sciendo

Tatra Mountains
Mathematical Publications
DOI: 10.2478/tmmp-2021-0022
Tatra Mt. Math. Publ. **79** (2021), 101–118

OSCILLATION BEHAVIOUR OF SOLUTIONS FOR A CLASS OF A DISCRETE NONLINEAR FRACTIONAL-ORDER DERIVATIVES

George. E. Chatzarakis[1] — A. George Maria Selvam[2] — Rajendran Janagaraj[3] — George. N. Miliaras[1]

[1]School of Pedagogical and Technological Education (ASPETE), Athens, GREECE

[2]Sacred Heart College(Autonomous), Tamil Nadu, INDIA

[3]Karpagam Academy of Higher Education, Tamil Nadu, INDIA

ABSTRACT. Based on the generalized Riccati transformation technique and some inequality, we study some oscillation behaviour of solutions for a class of a discrete nonlinear fractional-order derivative equation

$$\Delta\big[\gamma(\ell)[\alpha(\ell)+\beta(\ell)\Delta^{\mu}u(\ell)]^{\eta}\big]+\phi(\ell)f[G(\ell)]=0, \ell\in N_{\ell_0+1-\mu},$$

where

$$\ell_0>0,\quad G(\ell)=\sum_{j=\ell_0}^{\ell-1+\mu}(\ell-j-1)^{(-\mu)}u(j)$$

and Δ^{μ} is the Riemann-Liouville (R-L) difference operator of the derivative of order μ, $0<\mu\leq 1$ and η is a quotient of odd positive integers. Illustrative examples are given to show the validity of the theoretical results.

1. Introduction

Fractional calculus has been a hot topic in nonlinear science because it can describe physical phenomena more accurately. Continuous time fractional calculus has been around for centuries (see [9], [25]). However, theory regrading the discrete counterpart is relatively a young field. Recently, researchers have

2010 Mathematics Subject Classification: 26A33, 34A08, 39A12, 39A13, 39A21.
Keywords: oscillation, Riemann-Liouville fractional derivatives, difference equations.

diverted their attention to the discrete fractional calculus and attempted to put together a complete theoretical framework for the subject. Very recently, interest has been shown by the research community in the study and applications of fractional discrete calculus. Fractional discrete systems have a major advantage over their conventional counterparts due to the infinite memory.

The discrete version of the fractional calculus is a new direction which has potential applications in many areas of science and engineering. The application area has gained more attention during the last few years. The definition of a fractional difference operator was first made by Diaz and Olser in 1974 (see [17]). Some of the most interesting and relevant works on discrete fractional calculus in the last decade are available in [6], [7], [22]. In [1], the author discussed the discrete difference counterparts of conventional Riemann and Caputo derivatives. Recent years have witnessed a surge in the study of qualitative properties of the solutions of fractional difference equations such as oscillation, boundedness, existence, uniqueness and other asymptotic behaviours and some excellent results have been published, see [3], [6], [15], [16] and the references cited therein.

This research work is organized as follows. Some significant previous works are discussed in Section 2. Basic definitions of fractional derivative and lemmas are provided in Section 3. Section 4 establishes some oscillation behaviour of solutions for a class of a discrete nonlinear fractional-order equation by using generalized Riccati Transformation technique and inequalities. Further in Section 5, numerical applications are presented to emphasize the applicability of the theoretical results. Finally, this work ends with a brief conclusion.

2. Some significant previous works

Recently, many researchers have come up with new findings in the study on the oscillatory and nonoscillatory behaviour solutions of difference equations/fractional order difference equations (see [4], [10], [11], [13], [14], [19], [21], [27], [29]).

H. Adiguzel (see [2]) investigated the oscillatory behaviour of the solutions of fractional difference equations of the form

$$\Delta\Big(c(t)\Delta\Big(a(t)\Delta\big(r(t)\Delta^{\mu}x(t)\big)\Big)\Big) + q(t)G(t) = 0,$$

where

$$t \in N_{t_0+1-\alpha}, \quad G(t) = \sum_{s=t_0}^{t-1+\alpha} (t-s-1)^{(-\alpha)} x(s).$$

J. Alzabut et al. (see [5]) established the oscillation of nonlinear fractional difference equations with damping form

$$\Delta\big(a(t)\Delta^{\alpha}y(t)\big)+p(t)\Delta^{\alpha}y(t)+q(t)f\big(G(t)\big)=0,$$

where

$$t\in N_{t_0+1-\alpha},\quad G(t)=\sum_{s=t_0}^{t-1+\alpha}(t-s-1)^{(-\alpha)}y(s)$$

and Δ^{α} denotes the Riemann-Liouville difference operator of order $0<\alpha\le 1$.

Selvam et al. (see [20]) investigated the oscillation of a class of fractional difference equations with damping term by using Riccati transformation technique

$$\Delta^{1+\alpha}u(t)+p(t)\Delta^{\alpha}u(t)+q(t)F[G(t)]=0, t\ge t_0>0,$$

where

$$G(t)=\sum_{s=t_0}^{t-1+\alpha}(t-s-1)^{(-\alpha)}u(s)$$

and Δ^{α} denotes the Riemann-Liouville difference operator of order $0<\alpha\le 1$.

W. Li discussed the oscillation of forced fractional difference equations with damping term (see [24])

$$\big(1+p(t)\big)\Delta\big(\Delta^{\mu}x(t)\big)+p(t)\Delta^{\mu}x(t)+f\big(t,x(t)\big)=g(t),\quad t\in\mathbb{N}_0.$$

A. Secer and H. Adiguzel in [28] studied and obtained some oscillatory criteria for the given below fractional difference equation of order μ with $0<\mu\le 1$

$$\Delta\left(a(t)\Big[\Delta\Big(r(t)\big(\Delta^{\mu}x(t)\big)^{\gamma_1}\Big)\Big]^{\gamma_2}\right)+q(t)f\big(G(t)\big)=0,\quad t\in N_{t_0+1-\mu},$$

where

$$G(t)=\sum_{s=t_0}^{t-1+\mu}(t-s-1)^{(-\mu)}x(s)$$

and Δ^{μ} denotes the Riemann-Liouville difference operator of order $0<\mu\le 1$.

