PROPERTIES OF THE KATUGAMPOLA FRACTIONAL OPERATORS

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ABSTRACT. In this work, there are considered higher order fractional operators defined in the sense of Katugampola. There are proved some fundamental properties of the Katugampola fractional operators of any arbitrary real order. Moreover, there are given conditions ensuring existence of the higher order Katugampola fractional derivative in space of the absolutely continuous functions.

1. Introduction

In 2011 Udita N. Katugampola introduced (see [1], [2]) new fractional operators, which were named after his surname, i.e., the Katugampola fractional integral and the Katugampola fractional derivative. Such operators depend on some extra parameter $\rho > 0$, which by taking $\rho \to 0^+$ reduce to the Hadamard fractional operators, and for parameter $\rho = 1$ become the Riemann-Liouville fractional operators. Thanks to this, the use of the Katugampola fractional operators simplifies the theory. If we prove something for the Katugampola derivative, we get this fact for both the Riemann-Liouville derivative and the Hadamard derivative.

Nowadays, these operators are gaining more and more popularity. The Katugampola fractional derivative is widely discussed in the literature. They have applications in fields such as probability theory [3], theory of inequalities [4,5], differential equations [6,7], Mellin transforms [8], maximum principle [9] and chaos and stability [10]. So far, in the literature, the Katugampola fractional operators occur only up to the order 2. In this work, the author is interested in the Katugampola derivative of the higher order. There is considered the existence of the Katugampola fractional derivative of an arbitrary real order $\alpha > 0$.

Moreover, in the last section, there are described some useful properties of these operators, which can be used to further research on the subject, where is still much to be done.

2. Preliminaries

In this paper, we denote by $\mathbb{N}$ a set of natural numbers $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

In this section, we give the definitions of the Katugampola fractional integrals and fractional derivatives on a finite positive interval of the real line $[a, b]$ $(0 < a < b < \infty)$ with examples.

To formulate it, we need the following special functions:

- Gamma function defined for $x > 0$

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt,$$

  which, for $n \in \mathbb{N}$, has the following properties

$$\Gamma(n) = (n-1)!, \tag{1}$$

$$\Gamma(n) = n\Gamma(n-1).$$

- Beta function defined for $x, y > 0$

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} \, dt \tag{2}$$

which can be written as

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{3}$$

Moreover, let $L[a, b]$ be space of integrable functions $f : (a, b) \to \mathbb{R}$ such that

$$\int_a^b |f(t)| \, dt < \infty.$$

DEFINITION 2.1. Let $n \in \mathbb{N}$, $n - 1 < \alpha < n$, $\rho > 0$, $0 < a < b < \infty$. The operators

$$I_{a+}^{\alpha,\rho} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}} f(\tau) \, d\tau,$$

$$I_{b-}^{\alpha,\rho} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_t^b \frac{\tau^{\rho-1}}{(\tau^\rho - t^\rho)^{1-\alpha}} f(\tau) \, d\tau,$$

for $t \in (a, b)$ are called the left-sided and right-sided Katugampola integrals of fractional order α, provided they exist.

In particular,

$$I_{a+}^{\alpha,1} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} \, d\tau \qquad \text{(Riemman-Lioville fractional integral)}$$

$$\lim_{\rho \to 0^+} I_{a+}^{\alpha,\rho} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} f(\tau) \frac{d\tau}{\tau} \quad \text{(Hadamard fractional integral)},$$

and

$$I_{a+}^{n,1} f(t) = \int_a^t \int_a^{t_1} \int_a^{t_2} \cdots \int_a^{t_{n-1}} f(\tau) d\tau dt_{n-1} \ldots dt_1, \quad n \in \mathbb{N}.$$

Let us introduce the notation δ_ρ for δ_ρ-derivative defined by $\delta_\rho := t^{1-\rho} \frac{d}{dt}$. We can easily prove that for $n \in \mathbb{N}$

$$\delta_\rho^n I_{a+}^{n,\rho} f(t) = f(t), \tag{4}$$

provided $I_{a+}^{n,\rho} f$ exists.

