DE GRUYTER OPEN

TATRA MOUNTAINS
Mathematical Publications
DOI: 10.1515/tmmp-2016-0022
Tatra Mt. Math. Publ. **66** (2016), 81–90

THE q-GAMMA WHITE NOISE

Hakeem A. Othman

ABSTRACT. For $0 < q < 1$ and $0 < \alpha < 1$, we construct the infinite dimensional q-Gamma white noise measure $\gamma_{\alpha,q}$ by using the Bochner-Minlos theorem. Then we give the chaos decomposition of an L^2 space with respect to the measure $\gamma_{\alpha,q}$ via an isomorphism with the 1-mode type interacting Fock space associated to the q-Gamma measure.

1. Introduction and preliminaries

White noise can be informally regarded as a stochastic process which is independent at different times and is identically distributed with zero mean and infinite variance. T. Hida introduced the theory of white noise in 1975. Nowadays, this theory has been applied to stochastic integration, stochastic partial differential equation, stochastic variational equation, infinite dimensional harmonic analysis, Dirichlet forms, quantum field theory, Feynman integral and quantum probability. For the non-Gaussian white noise analysis, Y. Ito constructed the Poissonian counterpart of Hida's theory and Kondratiev et al. [8] established a purely non-Gaussian distribution theory in infinite dimensional analysis by means of a normalized Laplace transform, and Barhoumi et al. [1] developed the introduction of infinite dimensional Gegenbauer white noise.

On the other hand, a q-deformation of some orthogonal polynomials and their associated measure were introduced. We put for $n \in \mathbb{N}$,

$$[n]_q := \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1} \qquad ([0]_q := 0).$$

The q-factorial is

$$[n]_q! := [1]_q \times [2]_q \ldots [n]_q, \quad [0]_q! := 1.$$

2010 Mathematics Subject Classification: 60H40, 46A32, 46F25, 46G20.
Keywords: q-Gamma distribution, q-Laguerre polynomials, q-Gamma noise space, 1-mode type interacting Fock space.

Another quite frequently used symbol is the q-analogue of the Pochhammer symbol:

$$(a;q)_0 := 1 \quad \text{and} \quad (a;q)_n := \prod_{j=0}^{n-1}(1-aq^j), \qquad n = 1,2,\dots,\infty.$$

It is easy to see that for all integers n

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}.$$

(See [7], [9]). The q-gamma function

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \qquad 0<q<1, \quad x \neq 0,-1,-2,\dots$$

was introduced by Thomac in 1869 and later by Jackson in 1904. In 1847, Heine gave an equivalent definition, but without the factor $(1-q)^{1-x}$. When $x = n+1$ with n a nonnegative integer, this definition reduces to

$$\begin{aligned}\Gamma_q(n+1) &= 1(1+q)(1+q+q^2)+\dots+(1+q+q^2+\dots+q^{n-1})\\ &= [n]_q!,\end{aligned}$$

which clearly approaches $n!$ as $q \longrightarrow 1$. The q-Gamma distribution is given by

$$d\mu_{\alpha,q}(x) = \frac{\Gamma_q(-\alpha)}{\Gamma(\alpha+1)\Gamma(-\alpha)} x^\alpha e_q(-x)\mathbb{I}_{]0,+\infty[}(x)\,dx$$

and we can easily verify that its Fourier transform is given by:

$$\begin{aligned}\widehat{\mu}_{\alpha,q}(\lambda) &= \frac{\Gamma_q(-\alpha)}{\Gamma(\alpha+1)\Gamma(-\alpha)}\int_0^\infty e^{ix\lambda}\times x^\alpha \times e_q(-x)\,dx\\ &= \sum_{n\geq 0}\frac{\Gamma(\alpha+n+1)\Gamma(-\alpha-n)\Gamma_q(-\alpha)}{\Gamma(\alpha+1)\Gamma(-\alpha)\Gamma_q(-\alpha-n)}\frac{(i\lambda)^n}{n!}.\end{aligned}$$