In [12], Chatzarakis et al., provided conditions for oscillation of fractional order difference equations of the form

$$\Delta\big(\Delta^{\mu}x(t)\big)^{\gamma}+q(t)f\big(x(t)\big)=0,\quad t\in N_{t_0+1-\mu}.$$

The asymptotic behaviour of solutions for a class of non-linear fractional order difference equations with damping term of the form

$$\Delta\Big[c(t)\big[\Delta\big(r(t)\Delta^{\alpha}x(t)\big)\big]^{\eta}\Big]+p(t)\big[\Delta\big(r(t)\Delta^{\alpha}x(t)\big)\big]^{\eta}+q(t)f\left(\sum_{s=t_0}^{t-1+\alpha}(t-s-1)^{(-\alpha)}x(s)\right)=0,\quad t\in N_{t_0}$$

was presented by Bai and Xu (see [8]).

In this paper, we study the oscillation behaviour for a class of a discrete nonlinear fractional-order derivative equation of the form

$$\Delta\big[\gamma(\ell)[\alpha(\ell)+\beta(\ell)\Delta^{\mu}u(\ell)]^{\eta}\big]+\phi(\ell)f[G(\ell)]=0,\quad \ell\in N_{\ell_0+1-\mu}\,, \tag{1}$$

where

$$\ell_0>0,\quad G(\ell)=\sum_{j=\ell_0}^{\ell-1+\mu}(\ell-j-1)^{(-\mu)}u(j)$$

and $(\ell-j-1)^{(-\mu)}=\frac{\Gamma(\ell-j)}{\Gamma(\ell-j+\mu)}$ is the Kernel function (see [23]). Also the present status $u(\ell)$ depends on the past information $u(0),\dots,u(\ell-1)$ which is called the discrete memory effect and Δ^{μ} is the Riemann-Liouville (R-L) difference operator of the derivative of order μ, $0<\mu\le 1$, $N_{\ell_0}=\{\ell_0,\ell_0+1,\ell_0+2,\dots\}$.

The following conditions are assumed to hold throughout this paper.

C$_1$) $\gamma(\ell)\in C\big([\ell_0,\infty)\big),\phi(\ell)\in C\big([\ell_0,\infty)\big)$ are positive sequences of real numbers;

C$_2$) $\eta>0$ is a quotient of odd positive integers;

C$_3$) $\Delta\big[\frac{\alpha(\ell)}{\beta(\ell)}\big]\neq 0$ for $\ell\in[\ell_0,\infty)$ and $\lim_{\ell\to\infty}\sum_{j=\ell_0}^{\ell-1}\frac{\alpha(j)}{\beta(j)}<\infty$;

C$_4$) f is a monotone decreasing function satisfying $uf(u)>0$ such that $\frac{f(u)}{u^{\eta}}\ge k$ for $u\neq 0$ and a constant $k>0$;

C$_5$) $\alpha(\ell)$ is a nonnegative sequence on $[\ell_0,\infty)$ for a certain $\ell_0>0$, then there exists an integer $m>0$ such that $\beta(\ell)\ge m$ for $\ell\in[\ell_0,\infty)$.

3. Preliminaries and basic lemmas

Over the last decade, researchers have been working on a complete framework for the subject. Since the subject remains to be new topic and the notations and theory has not yet settled, let us start with a brief theory. In this section, the following definitions and lemmas provide the basis of the work carried out in our study.

Definition 3.1 (see [18]). A solution $u(\ell)$ of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is nonoscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

Definition 3.2 (see [7]). Let $\mu>0$, $\Delta^{-\mu}: N_a\to N_{a+\mu}$, f is defined for $j\equiv a \bmod (1)$ and $\Delta^{-\mu}f$ is defined for $\ell\equiv(a+\mu)\bmod(1)$. Then the $\mu-$th fractional sum is defined by

$$\Delta^{-\mu}f(\ell)=\frac{1}{\Gamma(\mu)}\sum_{j=a}^{\ell-\mu}(\ell-j-1)^{(\mu-1)}f(j)\quad\text{for}\quad \ell\in\mathbb{N}_{a+\mu}.$$

The falling factorial power function is

$$\ell^{(\mu)} = \frac{\Gamma(\ell+1)}{\Gamma(\ell-\mu+1)},$$

where Γ is the gamma function defined as $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\mathrm{d}t$, for $x > 0$.

Definition 3.3 (see [7]). Let $\nu > 0$ and $n-1 < \nu < n$, where n denotes a positive integer $n = \lceil \nu \rceil$. Set $\mu = n - \nu$. The $\nu-$th fractional difference is defined as

$$\Delta^\nu f(\ell) = \Delta^{n-\mu} f(\ell) = \Delta^n \Delta^{-\mu} f(\ell).$$

Lemma 3.4 (see [19]). *Let $u(\ell)$ be a solution of* (1) *and*

$$G(\ell) = \sum_{j=\ell_0}^{\ell-1+\mu} (\ell - j - 1)^{(-\mu)} u(j).$$

Then

$$\Delta G(\ell) = \Gamma(1-\mu)\Delta^\mu u(\ell).$$

Lemma 3.5 (see [18]). *The product and quotient rules of the difference operator Δ are as follows:*

$$\Delta[u(\ell)v(\ell)] = \Delta u(\ell) \cdot v(\ell+1) + u(\ell) \cdot \Delta v(\ell),$$

$$\Delta\left[\frac{u(\ell)}{v(\ell)}\right] = \frac{\Delta u(\ell) \cdot v(\ell+1) - u(\ell+1) \cdot \Delta v(\ell)}{v(\ell) \cdot v(\ell+1)},$$

where $\Delta u(\ell) = u(\ell+1) - u(\ell)$.

Lemma 3.6 (see [8]). *If η is a quotient of two odd positive integers, then the following two inequalities are true:*

$$\text{if } G(\ell+1) > G(\ell) > 0, \text{ then } \Delta G^\eta(\ell) \geq \big(\Delta G(\ell)\big)^\eta,$$

$$\text{if } G(\ell+1) < G(\ell) < 0, \text{ then } \Delta G^\eta(\ell) \leq \big(\Delta G(\ell)\big)^\eta.$$

Lemma 3.7 (see [8]). *Let $a > 0, b, \lambda \in R$; then $b\lambda - a\lambda^2 \leq \frac{b^2}{4a}$.*

