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ON STRONGLY WRIGHT-CONVEX STOCHASTIC PROCESSES

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ABSTRACT. Some characterizations of strongly Wright-convex stochastic processes are presented. Furthermore, the stochastic version of a theorem on strongly J-convex functions majorized by strongly J-concave functions is given.

1. Introduction

In 1980 K. Nikodem [7] introduced the notion of convex (Jensen-convex) stochastic processes and proved some basic properties of them. In particular, he gave conditions under which Jensen-convex stochastic processes are continuous. Next, A. Skowroński in [9] and [10] obtained some further properties of Jensen-convex and Wright-convex stochastic processes.

Strongly convex stochastic processes were investigated by the author in [2]. The aim of this note is to introduce the notion of strongly Wright-convex stochastic processes and to present some properties of them. In particular, we give a characterization of strongly Wright-convex stochastic processes, which is a counterpart of the celebrated C. T. Ng's representation theorem [5] of Wright-convex functions. We prove also the theorem on Jensen-convex stochastic processes majorized by Jensen-concave stochastic processes. It is a stochastic version of the results on (strongly) midconvex functions with (strongly) midconcave bounds proved in [1], [3], [6] and [8].

Let $(\Omega, \mathcal{A}, P)$ be an arbitrary probabilistic space. A function $X : \Omega \to \mathbb{R}$ is called a *random variable*, if it is $\mathcal{A}$-measurable. Given an interval I, a function $X : I \times \Omega \to \mathbb{R}$ is called a *stochastic process*, if for every $t \in I$ the function $X(t, \cdot)$ is a random variable.

Let $C : \Omega \to \mathbb{R}$ denote a positive random variable. We say that a stochastic process $X : I \times \Omega \to \mathbb{R}$ is

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(i) *strongly convex with modulus* $C(\cdot)$, if

$$X\big(\lambda u + (1-\lambda)v, \cdot\big) \leqslant \lambda X(u,\cdot) + (1-\lambda)X(v,\cdot) - C(\cdot)\lambda(1-\lambda)(u-v)^2 \quad \text{(a.e.)}$$

for all $\lambda \in [0,1]$ and $u, v \in I$,

(ii) *strongly midconvex (or strongly Jensen-convex) with modulus* $C(\cdot)$, if the above inequality is assumed only for $\lambda = \frac{1}{2}$ and all $u, v \in I$, i.e.,

$$X\Big(\frac{u+v}{2}, \cdot\Big) \leqslant \frac{1}{2}X(u,\cdot) + \frac{1}{2}X(v,\cdot) - \frac{C(\cdot)}{4}(u-v)^2 \quad \text{(a.e.)},$$

(iii) *strongly Wright-convex with modulus* $C(\cdot)$, if

$$X\big(\lambda u + (1-\lambda)v, \cdot\big) + X\big((1-\lambda)u + \lambda v, \cdot\big) \leqslant X(u,\cdot) + X(v,\cdot) - 2C(\cdot)\lambda(1-\lambda)(u-v)^2 \quad \text{(a.e.)}$$

holds for all $\lambda \in [0,1]$ and $u, v \in I$.

Obviously, by omitting the term $C(\cdot)\lambda(1-\lambda)(u-v)^2$ in cases (i) and (iii), we immediately get the definition of a convex, or Wright-convex stochastic processes introduced by K. Nikodem in [7], and A. Skowroński in [10], respectively. Simple computation shows that every strongly convex stochastic process is strongly Wright-convex, and every strongly Wright-convex stochastic process is strongly midconvex with the same modulus, but not the converse. More properties of strongly convex and strongly midconvex stochastic processes can be found in [2].

A stochastic process $A\colon \mathbb{R} \times \Omega \to \mathbb{R}$ is called *additive* if

$$A(u+v,\cdot) = A(u,\cdot) + A(v,\cdot) \quad \text{(a.e.)},$$

for all $u, v \in \mathbb{R}$. This definition was introduced by B. Nagy (see [4]).

A stochastic process $X : I \times \Omega \to \mathbb{R}$ is called *continuous* in the interval I, if for all $t_0 \in I$ we have

$$P - \lim_{t \to t_0} X(t,\cdot) = X(t_0,\cdot),$$

where $P - \lim$ denotes the limit in probability. The notion was introduced by B. Nagy in [4].

