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APPLICATION OF THE EXTENDED FAN SUB-EQUATION METHOD TO TIME FRACTIONAL BURGERS-FISHER EQUATION

Djouaher Abbas[1] — Abdelouahab Kadem[2]

[1]Bachir El-Ibrahimi University of Bordj Bou Arraridj, ALGERIA

[2]Ferhat Abbas Sétif University 1, ALGERIA

ABSTRACT. In this paper, the extended Fan sub-equation method to obtain the exact solutions of the generalized time fractional Burgers-Fisher equation is applied. By applying this method, we obtain different solutions that are benefit to further comprise the concepts of complex nonlinear physical phenomena. This method is simple and can be applied to several nonlinear equations. Fractional derivatives are taken in the sense of Jumarie's modified Riemann-Liouville derivative. A comparative study with the other methods approves the validity and effectiveness of the technique, and on the other hand, for suitable parameter values, we plot 2D and 3D graphics of the exact solutions by using the extended Fan sub-equation method. In this work, we use Mathematica for computations and programming.

1. Introduction

Fractional calculus, defined by the generalization of the order of classical (traditional) calculus to the arbitrary real or complex order, is like an ancient idea that is as old and deep-rooted as classic calculus. Towards the end of the 17th century, it has arisen with some specification between L'Hospital and Leibnitz, then it has been developed with the studies of the renown mathematicians such as Laplace, Abel, Fourier, Liouville, Riemann, Grunwald and Letnikov [18]. Recently, Atangana and Baleanu have suggested a new fractional order derivative, the new derivative based on the generalized Mittag-Leffler function and

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the fractional derivative that has non-singular and nonlocal kernel. With the help of the new derivative, many new papers have appeared in the literature, some of these are, Alkahtani analyse chaotic behaviour on the Chuas circuit model [1].

In recent years, for solving time fractional differential equations, many researchers have proposed powerful techniques to get an exact solution, such as the sine-cosine method [3, 14], (G'/G)-expansion method [10], the Exp-function method [8], the tanh method [23], the sub equation method [2, 7], the improved (G'/G)- expansion method [17], the invariant subspace method, the generalized Riccati equation method [16], the Kudryashov method [20], fractional residual power series method [12].

These numerical methods are good mathematical techniques to obtain numerical/approximate solutions for non-linear models. It has been observed that the numerical solutions are closer to the exact solution. In addition to this, both methods can gain different meanings depending on the field.

Burgers-Fisher (BF) equation is very important in many areas such as fluid dynamics, physics, engineering, etc. The study of the BF model has been investigated by many authors both for conceptual understanding of physical laws and testing various numerical methods. Also, some applications in fields as gas dynamics, number theory, heat conduction, elasticity, etc., have been found. It is a highly nonlinear equation because it is a combination of reaction, convection, and diffusion mechanisms. This equation is called Burgers-Fisher because it has the properties of a convective phenomenon from Burgers equation, and diffusion transport as well as reactions kind of characteristics from Fisher equation.

Some researchers have studied Burgers-Fisher equation in the numerical or analytical sense. Among them, Wazwaz [24] formally derived a variety of exact travelling wave solutions of distinct physical structures. In [4], Chandraker et al. investigated the numerical treatment of Burgers-Fisher equation. Zhu et al. [25] obtained the numerical solution of BF equation via cubic B-spline quasi-interpolation method and so on.

The form of the time fractional generalized Burgers-Fisher equation is

$$\frac{d^{\alpha}u(x,t)}{dt^{\alpha}}+\beta u^{\delta}\left(x,t\right)\frac{du\left(x,t\right)}{dx}-\frac{d^{2}u\left(x,t\right)}{dx^{2}}=\gamma u\left(x,t\right)\left(1-u^{\delta}\left(x,t\right)\right). \tag{1}$$

Here the coefficient α , $0<\alpha\leq 1$ is the order of the fractional time derivative and δ,γ,β are arbitrary constants. The extended Fan sub-equation method is used to construct the exact solitons solutions of Eq. (1). The rest of this paper is organized as follows. In Section 2, we describe the extended Fan sub-equation method for solving nonlinear FDEs. In Section 3, we give an application of the proposed method to the time fractional generalized Burgers-Fisher equation. In Section 4, some conclusions are given.