4. Main results

In this section, we establish some new sufficient conditions for the oscillation of solutions of fractional order difference equations making use of Riccati technique and some inequalities. For our convenience, let us define

$$z(\ell) = \alpha(\ell) + \beta(\ell)\Delta^\mu u(\ell).$$

Theorem 4.1. *Assume that* $C_1) - C_5)$ *hold. If there is a positive sequence* $\rho(\ell)$ *such that*

$$\lim_{\ell\to\infty} \sum_{j=\ell_0}^{\ell-1} \left[\frac{1}{\gamma(j)}\right]^{\frac{1}{\eta}} = \infty \tag{2}$$

and

$$\limsup_{\ell\to\infty} \sum_{j=\ell_2}^{\ell-1} \left[k\rho(j)\phi(j) - \frac{(\Delta\rho(j))^2\gamma(j+1)m^\eta}{4\rho(j+1)}\right] = \infty, \tag{3}$$

then every solution of (1) *is oscillatory.*

P r o o f. Suppose that $u(\ell)$ is a nonoscillatory solution of (1). Without loss of generality, we may assume that $u(\ell)$ is an eventually positive solution of (1). Then there exists $\ell_1 \in [\ell_0, \infty)$, such that $u(\ell) > 0$, $G(\ell) > 0$ for $\ell \in [\ell_1, \infty)$, where $G(\ell) = \sum_{j=\ell_0}^{\ell-1+\mu} (\ell - j - 1)^{(-\mu)} u(j)$. Therefore, it follows from (1) and $C_4)$ that

$$\Delta\left[\gamma(\ell)[z(\ell)]^\eta\right] = -\phi(\ell) f[G(\ell)],$$

or

$$\Delta\left[\gamma(\ell) z^\eta(\ell)\right] \le -k\phi(\ell) G^\eta(\ell) < 0. \tag{4}$$

Thus $\gamma(\ell)z^\eta(\ell)$ is strictly decreasing on $[\ell_1, \infty)$ and is eventually of one sign. Since $\alpha(\ell) > 0$ for $\ell \in [\ell_0, \infty)$ and $\eta > 0$ is a quotient of odd positive integers, we see that $z(\ell)$ is eventually of one sign. We will first show that

$$z(\ell) > 0, \quad \text{for} \quad \ell \in [\ell_1, \infty).$$

Indeed, assume that there exists $\ell_2 \ge \ell_1$ such that $z(\ell) < 0$. Then for $\ell \in [\ell_2, \infty)$, we get

$$\gamma(\ell) z^\eta(\ell) < \gamma(\ell_2) z^\eta(\ell_2) = c < 0,$$

i.e.,

$$z(\ell) < \left[\frac{c}{\gamma(\ell)}\right]^{\frac{1}{\eta}} < 0. \tag{5}$$

So we can see that $z(\ell) < 0$ on $[\ell_2, \infty)$. From these terms, for $\ell \in [\ell_2, \infty)$, we have

$$\alpha(\ell) + \beta(\ell)\Delta^\mu u(\ell) < \left[\frac{c}{\gamma(\ell)}\right]^{\frac{1}{\eta}},$$

or

$$\frac{\alpha(\ell)}{\beta(\ell)} + \Delta^\mu u(\ell) < \left[\frac{c}{\gamma(\ell)}\right]^{\frac{1}{\eta}} \frac{1}{\beta(\ell)}.$$

From C_5), we obtain

$$\frac{\alpha(\ell)}{\beta(\ell)} + \Delta^\mu u(\ell) < \frac{1}{m}\left[\frac{c}{\gamma(\ell)}\right]^{\frac{1}{\eta}}.$$

Apply Lemma 3.4 to get

$$\frac{\alpha(\ell)}{\beta(\ell)} + \frac{\Delta G(\ell)}{\Gamma(1-\mu)} < \frac{1}{m}\left[\frac{c}{\gamma(\ell)}\right]^{\frac{1}{\eta}}.$$

Summing up the last inequality from ℓ_2 to $\ell-1$, we get

$$\sum_{j=\ell_2}^{\ell-1}\left[\frac{\alpha(j)}{\beta(j)} + \frac{\Delta G(j)}{\Gamma(1-\mu)}\right] < \sum_{j=\ell_2}^{\ell-1}\frac{1}{m}\left[\frac{c}{\gamma(j)}\right]^{\frac{1}{\eta}},$$

or

$$\sum_{j=\ell_2}^{\ell-1}\Delta G(j) < \Gamma(1-\mu)\sum_{j=\ell_2}^{\ell-1}\left(\frac{1}{m}\left[\frac{c}{\gamma(j)}\right]^{\frac{1}{\eta}} - \frac{\alpha(j)}{\beta(j)}\right),$$

i.e.,

$$G(\ell) < G(\ell_2) + \Gamma(1-\mu)\sum_{j=\ell_2}^{\ell-1}\left(\frac{1}{m}\left[\frac{c}{\gamma(j)}\right]^{\frac{1}{\eta}} - \frac{\alpha(j)}{\beta(j)}\right).$$

Thus

$$\lim_{\ell\to\infty} G(\ell) < G(\ell_2) + \lim_{\ell\to\infty}\left[\Gamma(1-\mu)\sum_{j=\ell_2}^{\ell-1}\left(\frac{1}{m}\left[\frac{c}{\gamma(j)}\right]^{\frac{1}{\eta}} - \frac{\alpha(j)}{\beta(j)}\right)\right] = -\infty,$$

which contradicts $G(\ell) > 0$, $\ell \in [\ell_1, \infty)$. Hence $z(\ell) > 0$ for $\ell \in [\ell_2, \infty)$.

From $z(\ell)$, we get

$$\frac{z(\ell)}{\beta(\ell)} = \frac{\alpha(\ell)}{\beta(\ell)} + \Delta^\mu u(\ell).$$

Using Lemma 3.4, we obtain

$$\frac{z(\ell)}{\beta(\ell)} = \frac{\alpha(\ell)}{\beta(\ell)} + \frac{\Delta G(\ell)}{\Gamma(1-\mu)}.$$

Condition C_5) yields

$$\frac{\Delta G(\ell)}{\Gamma(1-\mu)} = \frac{z(\ell)-\alpha(\ell)}{\beta(\ell)} < \frac{z(\ell)}{m},$$

i.e.,

$$\Delta G(\ell) < \frac{z(\ell)}{m}. \tag{6}$$

Define the Riccati transformation $\omega(\ell)$ by

$$\omega(\ell) = \rho(\ell)\frac{\gamma(\ell)z^\eta(\ell)}{G^\eta(\ell)}, \quad \ell \in [\ell_1, \infty). \tag{7}$$