DEFINITION 2.2. Let $n \in \mathbb{N}$, $\rho > 0$, $0 < a < b < \infty$, $n-1 < \alpha < n$. The operators

$$D_{a+}^{\alpha,\rho} f(t) = \delta_\rho^n I_{a+}^{n-\alpha,\rho} f(t)$$

$$D_{b-}^{\alpha,\rho} f(t) = (-1)^n \delta_\rho^n I_{b-}^{n-\alpha,\rho} f(t)$$

for $t \in (a, b)$ are called the left-sided and right-sided Katugampola derivatives of fractional order α, provided they exist.

In particular, for $\rho = 1$ and for $\rho \to 0^+$

$$D_{a+}^{\alpha,1} f(t) = \left(\frac{d}{dt}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} \, d\tau,$$

$$\lim_{\rho \to 0^+} D_{a+}^{\alpha,\rho} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(t \frac{d}{dt}\right)^n \int_a^t \left(\log \frac{t}{\tau}\right)^{n-\alpha-1} f(\tau) \frac{d\tau}{\tau},$$

we obtain the Riemman-Liouville fractional derivative and the Hadamard fractional derivative, respectively. Moreover,

$$D_{a+}^{0,\rho} f(t) = D_{b-}^{0,\rho} f(t) = f(t),$$

and when $\alpha = n \in \mathbb{N}$, then

$$D_{a+}^{n,\rho} f(t) = \delta_\rho^{n+1} I_{a+}^{1,\rho} f(t) = \left(t^{1-\rho} \frac{d}{dt} \right)^{n+1} \int_a^t \tau^{\rho-1} f(\tau)\, d\tau = \delta_\rho^n f(t),$$

and, analogously,

$$D_{b-}^{n,\rho} f(t) = (-1)^n \delta_\rho^n f(t).$$

Now, we present the examples which show that Katugampola fractional integral and differential operators of power functions return power functions.

EXAMPLE. If $\alpha > 0$, $\rho > 0$ and $\lambda > -1$ then

$$I_{a+}^{\alpha,\rho} \left(\frac{t^\rho - a^\rho}{\rho} \right)^\lambda = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + \alpha + 1)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha + \lambda},$$

$$I_{b-}^{\alpha,\rho} \left(\frac{b^\rho - t^\rho}{\rho} \right)^\lambda = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + \alpha + 1)} \left(\frac{b^\rho - t^\rho}{\rho} \right)^{\alpha + \lambda}.$$

EXAMPLE. For $\rho > 0$, $\alpha > 0$, $\lambda > \alpha - 1$, we have

$$D_{a+}^{\alpha,\rho} \left(\frac{t^\rho - a^\rho}{\rho} \right)^\lambda = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + 1 - \alpha)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\lambda - \alpha}, \tag{5}$$

and

$$D_{b-}^{\alpha,\rho} \left(\frac{b^\rho - t^\rho}{\rho} \right)^\lambda = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + 1 - \alpha)} \left(\frac{b^\rho - t^\rho}{\rho} \right)^{\lambda - \alpha}. \tag{6}$$

The equality (5) was proved in [14]. Formula (6) can be proved similarly by using substitution of the form $u = \dfrac{b^\rho - \tau^\rho}{b^\rho - t^\rho}$.

From the above, we have the following remark.

Remark 1. If $\alpha > 0$, $n = [\alpha] + 1$ and $\rho > 0$, then

$$D_{a+}^{\alpha,\rho} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha - j} = 0, \quad \text{for } j = 1, \ldots, n.$$

In this work, we are interested in the Katugampola fractional derivatives of order $\alpha \in (n-1, n]$, $n \in \mathbb{N}$. Therefore, in the following, we give conditions ensuring existence of the higher order Katugampola fractional derivatives in the space $AC_\rho^n[a, b]$ defined below.