The present paper elaborates an introduction of the q-Gamma white noise. The outline of our paper is as follows. In Section 2, we use the q-Gamma function to construct a standard nuclear triplet

$$E_{\alpha,q} \subset H_{\alpha,q} := L^2(I,\mu_{\alpha,q}) \subset E'_{\alpha,q}$$

by the Bochner-Minlos theorem. In Section 3, we define the q-Gamma white noise measure $\gamma_{\alpha,q}$ on $\big(E'_{\alpha,q},\mathcal{B}(E'_{\alpha,q})\big)$ and we define the q-type wick product by means of an orthogonal system of infinite dimensional q-Laguerre polynomials, and we investigate the chaos decomposition of the space $L^2\big(E'_{\alpha,q},\mathcal{B}(E'_{\alpha,q}),\gamma_{\alpha,q}\big)$.

2. q-Gamma white noise measure

Let

$$d\mu_{\alpha,q}^{\sigma}(x) = \begin{cases} \frac{\Gamma_q(-\alpha)}{\Gamma(\alpha+1)\Gamma(-\alpha)|\sigma|^{\alpha+1}} x^{\alpha} e_q(\frac{-x}{|\sigma|}) \mathbb{I}_{]0,+\infty[}(x)\, dx, & if\ \sigma \neq 0, \\ \mu_{\alpha,q}^{0}(dx) = \delta_0(x) & if\ \sigma = 0 \end{cases}$$

and $0 < \alpha < 1$. Note that for $\sigma = 1$, we have the standard q-Gamma distribution:

$$\mu_{\alpha,q}(dx) = \frac{\Gamma_q(-\alpha)}{\Gamma(\alpha+1)\Gamma(-\alpha)} x^{\alpha} e_q(-x) \mathbb{I}_{]0,+\infty[}(x)\, dx.$$

In [7], the q-Laguerre polynomials are defined by

$$L_n^{\alpha}(x;q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n} \times_1 \phi_1(q^{-n}; q^{\alpha+1}; q; -xq^{n+\alpha+1}),$$

where the basic hypergeometric series ${}_r\phi_s$ have the following forms:

$$r\phi_s(a_1, a_2, \ldots, a_r; b_1, b_2, \ldots, b_s; q, z) \equiv_r \phi_s \begin{bmatrix} a_1, a_2, \ldots, a_r & \\ & ;\ q, z \\ b_1, b_2, \ldots, b_s & \end{bmatrix}$$

$$= \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n, \ldots, (a_r;q)_n}{(q;q)_n (b_1;q)_n (b_2;q)_n, \ldots, (b_s;q)_n} \left[(-1)^n q^{\begin{pmatrix} n \\ 2 \end{pmatrix}} \right]^{1+s-r} z^n$$

with

$$\begin{pmatrix} n \\ 2 \end{pmatrix} = \frac{n(n-1)}{2}.$$

They are orthogonal with respect to $\mu_{\alpha,q}$ since they verify

$$\int_0^{\infty} L_m^{\alpha}(x;q) L_n^{\alpha}(x;q)\, d\mu_{\alpha,q}(x) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n q^n} \delta_{m,n}.$$

Therefore, if we put

$$h_{n,q}(x) = L_n^{\alpha}(x;q) \left(\frac{(q^{\alpha+1};q)_n}{(q;q)_n q^n} \right)^{\frac{-1}{2}},$$

we obtain an orthonormal basis $\{h_{n,q},\ n \in \mathbb{N}\}$ for the Hilbert space $H_{\alpha,q} = L^2(I, \mu_{\alpha,q})$, with $I =]0, +\infty[$. Now, define the operator A_q, on $H_{\alpha,q}$ by

$$\big(A_q L_n^{\alpha}(.,q)\big)(x) = x D_q^2 L_n^{\alpha}\left(\frac{x}{q}; q\right) - \left(\frac{1-q^{\alpha+1}}{1-q} - q^{\alpha+1} x \right) D_q L_n^{\alpha}(x;q),$$