2. Theoretical background and preliminaries

The Jumarie's modified Riemann-Liouville derivative of αth order, see Jumarie (2009) [11], can be defined as

$$\frac{d^{\alpha}f(t)}{dt^{\alpha}} = \begin{cases} \frac{1}{\Gamma(-\alpha)}\int\limits_0^t (t-\xi)^{-\alpha-1} f(\xi)d\xi, & \alpha<0, \\ \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_0^t (t-\xi)^{-\alpha}\big(f(\xi)-f(0)\big)d\xi, & 0<\alpha<1, \\ \left[\frac{d^{\alpha-n}f(t)}{dt^{\alpha}}\right]^{(n)}, & n\le\alpha<n+1,\ n\ge 1. \end{cases} \tag{2}$$

Some useful formulas and properties of the modified Riemann-Liouville derivative can be summarized, as

$$\frac{d^{\alpha}t^{\beta}}{dt^{\alpha}} = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)}t^{\beta-\alpha}, \qquad \beta>0, \tag{3}$$

$$\frac{d^{\alpha}\left(u(t)v(t)\right)}{dt^{\alpha}} = v(t)\frac{d^{\alpha}u(t)}{dt^{\alpha}} + u(t)\frac{d^{\alpha}v(t)}{dt^{\alpha}}, \tag{4}$$

$$\frac{d^{\alpha}u(v(t))}{dt^{\alpha}} = \frac{d^{\alpha}u(v(t))}{dv^{\alpha}}(v_t')^{\alpha}, \tag{5}$$

$$\frac{d^{\alpha}u\left(v(t)\right)}{dt^{\alpha}} = u'\left(v(t)\right)\frac{d^{\alpha}v\left(t\right)}{dt^{\alpha}}. \tag{6}$$

Eqs. (2)–(6) which will be used in the application of Extended Fan sub-equation method are important tools for fractional calculus. In the rest of this section, basic steps of the method will be presented. In the third section, application of the method and obtained results will take place.

Li and He (2010) [9, 13] defined the fractional complex transformation as

$$u\left(x,y,z,t\right) = u\left(\xi\right), \quad \xi = \frac{kx^{\delta}}{\Gamma(1+\delta)} + \frac{py^{\gamma}}{\Gamma(1+\gamma)} + \frac{lz^{\lambda}}{\Gamma(1+\lambda)} + \frac{ct^{\alpha}}{\Gamma(1+\alpha)}. \tag{7}$$

It reduces the nonlinear partial differential equation of fractional order into a nonlinear ordinary differential equation (ODE), where k, p, l are arbitrary constants, the localized wave solution $u = u\left(\xi\right)$ travels with speed c and ξ is the amplitude of the travelling wave.

3. Extended Fan sub-equation method for finding the exact solutions of non-linear FDEs

In this section, we illustrate the basic idea of the extended Fan sub-equation method for solving nonlinear fractional partial differential equation.

Let us assume a fractional order partial differential equation with presented polynomial P including various order derivatives as

$$P\left(u, D_t^{\alpha}u, D_x^{\alpha}u, u_t, u_x, u_{xx}, \dots\right) = 0, \tag{8}$$

where x, t are independent variables and $u\left(x,t\right)$ is an unknown function and polynomial P includes the highest order derivative and nonlinear term of $u\left(x,t\right)$. Also, $D^{\alpha}\left(\cdot\right)$, symbolizes the modified Riemann Liouville fractional derivation.

We illustrate the main steps of Extended Fan Sub-Equation method as follows:

In the first step, we obtain the traveling wave solution of Eq. (1) of the form:

$$u\left(x,t\right) = u\left(\xi\right), \qquad \xi = kx - \frac{\lambda t^{\alpha}}{\Gamma\left(1+\alpha\right)}, \tag{9}$$

where k and λ are arbitrary constants. As a result, we obtain a nonlinear ODE in the following form

$$N\left(u, u', u'', u''', \dots\right) = 0, \tag{10}$$

where the prime indicates differentiation with respect to ξ. Exact solution of this equation can be constructed as follows [5]

$$u\left(\xi\right) = \frac{A_{-n}}{\Psi(\xi)^n} + \cdots + \frac{A_{-1}}{\Psi(\xi)} + A_0 + A_1\Psi(\xi) + \cdots + A_n\Psi(\xi)^n, \quad A_n \neq 0. \tag{11}$$