2. Strongly Wright-convex stochastic processes

In [10] A. Skowroński proved that a stochastic process $X : I \times \Omega \to \mathbb{R}$ defined on an open interval I is Wright-convex if and only if it can be presented in the form $X = X_1 + A$, where X_1 is a convex stochastic process and A is an additive process. It is a stochastic version of the celebrated theorem of C. T. Ng [5] characterizing Wright-convex functions. In this section we will give a counterpart of those results for strongly Wright-convex stochastic processes. We start with the following lemma.

Lemma 1. *If a stochastic process $X\colon I\times\Omega\to\mathbb{R}$ defined on an open interval I is convex and strongly midconvex with modulus $C(\cdot)$, then it is strongly convex with the same modulus.*

Proof. Since X is convex, then by K. Nikodem's result [7, Theorem 5] we arrive that X is continuous. Using of [2, Theorem 7], we infer that X is strongly convex with modulus $C(\cdot)$. □

Theorem 2. *Let I be an open interval. A process $X : I\times\Omega\to\mathbb{R}$ is strongly Wright-convex with modulus $C(\cdot)$ if and only if there exist a stochastic process $X_1 : I\times\Omega\to\mathbb{R}$ strongly convex with modulus $C(\cdot)$ and an additive stochastic process $A\colon\mathbb{R}\times\Omega\to\mathbb{R}$ such that*

$$X(u,\cdot)=X_1(u,\cdot)+A(u,\cdot)\quad (a.e.),\quad u\in I. \tag{1}$$

Proof. For the proof of the "only if" part, we assume that X is strongly Wright-convex with modulus $C(\cdot)$. Then X is also Wright-convex and by A. Skowroński theorem (see [10]) it can be represented in the following form

$$X(u,\cdot)=X_1(u,\cdot)+A(u,\cdot)\quad \text{(a.e.)},\quad u\in I,$$

where $X_1\colon I\times\Omega\to\mathbb{R}$ is a convex stochastic process and $A\colon\mathbb{R}\times\Omega\to\mathbb{R}$ is an additive stochastic process. Since X is strongly Wright-convex with modulus $C(\cdot)$, simple computation shows that the process $X-A$ is also strongly Wright-convex with modulus $C(\cdot)$, and consequently it is strongly midconvex with modulus $C(\cdot)$. By Lemma 1 the process $X_1(u,\cdot)=X(u,\cdot)-A(u,\cdot)$ (a.e.) is strongly convex with modulus $C(\cdot)$, which proves that X has the representation (1). The "if" part is obvious. □

By Lemma 2 proved in [2, Theorem 2] can be written in the following way.

Corollary 3. *By Theorem 2 a strongly Wright-convex stochastic process $X\colon I\times\Omega\to\mathbb{R}$ with modulus $C(\cdot)$ has representation* (1) *with a strongly convex with modulus $C(\cdot)$ process $X_1\colon I\times\Omega\to\mathbb{R}$ and an additive process $A\colon\mathbb{R}\times\Omega\to\mathbb{R}$. By mentioned above Lemma 2 from [2] the process X_1 is of the form*

$$X(u,\cdot)=X_2(u,\cdot)+C(\cdot)u^2+A(u,\cdot)\quad (a.e.),\quad u\in I. \tag{2}$$

Proof. By Theorem 2 the strongly Wright-convex stochastic process with modulus $C(\cdot)$ has the representation (1). By the mentioned above Lemma 2 (see [2]) every strongly convex stochastic process X_1 with modulus $C(\cdot)$ is of the form

$$X_1(u,\cdot)=X_2(u,\cdot)+C(\cdot)u^2\quad \text{(a.e.)}\quad u\in I, \tag{3}$$

where $X_2 : I \times \Omega \to \mathbb{R}$ is a convex stochastic process. Using (1) and (3) we get (2). The converse part is trivial. $\square$

Remark 4. An analogous characterization of strongly Wright-convex function with modulus c was obtained by N. Merentes, K. Nikodem and S. Rivas in [3].