where $D_q f(x) = \frac{f(qx)-f(x)}{x(q-1)}$. It is easy to show that for all $n \in \mathbb{N}$, the function $h_{n,q}$ is an eigenvector of A_q, namely

$$A_q h_{n,q} = \lambda_{q,n} h_{n,q},$$

with $\lambda_{q,n} = q^{\alpha+1-n/2} \times [n]_q$. Moreover, for any $p > 0$, A_q^{-p} is the Hilbert-Schmidt operator satisfying:

$$\| A_q^{-p} \|_{HS}^2 = \sum_{n\geq 0} \lambda_{q,n}^{-2p} < \infty.$$

For each $p \in \mathbb{R}$, define the norm $| \, . \, |_p$ on $H_{\alpha,q}$ by

$$| f |_p = | A_q^p f |_0 = \left(\sum_{n\geq 0} \lambda_{q,n}^{2p} \langle f, h_{n,q} \rangle^2 \right)^{1/2}, \qquad f \in H_{\alpha,q}\,,$$

where $| \, . \, |_0$ and $\langle ., . \rangle$ are the norm and the inner product of $H_{\alpha,q}$, respectively. For $p \geq 0$, let $E_{p,\alpha,q}$ be the Hilbert space consisting of all $f \in H_{\alpha,q}$ with $| f |_q < \infty$, and $E_{-p,\alpha,q}$, the completion of $H_{\alpha,q}$ with respect to $| \, . \, |_{-p}$. Since A_q^{-1} is of the Hilbert-Schmidt type, identifying $H_{\alpha,q}$ with its dual space we come to real standard nuclear triplet

$$E_{\alpha,q} := \bigcap_{p\geq 0} E_{p,\alpha,q} \subset H_{\alpha,q} \subset \bigcup_{p\geq 0} E_{-p,\alpha,q} := E'_{\alpha,q}\,.$$

Lemma 2.1. *The function $C(\xi) = \hat{\mu}_{\alpha,q}(\langle \xi \rangle)$, $\xi \in E_{\alpha,q}$ is a characteristic function, where $\langle \xi \rangle = \int_I \xi(x)\, dx$.*

P r o o f. Obviously, C is continuous on $E_{\alpha,q}$ and $C(0) = 1$. We shall prove that C is positive definite. For $z_1, \dots, z_n \in \mathbb{C}$ and $\xi_1, \dots, \xi_n \in E_{\alpha,q}$, we can see that

$$\begin{aligned}
\sum_{j,k=1}^{n} z_j \overline{z}_k \; C(\xi_j - \xi_k) &= \sum_{j,k=1}^{n} z_j \overline{z}_k \hat{\mu}_{\alpha,q}(\langle \xi_j - \xi_k \rangle) \\
&= \sum_{j,k=1}^{n} z_j \overline{z}_k \int_0^\infty e^{ix\langle \xi_j \rangle} e^{-ix\, \langle \xi_k \rangle} d_{\mu_{\alpha,q}(x)} \\
&= \int_0^\infty \sum_{j,k=1}^{n} l_j(x) \overline{l_k}(x)\, d_{\mu_{\alpha,q}(x)} \\
&= \int_0^\infty \left| \sum_{j=1}^{n} l_j(x) \right|^2 d_{\mu_{\alpha,q}(x)} \geq 0,
\end{aligned}$$

where $l_j(x) = z_j\, e^{ix\, \langle \xi_j \rangle}$. □

Definition 2.2. The probability measure over $E'_{\alpha,q}$, denoted by $\gamma_{\alpha,q}$, is defined by its Fourier transform (via the Bochner-Minlos Theorem) given by:

$$\int_{E'_{\alpha,q}} e^{i\langle \omega, \varphi \rangle} d\gamma_{\alpha,q}(\omega) = \sum_{n\geq 0} \frac{\Gamma(\alpha+n+1)\Gamma(-\alpha-n)\Gamma_{\alpha,q}(-\alpha)}{\Gamma(\alpha+1)\Gamma(-\alpha)\Gamma_q(-\alpha-n)} \frac{(i\langle \varphi \rangle)^n}{n!}.$$

We denote $\gamma_{\alpha,q}$ the q-Gamma noise measure and the space $(E_{\alpha,q}, \gamma_{\alpha,q})$ by the q-Gamma space.