Here $A_i\left(i = 0, 1, 2, 3, \dots, n\right)$ are constants to be determined later. Also, $\Psi = \Psi(\xi)$ satisfies the following ODE

$$\Psi'\left(\xi\right) = \epsilon\sqrt{\sum_{i=0}^{4} w_i \Psi^i}, \tag{12}$$

where $\epsilon = \pm 1$ and w_i are constants. Thus, the derivatives with respect to ξ can be calculated with respect to the variable Ψ as follows

$$\frac{du}{d\xi} = \epsilon\sqrt{\sum_{i=0}^{4} w_i \Psi^i}\,\frac{du}{d\Psi}, \tag{13}$$

$$\frac{d^2u}{d\xi^2} = \frac{1}{2}\sum_{i=0}^{4} i w_i \Psi^{i-1}\frac{du}{d\Psi} + \sum_{i=0}^{4} w_i \Psi^i \frac{d^2u}{d\Psi^2}, \dots \tag{14}$$

The solutions of equation (12) are as follows [6]
First solution:

$$\Psi = \varrho\left(\frac{\sqrt{w_3}}{2}\xi, g_2, g_3\right), \quad w_2 = w_4 = 0, \quad w_3 > 0, \quad g_2 = \frac{-4w_1}{w_3}, \quad g_3 = \frac{-4w_0}{w_3}. \tag{15}$$

Second solution:

$$\Psi = \frac{-w_1}{2w_2} + \frac{\epsilon w_1}{2w_2}\sinh\left(2\sqrt{w_2}\xi\right), \qquad w_0 = w_3 = w_4 = 0, w_2 > 0, \tag{16}$$

$$\Psi = \frac{-w_1}{2w_2} + \frac{\epsilon w_1}{2w_2}\sin\left(2\sqrt{-w_2}\xi\right), \qquad w_0 = w_3 = w_4 = 0, w_2 < 0, \tag{17}$$

$$\Psi = \frac{-w_2}{w_3}\sec h^2\left(\frac{\sqrt{w_2}}{2}\xi\right), \qquad w_0 = w_1 = w_4 = 0, w_2 > 0, \tag{18}$$

$$\Psi = \frac{-w_2}{w_3}\sec^2\left(\frac{\sqrt{-w_2}}{2}\xi\right), \qquad w_0 = w_1 = w_4 = 0, w_2 < 0, \tag{19}$$

$$\Psi = \frac{1}{w_3\xi^2}, \qquad w_0 = w_1 = w_2 = w_4 = 0. \tag{20}$$

Third solution

$$\Psi = \frac{-w_1}{2w_2}e^{(\epsilon\sqrt{w_2}\xi)}, \qquad w_3 = w_4 = 0, w_0 = \frac{w_1^2}{4w_2}, \quad w_2 > 0, \tag{21}$$

$$\Psi = \frac{-w_0}{w_1} + \frac{1}{4}w_1\xi^2, \qquad w_2 = w_3 = w_4 = 0, \quad w_1 \neq 0. \tag{22}$$

Substituting (11)–(14) into equation (10) and collecting all terms with the same powers of together, the left-hand side of equation (10) is converted into a polynomial. After setting each coefficients of this polynomial to zero, we obtain a set of algebraic equations in terms of A_n $(n = 0, 1, 2 \ldots, n)$. Solving the system of algebraic equations and then substituting the results and the general solutions of (15)–(22) into equation (11), give solutions of equation (10).

4. Application of the extended Fan sub-equation method

In this section, we apply the extended Fan sub-equation method for solving the time fractional generalized Burgers-Fisher equation as follows.

Example. We consider the time fractional generalized Burgers-Fisher equation in the form

$$u_t^\alpha + \beta u^\delta u_x - u_{xx} = \gamma u\left(1 - u^\delta\right), \tag{23}$$

where $0 < \alpha \leq 1$, α is the order of the fractional time derivative and β, γ, δ, are arbitrary constants.