Then we have $\omega(\ell) > 0$ for $\ell \in [\ell_1, \infty)$. Using Lemma 3.5, we take

$$\begin{aligned}\Delta\omega(\ell) &= \Delta\left[\rho(\ell)\frac{\gamma(\ell)z^\eta(\ell)}{G^\eta(\ell)}\right]\\ &= \Delta\rho(\ell)\frac{\gamma(\ell+1)z^\eta(\ell+1)}{G^\eta(\ell+1)} + \rho(\ell)\Delta\left[\frac{\gamma(\ell)z^\eta(\ell)}{G^\eta(\ell)}\right].\end{aligned}$$

From (1), (7) and condition C_4), we have

$$\Delta\omega(\ell) < \frac{\Delta\rho(\ell)}{\rho(\ell+1)}\omega(\ell+1) - k\rho(\ell)\phi(\ell) - \frac{\rho(\ell)\left[\gamma(\ell)z^\eta(\ell)\right]\Delta G^\eta(\ell)}{G^\eta(\ell)G^\eta(\ell+1)}.$$

Lemma 3.6 yields

$$\Delta\omega(\ell) < \frac{\Delta\rho(\ell)}{\rho(\ell+1)}\omega(\ell+1) - k\rho(\ell)\phi(\ell) - \frac{\rho(\ell)\left[\gamma(\ell)z^\eta(\ell)\right]}{G^\eta(\ell)G^\eta(\ell+1)}(\Delta G(\ell))^\eta.$$

Using (6) in the above inequality leads to

$$\Delta\omega(\ell) < \frac{\Delta\rho(\ell)}{\rho(\ell+1)}\omega(\ell+1) - k\rho(\ell)\phi(\ell) + \frac{\rho(\ell+1)\left[\gamma(\ell+1)z^\eta(\ell+1)\right]}{-G^\eta(\ell+1)G^\eta(\ell+1)}\frac{z^\eta(\ell+1)}{m^\eta},$$

or

$$\Delta\omega(\ell) < -k\rho(\ell)\phi(\ell) + \frac{\Delta\rho(\ell)}{\rho(\ell+1)}\omega(\ell+1) - \frac{\omega^2(\ell+1)}{\gamma(\ell+1)\rho(\ell+1)m^\eta}. \tag{8}$$

Apply Lemma 3.7, taking $a = \frac{1}{\gamma(\ell+1)\rho(\ell+1)m^\eta}$, $b = \frac{\Delta\rho(\ell)}{\rho(\ell+1)}$ and $X = \omega(\ell+1)$. Then we obtain

$$\Delta\omega(\ell) < -k\rho(\ell)\phi(\ell) + \frac{(\Delta\rho(\ell))^2\gamma(\ell+1)m^\eta}{4\rho(\ell+1)}.$$

Summing up from ℓ_2 to $\ell - 1$ leads to

$$\sum_{j=\ell_2}^{\ell-1}\Delta\omega(j) < \sum_{j=\ell_2}^{\ell-1}\left[-k\rho(j)\phi(j) + \frac{(\Delta\rho(j))^2\gamma(j+1)m^\eta}{4\rho(j+1)}\right],$$

or

$$\sum_{j=\ell_2}^{\ell-1}\left[k\rho(j)\phi(j) - \frac{(\Delta\rho(j))^2\gamma(j+1)m^\eta}{4\rho(j+1)}\right] < \omega(\ell_2) - \omega(\ell) < \omega(\ell_2).$$

Thus

$$\limsup_{\ell\to\infty}\sum_{j=\ell_2}^{\ell-1}\left[k\rho(j)\phi(j) - \frac{(\Delta\rho(j))^2\gamma(j+1)m^\eta}{4\rho(j+1)}\right] < \omega(\ell_2),$$

which contradicts (3). This completes the proof. □

Notation. Through the proofs of the theorems that follow, we use $H(\ell, j)$: $\ell, j \in N, \ell \geq j \geq 0$ to denote the double sequence satisfying [26]

$$\begin{aligned} H(\ell,\ell) &= 0 \quad \text{for} \quad \ell \geq \ell_0; \\ H(\ell,j) &> 0 \quad \text{for} \quad \ell > j \geq \ell_0; \\ \Delta_2 H(\ell,j) = H(\ell,j+1) - H(\ell,j) &< 0 \quad \text{for} \quad \ell > j \geq \ell_0. \end{aligned}$$

Theorem 4.2. *Assume that* $C_1) - C_5)$ *hold and there is a positive sequence* $\rho(\ell)$ *such that*

$$\limsup_{\ell\to\infty} \frac{1}{H(\ell,\ell_0)} \sum_{j=\ell_0}^{\ell-1} \left[k\rho(j)\phi(j)H(\ell,j) - \frac{h_+^2(\ell,j)\gamma(j+1)\rho(j+1)m^\eta}{4H(\ell,j)} \right] = \infty, \quad (9)$$

where

$$h_+(\ell,j) = \Delta_2 H(\ell,j) + \frac{\Delta\rho(j)}{\rho(j+1)} H(\ell,j).$$

Then every solution of (1) *is oscillatory.*

Proof. Suppose that $u(\ell)$ is a nonoscillatory solution of (1). Without loss of generality, we may assume that $u(\ell)$ is an eventually positive solution of (1). Then there exists $\ell_1 \in [\ell_0, \infty)$, such that $u(\ell) > 0$, $G(\ell) > 0$ for $\ell \in [\ell_1, \infty)$. According to the proof of Theorem 4.1, equation (8) holds. Multiplying both sides by $H(\ell, j)$ and then summing up from ℓ_2 to $\ell - 1$ yields

$$\begin{aligned} \sum_{j=\ell_2}^{\ell-1} \omega(j)H(\ell,j) < &- \sum_{j=\ell_2}^{\ell-1} k\rho(j)\phi(j)H(\ell,j) + \sum_{j=\ell_2}^{\ell-1} \frac{\Delta\rho(j)}{\rho(j+1)}\omega(j+1)H(\ell,j) \\ &- \sum_{j=\ell_2}^{\ell-1} \frac{\omega^2(j+1)}{\gamma(j+1)\rho(j+1)m^\eta} H(\ell,j), \end{aligned}$$

or

$$\begin{aligned} \sum_{j=\ell_2}^{\ell-1} k\rho(j)\phi(j)H(\ell,j) < &- \sum_{j=\ell_2}^{\ell-1} \omega(j)H(\ell,j) + \sum_{j=\ell_2}^{\ell-1} \frac{\Delta\rho(j)}{\rho(j+1)}\omega(j+1)H(\ell,j) \\ &- \sum_{j=\ell_2}^{\ell-1} \frac{\omega^2(j+1)}{\gamma(j+1)\rho(j+1)m^\eta} H(\ell,j). \end{aligned} \quad (10)$$