Moreover, in Section 4 we describe some properties of the Katugampola fractional operators.

3. Existence of the Katugampola fractional derivative

Let $AC[a, b]$ be the space of functions f which are absolutely continuous on $[a, b]$ i.e.,

$$AC[a, b] := \left\{ f : [a, b] \to \mathbb{R} : f(t) = c + \int_a^t \varphi(s)\, ds \right\}, \qquad (7)$$

where c is an arbitrary real constant and $\varphi \in L[a, b]$. Definition (7) can be written as

$$AC[a, b] := \left\{ f : [a, b] \to \mathbb{R} : f(t) = \tilde{c} - \int_t^b \varphi(s)\, ds \right\}. \qquad (8)$$

The existence of the Katugampola fractional derivative for $\alpha \in (0, 1]$ was considered in [13], and case for $\alpha \in (1, 2]$ in [4]. In this work, we consider the Katugampola fractional derivative for any positive real α. For this purpose, let us introduce the space $AC_\rho^n[a, b]$, for $n \in \mathbb{N}$ which consists of those real-valued Lebesgue measurable functions f on (a, b) which have δ_ρ-derivatives up to order $n - 1$ and $\delta_\rho^{n-1} f$ is absolutely continuous on $[a, b]$:

$$AC_\rho^n[a, b] := \left\{ f : [a, b] \to \mathbb{R} : \delta_\rho^{n-1} f \in AC[a, b], \ \delta_\rho = t^{1-\rho} \frac{d}{dt} \right\}. \qquad (9)$$

If $\rho = 1$ and $n = 1$, the space $AC_1^1[a, b]$ coincides with $AC[a, b]$.

The space $AC_\rho^n[a, b]$ is characterized by the following result.

LEMMA 3.1. *Let $0 < a < b < \infty$, $\rho > 0$ and $n \in \mathbb{N}$. The space $AC_\rho^n[a, b]$ consists of those and only those functions f which can be represented in the form*

$$f(t) = \sum_{i=0}^{n-1} c_i \left(\frac{t^\rho - a^\rho}{\rho} \right)^i + \frac{1}{(n-1)!} \int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{n-1} \varphi(\tau)\, d\tau, \qquad (10)$$

or

$$f(t) = \sum_{i=0}^{n-1} c_i (-1)^i \left(\frac{b^\rho - t^\rho}{\rho} \right)^i + \frac{(-1)^{n-1}}{(n-1)!} \int_t^b \left(\frac{\tau^\rho - t^\rho}{\rho} \right)^{n-1} \varphi(\tau)\, d\tau, \qquad (11)$$

where c_i $(i = 0, 1, \ldots, n - 1)$ are arbitrary real constants and $\varphi \in L[a, b]$.

P r o o f. We will show only the proof of the formula (10). First, we prove necessity. Let $f \in AC_\rho^n[a, b]$. Then by (9) we have

$$\delta_\rho^{n-1} f \in AC[a, b], \quad \text{where} \quad \delta_\rho = t^{1-\rho} \frac{d}{dt}.$$

From (7) we get

$$\delta_\rho^{n-1} f(t) = c_{n-1} + \int_a^t \varphi(\tau)\, d\tau, \tag{12}$$

where c_{n-1} is an arbitrary constant and $\varphi \in L[a,b]$. In order to prove the formula (11), we should use the above (8) instead of (7) and the rest of the proof will be analogous. We can rewrite (12) in the form

$$\frac{d}{dt}\delta_\rho^{n-2} f(t) = t^{\rho-1} c_{n-1} + t^{\rho-1}\int_a^t \varphi(\tau)\, d\tau. \tag{13}$$