Now substituting Eq. (9) in Eq. (23), we obtain

$$k^2u'' + \left(\lambda - k\beta u^{\delta}\right)u' + \gamma u\left(1 - u^{\delta}\right) = 0. \tag{24}$$

Applying the folding transformation

$$u(\xi) = v^{\frac{1}{\delta}}(\xi), \tag{25}$$

we obtain the following equation which is similar to the most general form of second order nonlinear oscillator equation [15, 19] with many arbitrary parameters. With some restrictions on the parameters, new integrable equations were found and discussed in [22]

$$k^2\delta vv'' + k^2(1-\delta)v'^2 + (\lambda - k\beta v)\delta vv' + \gamma\delta^2(1-v)v^2 = 0. \tag{26}$$

Balancing the highest power of nonlinear terms of $vv''$and $v'v^2$ gives n=1. Thus the extended Fan sub-equation method admits the following solution

$$v(\xi) = \frac{A_{-1}}{\Psi(\xi)} + A_0 + A_1\Psi(\xi), \tag{27}$$

where A_{-1}, A_0, A_1 are constants to be determined and Ψ satisfies equation (12).

By substituting equations (27) and (12) into equation (26), collecting the coefficients of Ψ^i and setting them to be zero, a set of algebraic equations is obtained. Solving this set of algebraic equations using Mathematica [21], we get

1.
$$\begin{aligned} A_{-1} &= w_4 = 0, & A_0 &= \tfrac{2+\delta}{3}, \\ A_1 &= \tfrac{k^2(2+\delta)w_3}{2\gamma\delta^2}, & w_3 &= \tfrac{2\gamma\delta^2 A_1}{k^2(2+\delta)}, \end{aligned} \tag{28}$$

with $A_0, A_1 \neq 0, w_0, w_2$ being arbitrary constants.

2.
$$\begin{aligned} A_{-1} &= A_0 = 0, & w_0 &= w_4 = 0, \\ A_1 &= \tfrac{\gamma\delta^2 + k^2 w_2}{k\beta\delta}, & w_2 &= \tfrac{-\gamma\delta^2 + k\beta\delta A_1}{k^2}, \end{aligned} \tag{29}$$

with $w_2, A_1 \neq 0, w_1, w_3$ being arbitrary constants.

3.
$$\begin{aligned} A_1 &= A_0 = 0, & w_3 &= w_4 = 0, \\ A_{-1} &= \tfrac{-kw_0(1+\delta)}{\beta\delta}, & w_1 &= \tfrac{2\left(\delta\lambda + \gamma\delta^2 A_{-1}\right)}{k^2(2+\delta)}, \end{aligned} \tag{30}$$

with $w_1, A_{-1} \neq 0, w_0, w_2$ being arbitrary constants.

Setting $w_2 = 0$ in (28), we get the Weirstrass elliptic doubly periodic type solution [6]

$$v_1(x,t) = A_0 + A_1\varrho\left(\sqrt{\frac{\gamma A_1\delta^2}{2k^2(2+\delta)}}\left(kx - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}\right), g_2, g_3\right), \tag{31}$$

where

$$g_2 = -\frac{4\left(2+\delta\right)\left[k\beta(-4+\delta^2)+3\lambda\right]}{3\gamma\left(-2+\delta\right)\delta A_1},$$

$$g_3 = \frac{2\left(2+\delta\right)^2\left[\gamma\delta(-4+\delta^2)-3\left(k\beta\left(-4+\delta^2\right)+6\lambda\right)A_1\right]}{27\gamma\left(-2+\delta\right)\delta A_1^3}.$$

Substituting (31) into (25), we have

$$u_1\left(x,t\right) = \left[A_0 + A_1\varrho\left(\sqrt{\frac{\gamma A_1\delta^2}{2k^2(2+\delta)}}\left(kx-\frac{\lambda t^\alpha}{\Gamma(1+\alpha)}\right), g_2, g_3\right)\right]^{\frac{1}{\delta}}.$$