Using the summation by parts formula, we obtain

$$-\sum_{j=\ell_2}^{\ell-1} \omega(j)H(\ell,j) = [-H(\ell,j)\omega(\ell)]_{\ell_2}^{\ell} + \sum_{j=\ell_2}^{\ell-1} \omega(j+1)\Delta_2 H(\ell,j),$$

i.e.,

$$-\sum_{j=\ell_2}^{\ell-1} \omega(j)H(\ell,j) = H(\ell,\ell_2)\,\omega(\ell_2) + \sum_{j=\ell_2}^{\ell-1} \omega(j+1)\Delta_2 H(\ell,j). \quad (11)$$

Therefore, using (11) in (10) leads to

$$\sum_{j=\ell_2}^{\ell-1} k\rho(j)\phi(j)H(\ell,j) < \sum_{j=\ell_2}^{\ell-1}\left[h_+(\ell,j)\omega(j+1) - \frac{\omega^2(j+1)H(\ell,j)}{\gamma(j+1)\rho(j+1)m^\eta}\right] + H(\ell,\ell_2)\omega(\ell_2),$$

where $h_+(\ell,j) = \Delta_2 H(\ell,j) + \dfrac{\Delta\rho(j)}{\rho(j+1)}H(\ell,j)$.

Lemma 3.7 yields

$$\sum_{j=\ell_2}^{\ell-1} k\rho(j)\phi(j)H(\ell,j) < H(\ell,\ell_2)\omega(\ell_2) + \sum_{j=\ell_2}^{\ell-1}\frac{h_+^2(\ell,j)\gamma(j+1)\rho(j+1)m^\eta}{4H(\ell,j)},$$

or

$$\sum_{j=\ell_2}^{\ell-1}\left[k\rho(j)\phi(j)H(\ell,j) - \frac{h_+^2(\ell,j)\gamma(j+1)\rho(j+1)m^\eta}{4H(\ell,j)}\right] < H(\ell,\ell_0)\omega(\ell_2). \quad (12)$$

Now

$$\begin{aligned}
&\sum_{j=\ell_0}^{\ell-1}\left[k\rho(j)\phi(j)H(\ell,j) - \frac{h_+^2(\ell,j)\gamma(j+1)\rho(j+1)m^\eta}{4H(\ell,j)}\right] \\
&= \sum_{j=\ell_0}^{\ell_2-1}\left[k\rho(j)\phi(j)H(\ell,j) - \frac{h_+^2(\ell,j)\gamma(j+1)\rho(j+1)m^\eta}{4H(\ell,j)}\right] \\
&\quad + \sum_{j=\ell_2}^{\ell-1}\left[k\rho(j)\phi(j)H(\ell,j) - \frac{h_+^2(\ell,j)\gamma(j+1)\rho(j+1)m^\eta}{4H(\ell,j)}\right].
\end{aligned}$$

Using (12) in the above inequality, we get

$$\begin{aligned}
&\sum_{j=\ell_0}^{\ell-1}\left[k\rho(j)\phi(j)H(\ell,j) - \frac{h_+^2(\ell,j)\gamma(j+1)\rho(j+1)m^\eta}{4H(\ell,j)}\right] \\
&< \sum_{j=\ell_0}^{\ell_2-1}\left[k\rho(j)\phi(j)H(\ell,j) - \frac{h_+^2(\ell,j)\gamma(j+1)\rho(j+1)m^\eta}{4H(\ell,j)}\right] \\
&\quad + H(\ell,\ell_0)\omega(\ell_2) \\
&< \sum_{j=\ell_0}^{\ell_2-1} k\rho(j)\phi(j)H(\ell,j) + H(\ell,\ell_0)\omega(\ell_2) \\
&< H(\ell,\ell_0)\sum_{j=\ell_0}^{\ell_2-1} k\rho(j)\phi(j) + H(\ell,\ell_0)\omega(\ell_2).
\end{aligned}$$

Therefore,

$$\frac{1}{H(\ell,\ell_0)}\sum_{j=\ell_0}^{\ell-1}\left[k\rho(j)\phi(j)H(\ell,j)-\frac{h_+^2(\ell,j)\gamma(j+1)\rho(j+1)m^{\eta}}{4H(\ell,j)}\right]$$
$$<\sum_{j=\ell_0}^{\ell_2-1}k\rho(j)\phi(j)+\omega(\ell_2)<\infty,$$

which contradicts (9). This completes the proof. □

The following theorem is discussed under the condition

$$\lim_{\ell\to\infty}\sum_{j=\ell_0}^{\ell_1}\left[\frac{1}{\gamma(j)}\right]^{\frac{1}{\eta}}<\infty. \tag{13}$$

Theorem 4.3. *Assume that* C_1)–C_5) *hold and there is a positive sequence* $\rho(\ell)$ *such that,*

$$\lim_{\ell\to\infty}\sum_{i=\ell_2}^{\ell-1}\left[\left(\frac{1}{\gamma(i)}\sum_{j=i}^{\infty}\phi(j)\right)^{\frac{1}{\eta}}-\frac{\alpha(i)}{\beta(i)}\right]=\infty. \tag{14}$$

Then every solution of (1) *is oscillatory or satisfies* $\lim_{\ell\to\infty}G(\ell)=0$.

P r o o f. Suppose that $u(\ell)$ is a nonoscillatory solution of (1). Without loss of generality, we may assume that $u(\ell)$ is an eventually positive solution of (1). Proceeding as in the proof of Theorem 4.1, we get that (4) holds. Then there are two cases for the sign of $z(\ell)$. When $z(\ell)>0$ is eventually positive, then from the proof of Theorem 4.1, we get every solution of (1) is oscillatory.