Proposition 2.3. *For $\xi \in E'_{\alpha,q}$ s.t $\langle \xi \rangle \geq 0$ let X_ξ be the random variable defined on $(E'_{\alpha,q}, B(E'_{\alpha,q}), \gamma_{\alpha,q})$ by*

$$X_\xi(\omega) := \langle \omega, \xi \rangle,$$

where $B(E'_{\alpha,q})$ is the σ-algebra on $E'_{\alpha,q}$. Then X_ξ has a q-Gamma distribution with parameters α and $\langle \xi \rangle$.

Proof. We have

$$\int_{E'_{\alpha,q}} e^{i\lambda X_\xi(\omega)}\, d\gamma_{\alpha,q}(\omega) = C(\lambda\xi) = \widehat{\mu}(\lambda\langle\xi\rangle) = \widehat{\mu}^{\langle\xi\rangle}_{\alpha,q}(\lambda), \qquad \lambda \in \mathbb{R}.$$

It proves that the distribution of the random variable X_ξ is the probability measure $\mu^{\langle\xi\rangle}_{\alpha,q}$ on $\mathbb{R}$. □

Lemma 2.4. *For all $\xi \in H_{\alpha,q}$ we have*

$$\int_{E'_{\alpha,q}} \langle \omega, \xi \rangle^k \, d\gamma_{\alpha,q}(\omega) = \frac{\Gamma(\alpha+k+1)\Gamma(-\alpha-k)\Gamma_q(-\alpha)}{\Gamma(\alpha+1)\Gamma(-\alpha)\Gamma_q(-\alpha-k)k!} \mid \langle\xi\rangle \mid^k .$$

Proof. We know that

$$\int_0^\infty x^k d\mu_{\alpha,q}(x) = \frac{(\hat{\mu}_{\alpha,q}(\lambda))^{(k)}}{i^k}\Bigg|_{\lambda=0} = \frac{\Gamma(\alpha+k+1)\Gamma(-\alpha-k)\Gamma_q(-\alpha)}{\Gamma(\alpha+1)\Gamma(-\alpha)\Gamma_q(-\alpha-k)k!}.$$

Then, we get

$$\begin{aligned}
\int_{E'_{\alpha,q}} \langle \omega, \xi \rangle^k \, d\gamma_q(\omega) &= \int_0^\infty y^k \, d\mu^{\langle\xi\rangle}_{\alpha,q}(y) \\
&= \int_0^\infty \mid \langle\xi\rangle \mid^k x^k \, d\mu_{\alpha,q}(x) \\
&= \frac{\Gamma(\alpha+k+1)\Gamma(-\alpha-k)\Gamma_q(-\alpha)}{\Gamma(\alpha+1)\Gamma(-\alpha)\Gamma_q(-\alpha-k)k!} \mid \langle\xi\rangle \mid^k
\end{aligned}$$

and it completes the proof. □

3. q-Gamma isomorphism

Let us begin by introducing the q-type Gamma wick product.

Definition 3.1. For $\omega \in E'_{\alpha,q}$ and $n = 0, 1, 2, \ldots$, we define the q-type Gamma wick product: $:\omega^{\otimes n}: q \in (E'_{\alpha,q})^{\otimes n}$ as follows:

$$\langle : \omega^{\otimes n} : q, \varphi^{\otimes n} \rangle = | \varphi |_0^n \, L_n^\alpha \left(\tfrac{\langle \omega, |\varphi| \rangle}{\langle |\varphi| \rangle}; q \right), \qquad \varphi \in E_{\alpha,q} \,. \tag{3.2}$$