3. Strongly midconvex stochastic processes with strongly midconcave bounds

It is known that if a midconvex function f is bounded from above by a midconcave function g then f is Wright-convex and g is Wright-concave. Moreover, there can be found a convex function f_1, a concave function g_1 and an additive function a such that $f = f_1 + a$ and $g = g_1 + a$ (see [1], [6] and [8]). In 1995 A. Skowroński proved analogous theorem for J-convex and J-concave stochastic processes (see [9]). In this section we will present a counterpart of these results for strongly midconvex and strongly midconcave stochastic processes. We say that a stochastic process $X : I \times \Omega \to \mathbb{R}$ is strongly concave (strongly midconcave) with modulus $C(\cdot)$ if $-X$ is strongly convex (strongly midconvex) with modulus $C(\cdot)$.

Theorem 5. *Let I be an open interval. Assume that $X : I \times \Omega \to \mathbb{R}$ is strongly midconvex with modulus $C(\cdot)$, $Y : I \times \Omega \to \mathbb{R}$ is strongly midconcave with modulus $C(\cdot)$ and $X(u, \cdot) \leqslant Y(u, \cdot)$ (a.e.) for every $u \in I$. Then there exist an additive stochastic process $A : \mathbb{R} \times \Omega \to \mathbb{R}$, a continuous stochastic process $X_1 : I \times \Omega \to \mathbb{R}$ strongly convex with modulus $C(\cdot)$ and a continuous stochastic process $Y_1 : I \times \Omega \to \mathbb{R}$ strongly concave with modulus $C(\cdot)$ such that*

$$X(u, \cdot) = X_1(u, \cdot) + A(u, \cdot) \quad \text{(a.e.)}$$

and

$$Y(u, \cdot) = Y_1(u, \cdot) + A(u, \cdot) \quad \text{(a.e.)}, \tag{4}$$

for all $u \in I$.

Proof. Since X and Y are strongly midconvex and strongly midconcave with modulus $C(\cdot)$, respectively, then X and Y are midconvex and midconcave, respectively. By A. Skowroński's theorem (see [9]) there exist the following representations for X and Y

$$X(u, \cdot) = X_1(u, \cdot) + A(u, \cdot) \quad \text{(a.e.)}$$

and

$$Y(u, \cdot) = Y_1(u, \cdot) + A(u, \cdot) \quad \text{(a.e.)}$$

for $u \in I$, where $X_1 : I \times \Omega \to \mathbb{R}$ is a convex stochastic process, $Y_1 : I \times \Omega \to \mathbb{R}$ is a concave stochastic process and $A : \mathbb{R} \times \Omega \to \mathbb{R}$ is an additive stochastic process. By the assumptions the additivity of the process A we get that the process $X_1(u,\cdot) = X(u,\cdot) - A(u,\cdot)$ (a.e.) is strongly midconvex with modulus $C(\cdot)$ and the process $Y_1(u,\cdot) = Y(u,\cdot) - A(u,\cdot)$ (a.e.) is strongly midconcave with modulus $C(\cdot)$. Finally, by Lemma 1 X_1 is strongly convex with modulus $C(\cdot)$ and Y_1 is strongly concave with modulus $C(\cdot)$. It proves that X and Y have the representation (4) with X_1, Y_1 and A satisfies the desired properties. The continuity of X_1 and Y_1 follows from K. Nikodem's theorem (see [7]). □

Remark 6. In the deterministic case the above theorem reduces to the result obtained in [3] for strongly midconvex functions with strongly midconcave bounds.

As before, by [2, Theorem 5 and Lemma 2], we get the following corollary.

Corollary 7. *Let I be an open interval. Assume that $X : I \times \Omega \to \mathbb{R}$ is strongly midconvex with modulus $C(\cdot)$, $Y : I \times \Omega \to \mathbb{R}$ is strongly midconcave with modulus $C(\cdot)$ and $X(u,\cdot) \leqslant Y(u,\cdot)$ (a.e.) for every $u \in I$. Then there exist an additive stochastic process $A : \mathbb{R} \times \Omega \to \mathbb{R}$, a continuous, convex stochastic process $X_2 : I \times \Omega \to \mathbb{R}$ and a continuous, concave stochastic process $Y_2 : I \times \Omega \to \mathbb{R}$ such that*

$$X(u,\cdot) = X_2(u,\cdot) + C(\cdot)u^2 + A(u,\cdot) \quad \text{(a.e.)},$$

$$Y(u,\cdot) = Y_2(u,\cdot) + C(\cdot)u^2 + A(u,\cdot) \quad \text{(a.e.)},$$

for all $u \in I$.

The proof is similar to the proof of Corollary 3, so we omit it.

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