Now, assume that $z(\ell)$ is eventually negative. Then there exists $\ell_2>\ell_0$ such that $z(\ell)<0$ for $\ell\geq\ell_2$. Hence

$$z(\ell)<0,$$
$$\alpha(\ell)+\beta(\ell)\Delta^{\mu}(\ell)<0,$$
$$\alpha(\ell)+\beta(\ell)\frac{\Delta G(\ell)}{\Gamma(1-\mu)}<0,$$
$$\frac{\Delta G(\ell)}{\Gamma(1-\mu)}<-\frac{\alpha(\ell)}{\beta(\ell)},$$

or

$$\Delta G(\ell)<-\Gamma(1-\mu)\frac{\alpha(\ell)}{\beta(\ell)}. \tag{15}$$

Let condition C_3) hold. Then we get

$$\lim_{\ell\to\infty}\frac{\alpha(\ell)}{\beta(\ell)}=0.$$

Letting $\ell \to \infty$ in (15) leads to

$$\lim_{\ell\to\infty} \Delta G(\ell) \le 0.$$

Since $G(\ell) > 0$, $\ell \in [\ell_1, \infty)$, we have

$$\lim_{\ell\to\infty} \Delta G(\ell) = \lambda \ge 0.$$

We claim that $\lambda = 0$. If not, that is $\lambda > 0$, then $G(\ell) \ge \lambda$, $\ell \in [\ell_2, \infty)$.

By (4),

$$\Delta\left[\gamma(\ell) z^\eta(\ell)\right] \le -k\phi(\ell) G^\eta(\ell) \le -k\phi(\ell)\lambda^\eta.$$

Summing up both sides from ℓ to ∞ yields

$$\sum_{j=\ell}^{\infty} \Delta\left[\gamma(j) z^\eta(j)\right] \le \sum_{j=\ell}^{\infty} -k\phi(j)\lambda^\eta$$

$$-\gamma(\ell) z^\eta(\ell) \le -\sum_{j=\ell}^{\infty} k\phi(j)\lambda^\eta$$

or

$$z^\eta(\ell) > k\lambda^\eta \frac{1}{\gamma(\ell)} \sum_{j=\ell}^{\infty} \phi(j)$$

i.e.,

$$z(\ell) > k^{\frac{1}{\eta}} \lambda \left[\frac{1}{\gamma(\ell)} \sum_{j=\ell}^{\infty} \phi(j)\right]^{\frac{1}{\eta}}.$$

Now

$$\frac{\Delta G(\ell)}{\Gamma(1-\mu)} > k^{\frac{1}{\eta}} \lambda \left[\frac{1}{\gamma(\ell)} \sum_{j=\ell}^{\infty} \phi(j)\right]^{\frac{1}{\eta}} - \frac{\alpha(\ell)}{\beta(\ell)}.$$

Summing up from ℓ_2 to $\ell - 1$ we get

$$G(\ell) - G(\ell_2) > \Gamma(1-\mu) \left[k^{\frac{1}{\eta}} \lambda \left[\frac{1}{\gamma(\ell)} \sum_{j=\ell}^{\infty} \phi(j)\right]^{\frac{1}{\eta}} - \frac{\alpha(\ell)}{\beta(\ell)}\right],$$

or

$$G(\ell) > G(\ell_2) + \Gamma(1-\mu) \left[k^{\frac{1}{\eta}} \lambda \left[\frac{1}{\gamma(\ell)} \sum_{j=\ell}^{\infty} \phi(j)\right]^{\frac{1}{\eta}} - \frac{\alpha(\ell)}{\beta(\ell)}\right].$$

Letting $\ell \to \infty$ we have $\lim_{\ell\to\infty} G(\ell) = \infty$, which contradicts to

$$G(\ell) > 0 \quad \text{for} \ \ \ell \in [\ell_1, \infty).$$

Thus we get $\lambda = 0$. Therefore $\lim\limits_{\ell\to\infty} G(\ell) = 0$. This completes the proof. □

5. Applications

In this section, we present some applications to demonstrate the theoretical results.

Example. Consider the discrete fractional order nonlinear equation

$$\Delta\left[\ell^2\left(\ell+\ell^3\Delta^{\frac{1}{2}}\right)^3\right]+\ell^{-3}f\left(\sum_{j=1}^{\ell-1+\mu}(\ell-j-1)^{(-\mu)}\right)=0, \ell\geq 1. \tag{16}$$

This is a special case of (1) with

$$\mu=\frac{1}{2},\qquad \gamma(\ell)=\ell^2,\quad \alpha(\ell)=\ell,\quad \beta(\ell)=\ell^3,$$
$$\phi(\ell)=\frac{1}{\ell^3},\qquad \eta=3,\qquad \ell_0=1,\qquad f(u)=u,$$

$$\frac{f(u)}{u^\eta}=\frac{1}{u^2}>\epsilon=L>0,\quad k=\frac{1}{2},\quad m=2$$

and the positive function $\rho(\ell)=\ell^2$ which implies that $\Delta[\rho(\ell)]=2\ell+1$. Clearly, assumptions $C_1)-C_5)$ hold. Moreover,

$$\Delta\left[\frac{\alpha(\ell)}{\beta(\ell)}\right]=\Delta\left(\frac{1}{\ell^2}\right)=-\frac{2\ell+1}{\ell^2(\ell+1)^2}\neq 0,$$
$$\lim_{\ell\longrightarrow\infty}\sum_{j=1}^{\ell-1}\left[\frac{1}{\gamma(j)}\right]^{\frac{1}{\eta}}=\lim_{\ell\longrightarrow\infty}\sum_{j=1}^{\ell-1}j^{-\frac{2}{3}}=\infty,$$
$$\lim_{\ell\longrightarrow\infty}\sum_{j=1}^{\ell-1}\left[\frac{\alpha(j)}{\beta(j)}\right]=\lim_{\ell\longrightarrow\infty}\sum_{j=1}^{\ell-1}\frac{1}{j^2}<\infty,$$

that is, condition $C_3)$ and (2) are satisfied. Hence,

$$\limsup_{\ell\longrightarrow\infty}\sum_{j=1}^{\ell-1}\left[k\rho(j)\phi(j)-\frac{(\Delta\rho(j))^2\gamma(j+1)m^\eta}{4\rho(j+1)}\right]$$
$$=\limsup_{\ell\longrightarrow\infty}\sum_{j=1}^{\ell-1}\left[\frac{1}{2j}-2(2j+1)^2\right]=\infty.$$

Condition (3) is satisfied. From Theorem 4.1, we deduce that every solution of equation (16) is oscillatory.