Lemma 3.3. *For all $\xi \in E_{\alpha,q}$ such that $\xi \neq 0$, then*

$$\int_{E'_{\alpha,q}} \langle : \omega^{\otimes n} : q, \xi^{\otimes n} \rangle \langle : \omega^{\otimes m} : q, \xi^{\otimes m} \rangle \, d\gamma_{\alpha,q}(\omega) = | \xi |_0^{2n} \times \frac{(q^{\alpha+1}; q)_n}{(q;q)_n q^n} \, \delta_{m,n} \,. \tag{3.4}$$

Proof. For $\xi \in H_{\alpha,q}, \xi \neq 0$, the image of the q-Gamma white noise measure $\gamma_{\alpha,q}$ under the map

$$\omega \longmapsto \left\langle \omega, \frac{| \xi |}{| \langle \xi \rangle |} \right\rangle \in R, \qquad \omega \in E'_{\alpha,q}$$

is the q-Gamma distribution $\mu_{\alpha,q}$. Then we have

$$\begin{aligned} &\int_{E'_{\alpha,q}} \langle : \omega^{\otimes n} : q, \xi^{\otimes n} \rangle \langle : \omega^{\otimes m} : q, \xi^{\otimes m} \rangle \, d\gamma_{\alpha,q}(\omega) \\ &= | \xi |_0^{n+m} \int_0^\infty L_n^\alpha(x;q) L_m^\alpha(x;q) \, d\mu_{\alpha,q}(x) \\ &= | \xi |_0^{2n} \, \frac{(q^{\alpha+1}; q)_n}{(q;q)_n q^n} \, \delta_{m,n} \,. \end{aligned}$$

This gives the desired statement. □

Let us denote $\mathcal{F}_{\alpha,q}(H_{\alpha,q})$ the following $1-mode$ type interacting Fock space:

$$\mathcal{F}_{\alpha,q}(H_{\alpha,q}) := \bigoplus_{n=0}^{\infty} \frac{(q^{\alpha+1}; q)_n}{(q;q)_n q^n} \, H_{\alpha,q}^{\widehat{\otimes} n} \,.$$

Thus $\mathcal{F}_{\alpha,q}(H_{\alpha,q})$ consists of sequences $\overrightarrow{f} = (f_0, f_1, \ldots)$ such that for any $n \in \mathbb{N}$ $f_n \in H_{\alpha,q}^{\widehat{\otimes} n}$ and

$$\| \overrightarrow{f} \|^2_{\mathcal{F}_{\alpha,q}(H_{\alpha,q})} = \sum_{n=0}^{\infty} \frac{(q^{\alpha+1}; q)_n}{(q;q)_n q^n} \, \| f_n \|^2_{H_{\alpha,q} \widehat{\otimes} n} < \infty \,.$$

Theorem 3.5. *For each $F \in L^2(E'_{\alpha,q}, \gamma_{\alpha,q})$, there exists a unique sequence $\overrightarrow{f} = (f_n)_{n=0}^{\infty} \in \mathcal{F}_{\alpha,q}(H_{\alpha,q})$ such that*

$$F = \sum_{n=0}^{\infty} \langle : .^{\otimes n} : q, f_n \rangle \tag{3.6}$$

in the L^2-sense. On the other hand, for any $\overrightarrow{f} = (f_n)_{n=0}^{\infty} \in \mathcal{F}_{\alpha,q}(H_{\alpha,q})$, (3.6) defines a function on $L^2(E'_{\alpha,q}, \gamma_{\alpha,q})$. In that case,

$$\| F \|^2_{L^2(E'_{\alpha,q}, \gamma_{\alpha,q})} = \sum_{n=0}^{\infty} \frac{(q^{\alpha+1}; q)_n}{(q;q)_n q^n} \| f_n \|^2_{H_{\alpha,q} \widehat{\otimes} n} = \| \overrightarrow{f} \|^2_{\mathcal{F}_{\alpha,q}(H_{\alpha,q})} .$$