If we restrict $w_3 = 0$ in (29) by using (16) and (17) we obtain a triangular type solution and a hyperbolic type solution ($w_2 < 0, w_2 > 0$) successively

$$v_2\left(x,t\right) = \frac{\delta\lambda A_1\left[-1\pm\sin\left(\left(kx-\frac{\lambda t^\alpha}{\Gamma(1+\alpha)}\right)\sqrt{\frac{\gamma\delta^2-k\beta\delta A_1}{k^2}}\right)\right]}{\left(-2+\delta\right)\left(-\gamma\delta^2+k\beta\delta A_1\right)}, \quad (32)$$

$$v_3\left(x,t\right) = \frac{\delta\lambda A_1\left[-1\pm\sinh\left(2\left(kx-\frac{\lambda t^\alpha}{\Gamma(1+\alpha)}\right)\sqrt{\frac{-\gamma\delta^2+k\beta\delta A_1}{k^2}}\right)\right]}{\left(-2+\delta\right)\left(-\gamma\delta^2+k\beta\delta A_1\right)}. \quad (33)$$

Substituting (32) into (25), we have

$$u_2\left(x,t\right) = \left[\frac{\delta\lambda A_1\left[-1\pm\sin\left(\left(kx-\frac{\lambda t^\alpha}{\Gamma(1+\alpha)}\right)\sqrt{\frac{\gamma\delta^2-k\beta\delta A_1}{k^2}}\right)\right]}{\left(-2+\delta\right)\left(-\gamma\delta^2+k\beta\delta A_1\right)}\right]^{\frac{1}{\delta}},$$

$$u_3\left(x,t\right) = \left[\frac{\delta\lambda A_1\left[-1\pm\sinh\left(2\left(kx-\frac{\lambda t^\alpha}{\Gamma(1+\alpha)}\right)\sqrt{\frac{-\gamma\delta^2+k\beta\delta A_1}{k^2}}\right)\right]}{\left(-2+\delta\right)\left(-\gamma\delta^2+k\beta\delta A_1\right)}\right]^{\frac{1}{\delta}}.$$

If we restrict $w_1 = 0$ in (29) by using (18) and (19) we obtain a bell shaped solitary wave solution ($w_2 > 0$)

$$v_4\left(x,t\right) = \frac{\left(\gamma\delta-k\beta A_1\right)\left(2+\delta\right)\sec h^2\left[\left(kx-\frac{\lambda t^\alpha}{\Gamma(1+\alpha)}\right)\sqrt{\frac{-\gamma\delta^2+k\beta\delta A_1}{4k^2}}\right]}{2\gamma\delta}, \quad (34)$$

and a triangular type solution ($w_2 < 0$)

$$v_5\left(x,t\right) = \frac{\left(\gamma\delta-k\beta A_1\right)\left(2+\delta\right)}{2\gamma\delta}\sec^2\left[\left(kx-\frac{\lambda t^\alpha}{\Gamma(1+\alpha)}\right)\sqrt{\frac{\gamma\delta^2-k\beta\delta A_1}{4k^2}}\right]. \quad (35)$$

substituting (34) and (35) into (25), we have

$$u_4(x,t) = \left[\frac{(\gamma\delta - k\beta A_1)(2+\delta)\sec h^2 \left[\left(kx - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)} \right) \sqrt{\frac{-\gamma\delta^2 + k\beta\delta A_1}{4k^2}} \right]}{2\gamma\delta} \right]^{\frac{1}{\delta}},$$

$$u_5(x,t) = \left[\frac{(\gamma\delta - k\beta A_1)(2+\delta)}{2\gamma\delta} \sec^2 \left[\left(kx - \frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)} \right) \sqrt{\frac{\gamma\delta^2 - k\beta\delta A_1}{4k^2}} \right] \right]^{\frac{1}{\delta}}.$$

At the end of this example, graphical representations of exact analytical solution $u_2(x,t)$ are presented for values $\delta = \frac{1}{2}, \alpha = \frac{4}{5}$ and intervals $(x,t) \in \left(\frac{-1}{2}, \frac{1}{2}\right)$ in Fig. 1.