Example. Consider the fractional order discrete nonlinear equation

$$\Delta\left[\ell^{-1}\left(\ell^3+\ell^5\Delta^{\frac{1}{2}}u(\ell)\right)^3\right]+\ell^{-2}f\left(\sum_{j=1}^{\ell-1+\mu}(\ell-j-1)^{(-\mu)}\right)=0, \ell\geq 1. \tag{17}$$

This is a special case of (1) with

$$\mu=\frac{1}{2},\qquad \gamma(\ell)=\frac{1}{\ell},\quad \alpha(\ell)=\ell^3,\quad \beta(\ell)=\ell^5,$$
$$\phi(\ell)=\frac{1}{\ell^2},\qquad \eta=3,\qquad \ell_0=1,\qquad f(u)=u,$$

$$\frac{f(u)}{u^\eta}=\frac{1}{u^2}>\epsilon=L>0,\quad k=\frac{1}{4},\quad m=1$$

and the positive function $\rho(\ell)=\ell^2$ which implies that $\Delta[\rho(\ell)]=2\ell+1$. It is clear that assumptions $\mathrm{C}_1)-\mathrm{C}_5)$ hold. Moreover,

$$\Delta\left[\frac{\alpha(\ell)}{\beta(\ell)}\right]=\Delta\left(\frac{1}{\ell^2}\right)=-\frac{2\ell+1}{\ell^2(\ell+1)^2}\neq 0,$$

$$\lim_{\ell\longrightarrow\infty}\sum_{j=1}^{\ell-1}\left[\frac{1}{\gamma(j)}\right]^{\frac{1}{\eta}}=\lim_{\ell\longrightarrow\infty}\sum_{j=1}^{\ell-1}j^{\frac{1}{3}}=\infty,$$

$$\lim_{\ell\longrightarrow\infty}\sum_{j=1}^{\ell-1}\left[\frac{\alpha(j)}{\beta(j)}\right]=\lim_{\ell\longrightarrow\infty}\sum_{j=1}^{\ell-1}\frac{1}{j^2}<\infty,$$

that is, condition $\mathrm{C}_3)$ and (2) are satisfied. Furthermore, we define the double sequence as follows:

$$H(\ell,j)=\ell-j>0,\ \ \ell>j>1,$$
$$H(\ell,1)=\ell-1>0,\ \ \ell>j>1,$$

$$\begin{aligned}\Delta_2[H(\ell,j)]&=H(\ell,j+1)-H(\ell,j)\\&=-1<0,\ \ \ell>j\geq 1,\end{aligned}$$

$$\begin{aligned}h_+(\ell,j)&=\Delta_2H(\ell,j)+\frac{\Delta\rho(j)}{\rho(j+1)}H(\ell,j)\\&=\frac{1}{(j+1)^2}\left[(2j+1)(\ell-j)-(j+1)^2\right].\end{aligned}$$

Hence,

$$\limsup_{\ell\to\infty} \frac{1}{H(\ell,\ell_0)} \sum_{j=\ell_0}^{\ell-1} \left[k\rho(\ell)\phi(\ell)H(\ell,j) - \frac{h_+^2(\ell,j)\gamma(j+1)\rho(j+1)m^\eta}{4H(\ell,j)} \right]$$
$$= \limsup_{\ell\to\infty} \frac{1}{H(\ell,\ell_0)} \sum_{j=\ell_0}^{\ell-1} \left[\frac{\ell-j}{4} - \frac{\left[(2j+1)(\ell-j)-(j+1)^2\right]^2}{4(\ell-j)(\ell+1)^3} \right] = \infty.$$

Condition (9) is satisfied. From Theorem 4.2, we deduce that every solution of equation (17) is oscillatory.

EXAMPLE. Consider the fractional order nonlinear difference equation

$$\Delta\left[\ell^3\left(\ell^2+\ell^4\Delta^{\frac{1}{2}}\right)\right] + \ell^{-2} f\left(\sum_{j=1}^{\ell-1+\mu} (\ell-j-1)^{(-\mu)}\right) = 0, \ell \geq 1. \qquad (18)$$

This special case of (1) with

$$\mu = \frac{1}{2}, \quad \gamma(\ell) = \ell^3, \quad \alpha(\ell) = \ell^2, \quad \beta(\ell) = \ell^4,$$
$$\phi(\ell) = \frac{1}{\ell^2}, \quad \eta = 1, \quad \ell_0 = 1, \quad f(u) = u^{-1},$$

$$\frac{f(u)}{u^\eta} = \frac{1}{u^2} > \epsilon = L > 0, \quad k = 3, \quad m = 4$$

and the positive function $\rho(\ell) = \ell^2$. It is clear that assumptions $\mathrm{C}_1) - \mathrm{C}_5)$ hold. Moreover,

$$\Delta\left[\frac{\alpha(\ell)}{\beta(\ell)}\right] = \Delta\left(\frac{1}{\ell^2}\right) = -\frac{2\ell+1}{\ell^2(\ell+1)^2} \neq 0,$$
$$\lim_{\ell\longrightarrow\infty} \sum_{j=1}^{\ell-1} \left[\frac{1}{\gamma(j)}\right]^{\frac{1}{\eta}} = \lim_{\ell\longrightarrow\infty} \sum_{j=1}^{\ell-1} \frac{1}{j^3} < \infty,$$
$$\lim_{\ell\longrightarrow\infty} \sum_{j=1}^{\ell-1} \left[\frac{\alpha(j)}{\beta(j)}\right] = \lim_{\ell\longrightarrow\infty} \sum_{j=1}^{\ell-1} \frac{1}{j^2} < \infty,$$

that is, condition C_3) and (13) are satisfied. Hence,

$$\lim_{\ell\longrightarrow\infty} \sum_{i=1}^{\ell-1} \left[\left(\frac{1}{\gamma(i)} \sum_{j=i}^{\infty} \phi(j) \right)^{\frac{1}{\eta}} - \frac{\alpha(i)}{\beta(i)} \right] = \lim_{\ell\longrightarrow\infty} \sum_{i=1}^{\ell-1} \left[i^3 \sum_{j=i}^{\infty} \frac{1}{j^2} - i \right] = \infty.$$

Condition (14) is satisfied. From Theorem 4.3, we deduce that every solution of equation (18) is oscillatory.

6. Conclusion

This paper obtained new results to ensure the oscillation of solutions for a class of a discrete nonlinear fractional-order equations by using generalized Riccati transformation technique and some inequality. The results are original and an extension from continuous fractional-order derivatives. Finally, applications are discussed to illustrate the validity of the theoretical results. In near future, we intend to discuss the oscillatory behaviour for the same equation with forcing and delay terms.

Acknowledgement. The authors would like to thank the referee for the constructive remarks which greatly improved the paper.

REFERENCES

[1] ABDELJAWAD, T.: *On Riemann and Caputo fractional differences*, Comput. Math. Appl. **62** (2011), no. 3, 1602–1611.

[2] ADIGUZEL, H.: *Oscillatory behavior of solutions of certain fractional difference equations*, Adv. Difference Equ. **2018** paper no. 445, 13 pp.