The following unitary operator is called the q-Gamma isometry

$$I \colon \mathcal{F}_q(H_{\alpha,q}) \longrightarrow L^2(E'_{\alpha,q}, \gamma_{\alpha,q}) ,$$
$$(f_n)_{n=0}^{\infty} \longmapsto F.$$

P r o o f. It is easy to see that the set

$$\mathcal{P}(E'_{\alpha,q}) = \left\{ \phi, \phi(\omega) = \sum_{k=0}^{n} \langle : \omega^{\otimes k} : q, \phi_k \rangle,\ \phi_k \in E^{\otimes k}_{\alpha,q},\ \omega \in E'_{\alpha,q},\ n \in \mathbb{N} \right\}$$

of smooth continuous polynomials on $E'_{\alpha,q}$ is continuously and densely embedded in $L^2(E'_{\alpha,q}, \gamma_{\alpha,q})$. Then, for any $F \in L^2(E'_{\alpha,q}, \gamma_{\alpha,q})$ there exists a unique sequence $\overrightarrow{f} = (f_n)_{n=0}^{\infty} \in \mathcal{F}_{\alpha,q}(H_{\alpha,q})$ such that

$$F = \sum_{n=0}^{\infty} \langle : .^{\otimes n} : q, f_n \rangle .$$

We need to show for two polynomials ϕ, ψ given by, respectively,

$$\phi(\omega) = \sum_{n=0}^{\infty} \langle : \omega^{\otimes n} : q, \phi_n \rangle, \quad \psi(\omega) = \sum_{n=0}^{\infty} \langle : \omega^{\otimes n} : q, \psi_n \rangle$$

that

$$\int_{E'_{\alpha,q}} \phi(\omega) \psi(\omega) \, d\gamma_q(\omega) = \sum_{n=0}^{\infty} \frac{(q^{\alpha+1}; q)_n}{(q;q)_n q^n} \langle \phi_n, \psi_n \rangle . \tag{3.7}$$

In particular, the L^2-norm of ϕ with respect to $\gamma_{\alpha,q}$ is given by

$$\| \phi \|_0^2 = \sum_{n=0}^{\infty} \frac{(q^{\alpha+1}; q)_n}{(q;q)_n q^n} \mid \phi_n \mid_0^2 . \tag{3.8}$$

For this and since ϕ_n and ψ_n are linear combinations of elements of the form $\xi^{\otimes n}, \xi \in E_{\alpha,q}$, it suffices to show that for $\xi_1, \xi_2 \in E_{\alpha,q}$,

$$\int_{E'_{\alpha,q}} \langle : \omega^{\otimes n} : q, \xi_1^{\otimes n}\rangle\langle : \omega^{\otimes m} : q, \xi_2^{\otimes m}\rangle \, d\gamma_{\alpha,q}(\omega) = \langle \xi_1, \xi_2\rangle^n \times \frac{(q^{\alpha+1}; q)_n}{(q;q)_n q^n} \, \delta_{m,n} \, . \quad (3.9)$$

It is sufficient to prove the identity under the assumption $|\ \xi\ |_0 = |\ \eta\ |_0 = 1$. Taking a unit vector $\zeta \in E_{\alpha,q}$ such that $\langle \xi, \zeta\rangle = 0$, we may write

$$\eta = \beta\xi + \lambda\zeta, \quad \beta^2 + \lambda^2 = 1,$$

and we have

$$\langle : \omega^{\otimes n} : q, \eta^{\otimes n}\rangle = \sum_{k=0}^{n} \binom{n}{k} \beta^{n-k}\lambda^k \langle : \omega^{\otimes(n-k)} : q, \xi^{\otimes(n-k)}\rangle\langle : \omega^{\otimes k} : q, \zeta^{\otimes k}\rangle.$$