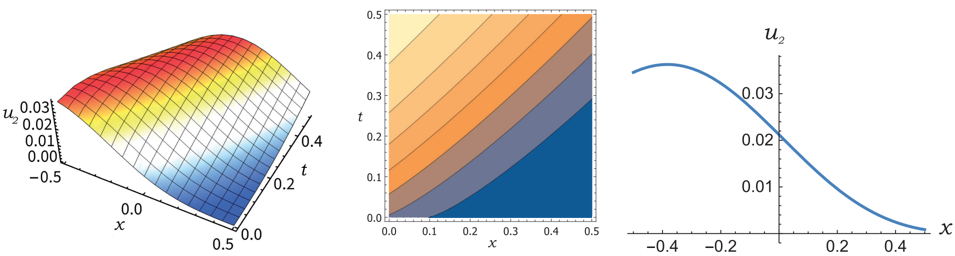

FIGURE 1. The triangular type solution $u_2(x,t)$ of Eq. (1) by substituting the values $\delta = \frac{1}{2},\ \alpha = \frac{4}{5}, A_1 = -1,\ \gamma = 4, \beta = 6$ in Eq. (32).

Graphical representations of exact analytical solution $u_4(x,t)$ are presented for values $\delta = \frac{1}{3}, \alpha = \frac{3}{5}$ and intervals $(x,t) \in \left(\frac{-1}{2}, \frac{1}{2}\right)$ in Fig. 2.

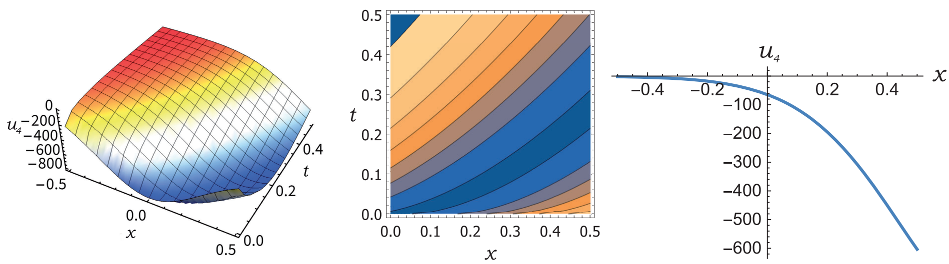

FIGURE 2. Bell shaped solitary wave solution $u_4(x,t)$ of Eq. (1) by substituting the values $\delta = \frac{1}{3},\ \alpha = \frac{3}{5},\ A_1 = 1, k = 2,\ \lambda = 2$ in Eq. (34).

If we restrict $w_0 = \frac{w_1^2}{4w_2}$ in (30) by using (21) we obtain exponential type solution as follows

$$v_6(x,t) = \frac{kw_1^2(1+\delta)}{2\beta\delta\left[w_1 - 2e^{\pm\sqrt{w_2}\left(kx-\frac{\lambda t^\alpha}{\Gamma(1+\alpha)}\right)}w_2\right]}, \qquad w_2 > 0, \tag{36}$$

$$u_6(x,t) = \left[\frac{kw_1^2(1+\delta)}{2\beta\delta\left[w_1 - 2e^{\pm\sqrt{w_2}\left(kx-\frac{\lambda t^\alpha}{\Gamma(1+\alpha)}\right)}w_2\right]}\right]^{\frac{1}{\delta}}, \qquad w_2 > 0.$$

Graphical representations of exact analytical solution $u_6(x,t)$ are presented for values $\delta = \frac{1}{2}, \alpha = \frac{1}{2}$ and intervals $(x,t) \in \left(\frac{-1}{2}, \frac{1}{2}\right)$ in Fig. 3.

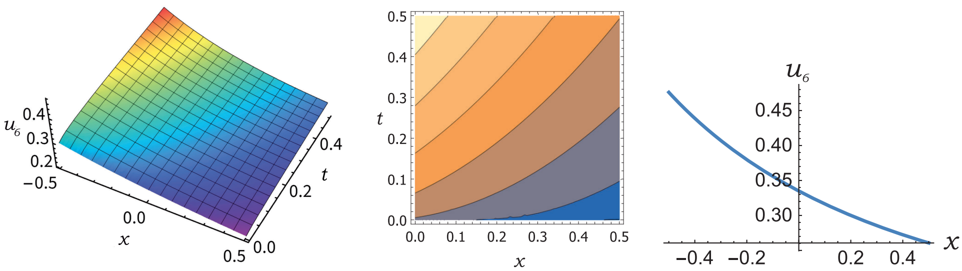

FIGURE 3. Exponential type solution as follows $u_6(x,t)$ of Eq. (1) by substituting the values $\delta = \frac{1}{2}$, $\alpha = \frac{1}{2}$, $\gamma = \frac{-1}{3}$, $k = 2, \lambda = 2$ in Eq. (36).

