{4}\, \rho \quad \text{and} \quad x(s) \in [\rho/4, \rho] \quad \text{or} \quad x(s) \in [r/5, r].$$

Therefore,

$$\|Tx\| = \max_{0 \le t \le 1} \int_0^1 G(t,s)a(s)f\big(x(s)\big)ds \ge \max_{0 \le t \le 1} \int_{1/14}^{1/2} G(t,s)a(s)f\big(x(s)\big)ds$$

$$\ge I_2 r \max_{0 \le t \le 1} \int_{1/14}^{1/2} G(t,s)a(s)ds = r \ge \rho. \qquad \square$$

Now, we are ready to give the main result of this section. Let:

- $\lfloor \alpha \rfloor = \max\{k \in \mathbb{Z} \big|\ k \le \alpha\}$ be the floor of a real number α and
- $\lceil \alpha \rceil = \min\{k \in \mathbb{Z} \big|\ k \ge \alpha\}$ be the ceiling of α.

Theorem 4.3. *Suppose that there exist constants $0 < r_1 < r_2 < \cdots < r_n (n \ge 3)$. If $f(x) \le I_1\, \frac{4}{5}\, r_{2i-1}$ for $x \in [0, r_{2i-1}]$, $1 \le i \le \left\lceil \frac{n}{2} \right\rceil$ and $f(x) \ge I_2\, r_{2j}$ for $x \in [r_{2j}/5, r_{2j}]$, $1 \le j \le \left\lfloor \frac{n}{2} \right\rfloor$, then boundary value problem (1), (2) has at least $(n-1)$ positive solutions $x_k(t)$ such that*

$$r_k \le \|x_k\| < r_{k+1}, \quad 1 \le k \le (n-1).$$

P r o o f. Let $l_m \in (4\, r_m/5, r_m)$ and $l_m > r_{m-1}$, $2 \le m \le n$. Define

$$\Omega_m^{l_m} = \left\{x \in C[0,1] \;\middle|\; \|x\| < l_m\right\}, \qquad 2 \le m \le n,$$

and

$$\Omega_s^{r_s} = \left\{x \in C[0,1] \;\middle|\; \|x\| < r_s\right\}, \qquad 1 \le s \le n.$$

Then, from Proposition 4.1 and Proposition 4.2, we have:

$$\begin{aligned}
\|Tx\| \le \|x\| \quad &\text{for} \quad x \in K \cap \partial\Omega_{2p-1}^{l_{2p-1}}, && 2 \le p \le \left\lceil \frac{n}{2} \right\rceil, \\
\text{and} \quad &\text{for} \quad x \in K \cap \partial\Omega_{2i-1}^{r_{2i-1}}, && 1 \le i \le \left\lceil \frac{n}{2} \right\rceil; \\
\|Tx\| \ge \|x\| \quad &\text{for} \quad x \in K \cap \partial\Omega_{2j}^{l_{2j}}, && 1 \le j \le \left\lfloor \frac{n}{2} \right\rfloor, \\
\text{and} \quad &\text{for} \quad x \in K \cap \partial\Omega_{2j}^{r_{2i}}, && 1 \le j \le \left\lfloor \frac{n}{2} \right\rfloor.
\end{aligned}$$

From Theorem 1.1, we see that T has fixed point in each of the sets $K \cap \left(\overline{\Omega}_m^{l_m} \setminus \Omega_{m-1}^{r_{m-1}}\right)$, $2 \le m \le n$. Thus, boundary value problem (1), (2) has at least $(n-1)$ positive solutions. □

5. Examples

In this section, we consider some examples to illustrate the applicability of the results established in Section 3 and Section 4.

Example. Consider boundary value problem consisting of the differential equation

$$x''' = 100\,\frac{x + 5\,x^3}{3 + x^2}, \qquad 0 < t < 1, \tag{10}$$

and boundary conditions (2). We have

$$a(t) = 100, \quad f(x) = \frac{x + 5x^3}{3 + x^2},$$

and therefore, $f_0 = \frac{1}{3}$, $f_\infty = 5$, $I_1 \approx 0.3776$, $4I_2 \approx 4.405$. Since $f_0 < I_1$ and $f_\infty > 4\,I_2$, by Theorem 3.2, boundary value problem (10), (2) has at least one positive solution, which is depicted in Figure 1. The initial conditions for this solution are:

$$x(0) = 0, \quad x'(0) \approx 6.8852, \quad x''(0) \approx -30.9744.$$

Example. Consider boundary value problem for the differential equation

$$x''' = 100\,t\left(\frac{3x}{5} + \frac{12x}{1 + x^2}\right), \quad 0 < t < 1, \tag{11}$$

with boundary conditions (2). Here

$$a(t) = 100\,t, \quad f(x) = \frac{3x}{5} + \frac{12x}{1 + x^2},$$

and therefore, $f_0 = 12.6$, $f_\infty = 0.6$, $I_1 \approx 0.6585$, $4\,I_2 \approx 12.253$. Since $f_0 > 4\,I_2$ and $f_\infty < I_1$, by Theorem 3.3, boundary value problem (11), (2) has at least one positive solution, which is depicted in Figure 2. The initial conditions for this solution are $x(0) = 0$, $x'(0) \approx 47.2081$, $x''(0) \approx -143.215$.