Then we get

$$\begin{aligned} &\int_{E'_{\alpha,q}} \langle : \omega^{\otimes m} : q, \xi^{\otimes m}\rangle\langle : \omega^{\otimes n} : q, \eta^{\otimes n}\rangle \, d\gamma_{\alpha,q}(\omega) \\ &= \sum_{k=o}^{n} \binom{n}{k} \beta^{n-k}\lambda^k \int_{E'_{\alpha,q}} \langle : \omega^{\otimes m} : q, \xi^{\otimes m}\rangle\langle : \omega^{\otimes(n-k)} : q, \xi^{\otimes(n-k)}\rangle \\ &\qquad\qquad \langle : \omega^{\otimes k} : q, \zeta^{\otimes k}\rangle \, d\gamma_{\alpha,q}(\omega). \end{aligned}$$

On the other hand, by using the independence of the two random variables $\langle ., \xi\rangle$ and $\langle ., \zeta\rangle$, we obtain

$$\begin{aligned} &\int_{E'_{\alpha,q}} \langle : \omega^{\otimes m} : q, \xi^{\otimes m}\rangle\langle : \omega^{\otimes(n-k)} : q, \xi^{\otimes(n-k)}\rangle\langle : \omega^{\otimes k} : q, \zeta^{\otimes k}\rangle \, d\gamma_{\alpha,q}(\omega) \\ &= \int_{E'_{\alpha,q}} \langle : \omega^{\otimes m} : q, \xi^{\otimes m}\rangle\langle : \omega^{\otimes(n-k)} : q, \xi^{\otimes(n-k)}\rangle \, d\gamma_{\alpha,q}(\omega) \\ &\quad \times \int_{E'_{\alpha,q}} \langle : \omega^{\otimes k} : q, \zeta^{\otimes k}\rangle \, d\gamma_{\alpha,q}(\omega). \end{aligned}$$

Therefore, the last integral is equal to 0 unless $k \neq 0$ and is 1 if k=0. Hence,

$$\begin{aligned} &\int_{E'_{\alpha,q}} \langle : \omega^{\otimes m} q, \xi^{\otimes m}\rangle\langle : \omega^{\otimes n} : q, \eta^{\otimes n}\rangle \, d\gamma_{\alpha,q}(\omega) \\ &= \beta^n \, \frac{(q^{\alpha+1}; q)_n}{(q;q)_n q^n} \int_{E'_{\alpha,q}} \langle : \omega^{\otimes m} : q, \xi^{\otimes m}\rangle\langle : \omega^{\otimes n} : q, \xi^{\otimes n}\rangle \, d\gamma_{\alpha,q}(\omega). \end{aligned}$$

We conclude that

$$\int\limits_{E'_{\alpha,q}} \langle :\omega^{\otimes m}: q, \xi^{\otimes m}\rangle \langle :\omega^{\otimes n}: q, \eta^{\otimes n}\rangle \, d\gamma_{\alpha,q}(\omega) = \beta^n \, \frac{(q^{\alpha+1};q)_n}{(q;q)_n q^n} \, \delta_{m,n} \, .$$

Since $\beta = \langle \xi, \eta \rangle$, we have completed the proof of (3.9). Then, because of this and ϕ_n and ψ_n are linear combinations of elements of the form $\xi^{\otimes n}$, $\xi \in E_{\alpha,q}$ for all $\phi_n, \psi_n \in E_{\alpha,q}^{\widehat{\otimes} n}$ we have:

$$\int\limits_{E'_{\alpha,q}} \langle :\omega^{\otimes n}: q, \phi_n\rangle \langle :\omega^{\otimes m}: q, \psi_m\rangle \, d\gamma_{\alpha,q}(\omega) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n q^n} \, \langle \phi_n, \psi_n \rangle \times \delta_{m,n} \, .$$