Now, integrating both sides of (13) we get

$$\delta_\rho^{n-2} f(t) = c_{n-1}\frac{t^\rho - a^\rho}{\rho} + c_{n-2} + \int_a^t \frac{t^\rho - \tau^\rho}{\rho}\varphi(\tau)\, d\tau,$$

where c_{n-1}, c_{n-2} are arbitrary constants. Repeating this process m ($1 \le m \le n-1$) times we obtain successively

$$\delta_\rho^{n-3} f(t) = c_{n-3} + c_{n-2}\frac{t^\rho - a^\rho}{\rho} + \frac{1}{2}c_{n-1}\left(\frac{t^\rho - a^\rho}{\rho}\right)^2$$
$$+ \frac{1}{2}\int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^2 \varphi(\tau)\, d\tau.$$

$$\delta_\rho^{n-4} f(t) = c_{n-4} + c_{n-3}\frac{t^\rho - a^\rho}{\rho} + \frac{1}{2}c_{n-2}\left(\frac{t^\rho - a^\rho}{\rho}\right)^2$$
$$+ \frac{1}{6}c_{n-1}\left(\frac{t^\rho - a^\rho}{\rho}\right)^3$$
$$+ \frac{1}{6}\int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^3 \varphi(\tau)\, d\tau.$$

$$\delta_\rho^{n-m} f(t) = \sum_{i=0}^{m-1} c_{n-m+i}\left(\frac{t^\rho - a^\rho}{\rho}\right)^i$$
$$+ \frac{1}{(m-1)!}\int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{m-1} \varphi(\tau)\, d\tau, \tag{14}$$

where $c_{n-1}, c_{n-2}, \ldots, c_{n-m}$ are arbitrary constants. Taking (14) with $m = n$, we obtain (10) and the necessity is proved.

Now, let f be represented by (10). Taking δ_ρ-derivative m times ($m \leq n-1$), we have, respectively,

$$\delta_\rho f(t) = \sum_{i=1}^{n-1} i c_i \left(\frac{t^\rho - a^\rho}{\rho} \right)^{i-1} + \frac{1}{(n-2)!} \int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{n-2} \varphi(\tau) \, d\tau,$$

$$\delta_\rho^2 f(t) = \sum_{i=2}^{n-1} i(i-1) c_i \left(\frac{t^\rho - a^\rho}{\rho} \right)^{i-2} + \frac{1}{(n-3)!} \int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{n-3} \varphi(\tau) \, d\tau,$$

$$\delta_\rho^m f(t) = \sum_{i=m}^{n-1} c_i \frac{i!}{(i-m)!} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{i-m}$$

$$+ \frac{1}{(n-m-1)!} \int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{n-m-1} \varphi(\tau) \, d\tau.$$

Thus, for $m = n-1$ we obtain

$$\delta_\rho^{n-1} f(t) = c_{n-1}(n-1)! + \int_a^t \varphi(\tau) \, d\tau,$$

which means that $f \in AC_\rho^n[a,b]$. $\qquad\qquad\square$

THEOREM 3.2. *Let $\rho > 0$, $n \in \mathbb{N}$, $n - 1 < \alpha < n$, $0 < a < b < \infty$ and $f \in AC_\rho^n[a,b]$ then the Katugampola fractional derivatives $D_{a+}^{\alpha,\rho} f$ and $D_{b-}^{\alpha,\rho} f$ exist almost everywhere on $[a,b]$ and may be represented in the form*

$$D_{a+}^{\alpha,\rho} f(t) = \sum_{i=0}^{n-1} c_i \frac{\Gamma(i+1)}{\Gamma(i+1-\alpha)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{i-\alpha}$$

$$+ \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{n-\alpha-1} \varphi(s) \, ds,$$

(15)

and

$$D_{b-}^{\alpha,\rho} f(t) = \sum_{i=0}^{n-1} c_i (-1)^i \frac{\Gamma(i+1)}{\Gamma(i+1-\alpha)} \left(\frac{b^\rho - t^\rho}{\rho} \right)^{i-\alpha}$$

$$+ \frac{(-1)^{n-1}}{\Gamma(n-\alpha)} \int_t^b \left(\frac{s^\rho - t^\rho}{\rho} \right)^{n-\alpha-1} \varphi(s) \, ds,$$

(16)

where c_i ($i = 0, 1, \ldots, n-1$) are some real constants and $\varphi \in L[a,b]$.