If we restrict $w_2 = 0$, $w_1 \neq 0$ in (30) by using (22) we obtain a polynomial type solution

$$v_7(x,t) = \frac{-2\delta\lambda + 2k^2w_1 + k^2\delta w_1}{2\gamma\delta^2\left[\frac{1}{4}w_1\left(kx - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}\right)^2 - \frac{w_0}{w_1}\right]}, \tag{37}$$

$$u_7(x,t) = \left[\frac{-2\delta\lambda + 2k^2w_1 + k^2\delta w_1}{2\gamma\delta^2\left[\frac{1}{4}w_1\left(kx - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}\right)^2 - \frac{w_0}{w_1}\right]}\right]^{\frac{1}{\delta}}.$$

If we restrict $w_0 = 0$ in (30), in this way (30) becomes

$$A_1 = A_0 = 0, w_0 = w_3 = w_4 = 0,$$

$$A_{-1} = \frac{-2\delta\lambda + 2k^2w_1 + k^2w_1\delta}{2\gamma\delta^2}, w_1 = \frac{2\left(\delta\lambda + \gamma\delta^2A_{-1}\right)}{k^2(2+\delta)},$$

by using (16) and (17) we obtain a triangular type solution and a hyperbolic type solution

$$v_8(x,t) = \frac{-2\delta\lambda + 2k^2 w_1 + k^2 w_1 \delta}{2\gamma\delta^2 \frac{w_1}{2w_2}\left[-1 \pm \sin\left(\left(kx - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}\right)\sqrt{-w_2}\right)\right]}, \quad w_2 < 0, \tag{38}$$

$$v_9(x,t) = \frac{-2\delta\lambda + 2k^2 w_1 + k^2 w_1 \delta}{2\gamma\delta^2 \frac{w_1}{2w_2}\left[-1 \pm \sinh\left(2\left(kx - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}\right)\sqrt{w_2}\right)\right]}, \quad w_2 > 0, \tag{39}$$

substituting (38) and (39) into (25), we have

$$u_8(x,t) = \left[\frac{-2\delta\lambda + 2k^2 w_1 + k^2 w_1 \delta}{2\gamma\delta^2 \frac{w_1}{2w_2}\left[-1 \pm \sin\left(\left(kx - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}\right)\sqrt{-w_2}\right)\right]}\right]^{\frac{1}{\delta}}, \quad w_2 < 0,$$

$$u_9(x,t) = \left[\frac{-2\delta\lambda + 2k^2 w_1 + k^2 w_1 \delta}{2\gamma\delta^2 \frac{w_1}{2w_2}\left[-1 \pm \sinh\left(2\left(kx - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}\right)\sqrt{w_2}\right)\right]}\right]^{\frac{1}{\delta}}, \quad w_2 > 0.$$

5. Conclusion

We have applied the extended Fan sub-equation method to solve time fractional generalized Burgers-Fisher equation. The execution of extended Fan subequation method in the sense of Jumarie's modified Riemann- Liouville derivative and many new exact solitons solutions including polynomial solutions, trigonometric periodic wave solutions, exponential solutions, rational solutions, hyperbolic and solitary wave solutions have been presented. This approach is useful when analytical solution of the mathematically defined problem is possible, but it is time-consuming and the error of approximation obtained with numerical solution is acceptable. In this case, the calculations are mostly made with use of computer, because otherwise it will be highly doubtful if any time is saved. Numerical methods are commonly used for solving mathematical problems that are formulated in science and engineering where it is difficult or impossible to obtain exact solutions.

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Department of Mathematics
Faculty of Mathematics et Informatique
University of Bachir El-Ibrahimi
34 Bordj Bou Arraridj
ALGERIA
E-mail: djouaherabbas@yahoo.com

Laboratory of Fundamental and
Numerical Mathematics
Department of Mathematics
Faculty of Sciences
Ferhat Abbas Sétif University 1
19000 Sétif
ALGERIA
E-mail: abdelouahabk@yahoo.fr