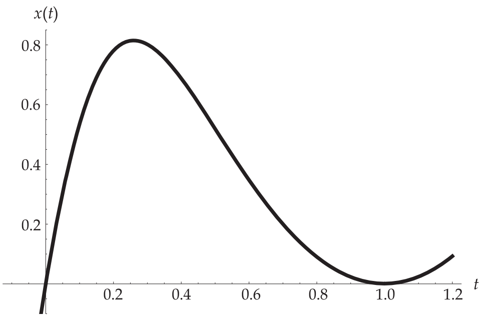

FIGURE 1.
Solution of (10), (2).

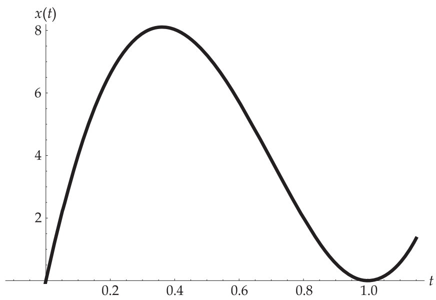

FIGURE 2.
Solution of (11), (2).

EXAMPLE. Consider boundary value problem for the differential equation

$$x''' = 10\, x^2\, e^{6-x}, \qquad 0 < t < 1, \tag{12}$$

with boundary conditions (2). We have:

$$a(t) = 10, \quad f(x) = x^2\, e^{6-x}, \quad \frac{4}{5}\, I_1 \approx 3.021, \quad I_2 \approx 11.013.$$

If we choose $r_1 = 0.006$, $r_2 = 5$, $r_3 = 73$, we get

$$\begin{aligned} f(x) &\le I_1\, \frac{4}{5}\, r_1 \quad \text{for} \quad x \in [0, r_1], \\ f(x) &\le I_1\, \frac{4}{5}\, r_3 \quad \text{for} \quad x \in [0, r_3], \\ f(x) &\ge I_2\, r_2 \qquad \text{for} \quad x \in [r_2/5, r_2]. \end{aligned}$$

Therefore, by Theorem (4.3), boundary value problem (12), (2) has at least two positive solutions $x_1(t)$ and $x_2(t)$ such that

$$0.006 \le \|x_1\| < 5 \le \|x_2\| < 73.$$

Solutions $x_1(t)$ and $x_2(t)$ are depicted in Figure 3 and Figure 4. The initial conditions for these solutions are $x_1(0) = 0,\ x_1'(0) \approx 0.2618675,\ x_1''(0) \approx -1.189665$ and $x_2(0) = 0,\ x_2'(0) \approx 57.148,\ x_2''(0) \approx -297.923$.

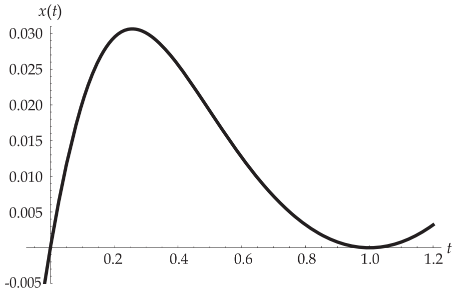

Figure 3.
Solution x_1 of (12), (2).

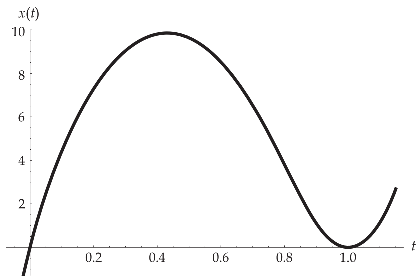

Figure 4.
Solution x_2 of (12), (2).

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Sergey Smirnov
Institute of Mathematics and Computer Science
University of Latvia
Raina bulvaris 29
LV-1459 Riga
LATVIA

Faculty of Physics, Mathematics and Optometry
University of Latvia
Jelgavas iela 3
LV-1004 Riga
LATVIA
E-mail: srgsm@inbox.lv