P r o o f. We show in detail only the proof of the formula (15). Since $f \in AC_\rho^n[a,b]$, we have the representation (10). Substituting it into the definition of the left-sided Katugampola fractional derivative and using (5), we obtain

$$D_{a+}^{\alpha,\rho} f(t) = \sum_{i=0}^{n-1} c_i \frac{\Gamma(i+1)}{\Gamma(i+1-\alpha)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{i-\alpha}$$
$$+ \frac{\rho^{\alpha-2n+1}}{(n-1)!\Gamma(n-\alpha)} \delta_\rho^n \left(\int_a^t \frac{\rho\tau^{\rho-1}}{(t^\rho-\tau^\rho)^{1-n+\alpha}} \int_a^\tau (\tau^\rho - s^\rho)^{n-1} \varphi(s)\, ds\, d\tau\right).$$

(17)

Changing the order of integration we have

$$\int_a^t \frac{\rho\tau^{\rho-1}}{(t^\rho-\tau^\rho)^{1-n+\alpha}} \int_a^\tau (\tau^\rho - s^\rho)^{n-1} \varphi(s)\, ds\, d\tau$$
$$= \int_a^t \varphi(s) \int_s^t \frac{\rho\tau^{\rho-1}}{(t^\rho-\tau^\rho)^{1-n+\alpha}} (\tau^\rho - s^\rho)^{n-1}\, d\tau\, ds.$$

The inner integral can be evaluated by the change of variable $u = \frac{\tau^\rho - s^\rho}{t^\rho - s^\rho}$. It gives

$$\int_s^t \frac{\rho\tau^{\rho-1}}{(t^\rho-\tau^\rho)^{1-n+\alpha}} (\tau^\rho - s^\rho)^{n-1}\, d\tau$$
$$= (t^\rho - s^\rho)^{2n-\alpha-1} \int_0^1 u^{n-1}(1-u)^{n-\alpha-1}\, du.$$

Using the Beta function and the fact (3), we obtain

$$\int_s^t \frac{\rho\tau^{\rho-1}}{(t^\rho-\tau^\rho)^{1-n+\alpha}} (\tau^\rho - s^\rho)^{n-1}\, d\tau$$
$$= (t^\rho - s^\rho)^{2n-\alpha-1} \frac{\Gamma(n)\Gamma(n-\alpha)}{\Gamma(2n-\alpha)}.$$

Thus

$$\int_a^t \frac{\rho\tau^{\rho-1}}{(t^\rho-\tau^\rho)^{1-n+\alpha}} \int_a^\tau (\tau^\rho - s^\rho)^{n-1} \varphi(s)\, ds\, d\tau$$
$$= \frac{\Gamma(n)\Gamma(n-\alpha)}{\Gamma(2n-\alpha)} \int_a^t (t^\rho - s^\rho)^{2n-\alpha-1} \varphi(s)\, ds.$$

Taking δ_ρ^n-differentiation and using properties of Gamma function we obtain

$$\delta_\rho^n \left(\frac{\Gamma(n)\Gamma(n-\alpha)}{\Gamma(2n-\alpha)} \int\limits_a^t (t^\rho - s^\rho)^{2n-\alpha-1} \varphi(s) \, ds \right)$$

$$= \frac{\rho\Gamma(n)\Gamma(n-\alpha)}{\Gamma(2n-\alpha-1)} \delta_\rho^{n-1} \left(\int\limits_a^t (t^\rho - s^\rho)^{2n-\alpha-2} \varphi(s) \, ds \right)$$

$$= \frac{\rho^2\Gamma(n)\Gamma(n-\alpha)}{\Gamma(2n-\alpha-2)} \delta_\rho^{n-2} \left(\int\limits_a^t (t^\rho - s^\rho)^{2n-\alpha-3} \varphi(s) \, ds \right)$$

$$= \cdots$$

$$= \rho^m \Gamma(n)\Gamma(n-\alpha)\Gamma(2n-\alpha-m)\delta_\rho^{n-m} \left(\int\limits_a^t (t^\rho - s^\rho)^{2n-\alpha-m-1} \varphi(s) \, ds \right).$$