[3] AGARWAL, R. P.—MARTIN BOHNER,—GRACE, S. R.—DONAL O'REGAN: *Discrete Oscillation Theory*, Hindawi Publishing Corporation, New York, NY 10022, USA, 2005.

[4] ALZABUT, J.—ABDELJAWAD, T.: *Sufficient conditions for the oscillation of nonlinear fractional difference equations*, J. Fract. Calc. Appl. **5** (2014), no. 1, 177–187.

[5] ALZABUT, J.—MUTHULAKSHMI, V.—OZBEKLER, A.—ADIGUZEL, H.: *On the oscillation of nonlinear fractional difference equations with damping*, Mathematics **7** (2019), paper no. 687, 14 pp.

[6] ANASTASSIOU, G. A.: *Discrete fractional calculus and inequalities*, arXiv:0911.3370v1 [math.CA] 17 Nov 2009, doi 10.1007/978-3-642-17098-0_35. 11 pp.

[7] ATICI, F. M.—ELOE, P. W.: *Initial value problems in discrete fractional calculus*, Proc. Amer. Math. Soc. **137** (2008), no. 3, 981–989.

[8] BAI, Z.—XU, R.: *The asymptotic behavior of solutions for a class of nonlinear fractional difference equations with damping term*, Discrete Dyn. Nat. Soc. **2018** (2018), paper no. 5232147, 11 pp.

[9] BALEANU, D.—REZAPOUR, S.—MOHAMMADI, H.: *Some existence results on nonlinear fractional differential equations*, Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **371** (2013), no. 1990, 20120144, 7 pp.

[10] BAYRAM, M.—ADIGUZEL, H.—SECER, A.: *Oscillation criteria for nonlinear fractional differential equation with damping term*, De Gruyter : Open Phys. **14** (2016), no. 1, 119–128, doi:10.1515/phys-2016-0012.

[11] BAYRAM, M.—SECER, A.: *Oscillation properties of solutions of fractional difference equations*, Thermal Science **23** (2019), no. 1, S 185–S 192.

[12] CHATZARAKIS, G. E.—GOKULRAJ, P.—KALAIMANI, T.: *Oscillation test for fractional difference equations*, Tatra Mt. Math. Publ. **71** (2018), no. 1, 53–64.

[13] CHATZARAKIS, G. E.—GEORGE MARIA SELVAM, A.—JANAGARAJ, R., MARIA DOUKA: *Oscillation theorems for certain forced nonlinear discrete fractional order equations*, Commun. Math. Appl. **10** (2019), no. 4, 763–772.

[14] CHATZARAKIS, G. E.—GEORGE MARIA SELVAM, A.—JANAGARAJ, R.— MILIARAS, G. N.: *Oscillation criteria for a class of nonlinear discrete fractional order equations with damping term*, Math. Slovaca **70** (2020), no. 5, 1165–1182.

[15] CHEN, F.—LUO, X.—ZHOU, Y.: *Existence results for nonlinear fractional difference equations*, Adv. Difference Equ. **2011** (2011), paper no. 713201, 12 pp.

[16] CHEN, F.: *Fixed points and asymptotic stability of nonlinear fractional difference equations*, Electron. J. Qual. Theory Differ. Equ. **39** (2011), 18 pp.

[17] DIAZ, J. B.—OSLER, T. J.: *Differences of fractional order*, Math. Comp. **28** (1974), no. 125, 185–202.

[18] ELAYDI, S.: *An Introduction to Difference Equations*, In: Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1996.

[19] GEORGE MARIA SELVAM, A.—JANAGARAJ, R.: *Oscillation theorems for damped fractional order difference equations*, AIP Conf. Proc. **2095** (030007) (2019), 7 pp, https://doi.org/10.1063/1.5097518

[20] GEORGE MARIA SELVAM, A.—JANAGARAJ, R.: *Oscillation criteria of a class of fractional order damped difference equations*, Int. J. Appl. Math. **32** (2019), no. 3, 433–441.

[21] GEORGE MARIA SELVAM, A.—ALZABUT, J.—JANAGARAJ, R.—ADIGUZEL, H.: *Oscillation analysis for nonlinear discrete fractional order delay and neutral equations with forcing term*, Dynam. Systems Appl. **29**(2) (2020), 327–342.

[22] GOODRICH, C. S.—PETERSON, A. C.: *Discrete Fractional Calculus*. Springer International Publishing, Switzerland, 2015.

[23] HOLM, M. T.: *The Theory of Discrete Fractional Calculus: Development and Application*. Ph.D. Thesis, The University of Nebraska-Lincoln, ProQuest LLC, nn Arbor, MI, 2011, 123 pp.

[24] LI, W. N.: *Oscillation results for certain forced fractional difference equations with damping term*, Adv. Difference Equ. **2016** (2016), paper no. 70, 9 pp.

[25] ROSS, B.: *Fractional Calculus and its Applications*, Springer-Verlag, Berlin, 1975.

[26] SAKER, S. H.: *Oscillation Theory of Delay Differential and Difference Equations, Second and Third Orders*. VDM Verlag Dr.Müller, 2010.

[27] SANTRA, S. S.: *Necessary and sufficient conditions for oscillation of second-order delay differential equations*, Tatra Mt. Math. Publ. **75** (2020), 135–146.

[28] SECER A.—ADIGUZEL, H.: *Oscillation of solutions for a class of nonlinear fractional difference equations*, J. Nonlinear Sci. Appl. **9** (2016), 5862–5869.

[29] TRIPATHY, A. K.—CHHATRIA, G. N.: *Oscillation criteria for forced first order nonlinear neutral impulsive difference system*, Tatra Mt. Math. Publ. **71** (2018), 175–193.

Received May 22, 2020

George. E. Chatzarakis
George. N. Miliaras
School of Pedagogical and Technological Education (ASPETE)
Department of Electrical and Electronic Engineering Educators
Marousi 15122, Athens
GREECE
E-mail: geaxatz@otenet.gr
gmiliara@yahoo.gr

A. George Maria Selvam
Department of Mathematics,
Sacred Heart College (Autonomous)
Tirupattur-635601, Tamil Nadu
INDIA
E-mail: agmshc@gmail.com

Rajendran Janagaraj
Department of Mathematics
Faculty of Engineering
Karpagam Academy of Higher Education
Coimbatore-641021, Tamil Nadu,
INDIA
E-mail: janagarajtk@gmail.com