It follows from (3.7) and (3.8) that

$$\begin{aligned} \| \, F \, \|^2_{L^2(E'_{\alpha,q}, \gamma_{\alpha,q})} &= \int\limits_{E'_{\alpha,q}} \left(\sum_{n=0}^{\infty} \langle :\omega^{\otimes n} : q, f_n \rangle \right)^2 d\gamma_{\alpha,q}(\omega) \\ &= \sum_{n=0}^{\infty} \frac{(q^{\alpha+1};q)_n}{(q;q)_n q^n} \, \langle f_n, f_n \rangle_{H_{\alpha,q} \widehat{\otimes} n} \\ &= \| \, \overrightarrow{f} \, \|^2_{\mathcal{F}_{\alpha,q}(H_{\alpha,q})} \, . \end{aligned}$$

The second part of the Theorem 3.5 is straightforward. □

Remark 3.10. In this paper we constructed the infinite dimensional q-Gamma white noise, in particular the q-Gamma Gel'fand triple was obtained. We can use it to develop a new theory of the q-Gamma white noise as analogue of Hida's theory in the white noise setting. Also a space of holomorphic functions on the complexification of $E'_{\alpha,q}$ with θ-exponential growth which is the analogue of the Gaussian case (where θ is the Young function) can be introduced. We expect to develop a new quantum q-Gamma white noise operator theory as an analogue of the quantum white noise operator theory, (see [2]–[6], [11]–[13]).

REFERENCES

[1] BARHOUMI, A.—OUERDIANE, H.—RIAHI, A.: *Infinite dimensional Gegenbauer functionals*, Banach Center Publ. **78** (2007), 35–45.

[2] BARHOUMI, A.—LANCONELLI, A.—RGUIGUI, H.: *QWN-convolution operators with application to differential equations*, Random Oper. Stochastic Equations **22** (2014), 195–211.

[3] BARHOUMI, A.—OUERDIANE, H.—RGUIGUI, H.: *QWN-Euler operator and associated Cauchy problem*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **15** (2012), 1250004, 20 p.

[4] BARHOUMI, A.—OUERDIANE, H.—RGUIGUI, H.: *Stochastic heat equation on algebra of generalized functions*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **15** (2012), 1250026, 18 p.
[5] BARHOUMI, A.—BEN AMMOU, B. K.—RGUIGUI, H.: *Operator theory: quantum white noise approach*, Quantum Stud. Math. Found. **2** (2015), 221–241.
[6] GANNOUN, R.—HCHAICHI, R.—OUERDIANE, H.—REZGUI, A.: *Un théorème de dualité entre espace de fonction holomorphes à croissance exponentielle,* J. Funct. Anal. **171** (2000), 1–14.
[7] GASPER, G.—RAHMAN, M.: *Basic Hypergeometric Series,* in: Encyclopedia Math. Appl., Vol. 34, Cambridge Universty Press, Cambridge, 1990.
[8] KONDRATIV, YU. G.—SILVA, J. L.—STREIT, L.—US, G. F.: *Analysis on Poisson and Gamma space.* Infin. Dimens. Anal. Quantum Probab. Relat. Top. **1** (1998), 91–118.
[9] VAN LEEUWEN, H.—MAASSEN, H.: *A q-deformation of the Gauss distribution.* J. Math. Phys. **36** (1995), 4743–4756.
[10] OBATA, N.: *White Noise Calculus and Fock Space,* in: Lecture Notes in Math., Vol. 1577, Springer, Berlin, 1994.
[11] OUERDIANE, H.—RGUIGUI, H.: *QWN-conservation operator and associated wick differential equation*, Commun. Stoch. Anal. **6** (2012), 437–450.
[12] RGUIGUI, H.: *Quantum Ornstein-Uhlenbeck semigroups,* Quantum Stud. Math. Found. **2** (2015), 159–175.
[13] RGUIGUI, H.: *Quantum λ-potentials associated to quantum Ornstein-Uhlenbeck semigroups*, Chaos Solitons Fractals **73** (2015), 80–89.

Received August 26, 2016

Department of Mathematics
Al-Qunfudah Center for
Scientific Research
Al-Qunfudhah University College
Umm Al-Qura University
KSA

Department of Mathematics
Rada'a College of Education and Science
Albida University
Albida
YEMEN
E-mail: hakim_albdoie@yahoo.com