Finally, taking $n = m$ we have

$$\delta_\rho^n \left(\frac{\Gamma(n)\Gamma(n-\alpha)}{\Gamma(2n-\alpha)} \int\limits_a^t (t^\rho - s^\rho)^{2n-\alpha-1} \varphi(s) \, ds \right)$$

$$= \rho^n \Gamma(n) \left(\int\limits_a^t (t^\rho - s^\rho)^{n-\alpha-1} \varphi(s) \, ds \right).$$

Substituting this into (17) and using the property of Gamma function (1), we obtain (15).

In order to prove the formula (16), we should start with substituting (11) into definition of the right-sided Katugampola fractional derivative. These details are left to the reader. □

COROLLARY 3.3. *If $0 < \alpha < 1$ and $f \in AC[a, b]$, then $D_{a+}^{\alpha;\rho} f$ exists almost everywhere in $[a, b]$ and*

$$D_{a+}^{\alpha;\rho} f(t) = c \left(\frac{\rho}{t^\rho - a^\rho} \right)^\alpha + \frac{1}{\Gamma(1-\alpha)} \int\limits_a^t \left(\frac{\rho}{t^\rho - \tau^\rho} \right)^\alpha f'(\tau) \, d\tau, \qquad (18)$$

where c is real constant.

Remark 2. Relation (18) was proved in [13].

These results in Theorem 3.2 for the Katugampola fractional derivatives are analogous to those which have been proven for the Riemann-Liouville and the Hadamard fractional derivatives. It means that if we take in Theorem 3.2 $\rho = 1$,

we obtain the form of the Riemann-Liouville fractional derivatives of the order α in space $AC_1^n[a,b]$ $(n = [\alpha] + 1)$, i.e.,

$$D_{a+}^\alpha f(t) = \sum_{i=0}^{n-1} c_i(t-a)^{i-\alpha} + \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{\varphi(s)}{(t-s)^{\alpha-n+1}} \, ds,$$

$$D_{b-}^\alpha f(t) = \sum_{i=0}^{n-1} (-1)^i c_i(b-t)^{i-\alpha} + \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b \frac{\varphi(s)}{(s-t)^{\alpha-n+1}} \, ds,$$

where $\varphi \in L[a,b]$ and c_i are arbitrary constants. These relations were established in [11, Lemma 2.2]. Moreover, taking $\rho \to 0^+$ in Theorem 3.2 we obtain result from [12, Theorem 3.4].

4. Properties of the Katugampola fractional operators

The Katugampola fractional integral satisfies several important properties including boundedness in the $L^p(a,b)$ space or linearity property. These facts were proved in [1]. The following assertion shows that Katugampola fractional integrals satisfy semigroup property.

LEMMA 4.1. *Let $\alpha > 0$, $\beta > 0$, $\rho > 0$ and $f \in L[a,b]$ then*

$$I_{a+}^{\alpha,\rho} I_{a+}^{\beta,\rho} f(t) = I_{a+}^{\alpha+\beta,\rho} f(t), \tag{19}$$

and

$$I_{b-}^{\alpha,\rho} I_{b-}^{\beta,\rho} f(t) = I_{b-}^{\alpha+\beta,\rho} f(t) \tag{20}$$

for all $t \in (a,b)$.

P r o o f. By definition of the Katugampola fractional integral, and next by changing order of integration we obtain

$$I_{a+}^{\alpha,\rho} I_{a+}^{\beta,\rho} f(t) = \frac{\rho^{1-\alpha-\beta}}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t s^{\rho-1} f(s) \int_s^t \frac{\rho \tau^{\rho-1}}{(t^\rho - \tau^\rho)^{1-\alpha}(\tau^\rho - s^\rho)^{1-\beta}} \, d\tau \, ds.$$

Substitution $u = \dfrac{\tau^\rho - s^\rho}{t^\rho - s^\rho}$ in the inner integral gives

$$I_{a+}^{\alpha,\rho} I_{a+}^{\beta,\rho} f(t) = \frac{\rho^{1-\alpha-\beta}}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t s^{\rho-1} f(s)(t^\rho - s^\rho)^{\alpha+\beta-1} \int_0^1 u^{\beta-1}(1-u)^{\alpha-1} \, du \, ds.$$

Using definition of the function Beta and its property (3), we obtain (19). The property (20) is proved by analogy. $\qquad\square$

LEMMA 4.2. *Let $\alpha > \beta > 0$, $\rho > 0$ and $f \in L[a, b]$ then*

$$D_{a+}^{\beta,\rho} I_{a+}^{\alpha,\rho} f(t) = I_{a+}^{\alpha-\beta,\rho} f(t), \qquad a < t < b.$$

P r o o f. Let $n = [\beta] + 1$. Using definitions of the Katugampola fractional operators and index rule (19) we obtain

$$D_{a+}^{\beta,\rho} I_{a+}^{\alpha,\rho} f(t) = \delta_\rho^n I_{a+}^{n-\beta,\rho} I_{a+}^{\alpha,\rho} f(t)$$

$$= \delta_\rho^n I_{a+}^{n,\rho} I_{a+}^{\alpha-\beta,\rho} f(t),$$

which ends the proof, because $\delta_\rho^n I_{a+}^{n,\rho} f(t) = f(t)$, as it is mentioned in (4). $\qquad\square$

The next result shows that, for certain classes of functions, the Katugampola fractional derivative is the left inverse operator of the Katugampola fractional integral.

THEOREM 4.3. *Let $\alpha > 0$, $\rho > 0$ and $f \in L[a, b]$, then*

$$D_{a+}^{\alpha,\rho} I_{a+}^{\alpha,\rho} f(t) = f(t).$$

However, as we show below, the Katugampola fractional derivative is not the right inverse of the Katugampola fractional integral.

THEOREM 4.4. *Let $n - 1 < \alpha < n$, $n \in \mathbb{N}$, $\rho > 0$, $I_{a+}^{n-\alpha,\rho} f \in AC_\rho^n[a, b]$, then*

$$I_{a+}^{\alpha,\rho} D_{a+}^{\alpha,\rho} f(t) = f(t) + \sum_{i=0}^{n-1} \tilde{c}_i \left(\frac{t^\rho - a^\rho}{\rho} \right)^{i-n+\alpha},$$

where $\tilde{c}_i$ are real constants.

P r o o f. From definition of the space $AC_\rho^n[a, b]$ (9) and Lemma 3.1, we can rewrite the assumption $I_{a+}^{n-\alpha,\rho} f \in AC_\rho^n[a, b]$ as

$$\delta_\rho^{n-1} I_{a+}^{n-\alpha,\rho} f(t) = c + \int_a^t \varphi(\tau)\, d\tau \tag{21}$$

and

$$I_{a+}^{n-\alpha,\rho} f(t) = \sum_{i=0}^{n-1} c_i \left(\frac{t^\rho - a^\rho}{\rho} \right)^i + \frac{1}{(n-1)!} \int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{n-1} \varphi(\tau)\, d\tau, \tag{22}$$

where c_i are real constants and $\varphi \in L[a, b]$.

Taking δ_ρ-derivative of (21) we obtain

$$D_{a+}^{\alpha,\rho} f(t) = t^{1-\rho} \varphi(t).$$

Thus

$$I_{a+}^{\alpha,\rho} D_{a+}^{\alpha,\rho} f(t) = I_{a+}^{\alpha,\rho} \left[t^{1-\rho} \varphi(t) \right]. \tag{23}$$

On the other hand, applying operator $D_{a+}^{n-\alpha,\rho}$ to both sides of (22) we have

$$
\begin{aligned}
f(t) = \sum_{i=0}^{n-1} c_i D_{a+}^{n-\alpha,\rho} \left(\frac{t^\rho - a^\rho}{\rho} \right)^i \\
+ \frac{1}{(n-1)!} D_{a+}^{n-\alpha,\rho} \left(\int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{n-1} \varphi(\tau)\, d\tau \right).
\end{aligned}
\tag{24}
$$

From (5) we have

$$
D_{a+}^{n-\alpha,\rho} \left(\frac{t^\rho - a^\rho}{\rho} \right)^i = \frac{\Gamma(i+1)}{\Gamma(i+1-n+\alpha)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{i-n+\alpha}.
\tag{25}
$$

Furthermore, we have

$$
\begin{aligned}
& D_{a+}^{n-\alpha,\rho} \left(\int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{n-1} \varphi(\tau)\, d\tau \right) \\
& = \delta_\rho I_{a+}^{1-n+\alpha,\rho} \left[\int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{n-1} \varphi(\tau)\, d\tau \right] \\
& = \frac{\rho^{-\alpha}}{\Gamma(1-n+\alpha)} \delta_\rho \left(\int_a^t \varphi(s) \int_s^t \frac{\rho \tau^{\rho-1}(\tau^\rho - s^\rho)^{n-1}}{(t^\rho - \tau^\rho)^{n-\alpha}}\, d\tau \right).
\end{aligned}
$$

Substitution $u = \dfrac{\tau^\rho - s^\rho}{t^\rho - s^\rho}$ gives

$$
\begin{aligned}
& D_{a+}^{n-\alpha,\rho} \left(\int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{n-1} \varphi(\tau)\, d\tau \right) \\
& = \frac{\rho^{-\alpha}}{\Gamma(1-n+\alpha)} \delta_\rho \left(\int_a^t \varphi(s)(t^\rho - s^\rho)^\alpha \int_0^1 u^{n-1}(1-u)^{\alpha-n}\, du\, ds \right).
\end{aligned}
$$

Finally, using properties of Beta and Gamma functions we obtain

$$
\begin{aligned}
D_{a+}^{n-\alpha,\rho} \left(\int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho} \right)^{n-1} \varphi(\tau) d\tau \right) & = \frac{\rho^{-\alpha}\Gamma(n)}{\Gamma(\alpha+1)} \delta_\rho \left(\int_a^t \varphi(s)(t^\rho - s^\rho)^\alpha\, ds \right) \\
& = \Gamma(n) I_{a+}^{\alpha,\rho} \left(t^{1-\rho}\varphi(t) \right).
\end{aligned}
\tag{26}
$$

Substituting (25) and (26) into (24) we have

$$f(t) = \sum_{i=0}^{n-1} c_i \frac{\Gamma(i+1)}{\Gamma(i+1-n+\alpha)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{i-n+\alpha} + I_{a+}^{\alpha,\rho} \left(t^{1-\rho} \varphi(t) \right). \qquad (27)$$

By (23) and (27) we get the thesis, and the proof is completed. $\qquad \square$

The above theorem allows to find a general solution of the following fractional equation

$$D_{a+}^{\alpha,\rho} u(t) = 0, \qquad (28)$$

where α, ρ are positive real constants. We have

THEOREM 4.5. *Let $n-1 < \alpha < n$, $n \in \mathbb{N}$ and $\rho > 0$, then equation (28) holds if, and only if,*

$$u(t) = \sum_{i=1}^{n} c_i \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-i}.$$

The proof of this theorem is trivial, so it is left to the reader.

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