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TATRA MOUNTAINS
Mathematical Publications
DOI: 10.1515/tmmp-2016-0026
Tatra Mt. Math. Publ. **66** (2016), 137–149

ON ALGEBRAIC PROPERTIES AND LINEARITY OF OWA OPERATORS FOR FUZZY SETS

Zdenko Takáč

ABSTRACT. We deal with an ordered weighted averaging operator (OWA operator) on the set of all fuzzy sets. Our starting point is OWA operator on any lattice introduced in Lizasoain, I.—Moreno, C.: *OWA operators defined on complete lattices,* Fuzzy Sets and Systems **224** (2013), 36–52; Ochoa, G.—Lizasoain, I.–Paternain, D.—Bustince, H.—Pal, N. R.: *Some properties of lattice OWA operators and their importance in image processing,* in: Proc. of the 16th World Congress of the Internat. Systems Assoc.—IFSA '15 and the 9th Conf. of the European Soc. for Fuzzy Logic and Technology—EUSFLAT '15 (J. M. Alonso et al., eds.), Atlantis Press, Gijón, Spain, 2015, pp. 1261–1265. We focus on a particular case of lattice, namely that of all normal convex fuzzy sets in [0,1], and study algebraic properties and linearity of the proposed OWA operator. It is shown that the operator is an extension of standard OWA operator for real numbers and it possesses similar algebraic properties as standard one, however, it is neither homogeneous nor shift-invariant, i.e., it is not linear in contrast to the standard OWA operator.

1. Introduction

The aggregation of fuzzy sets, that is fuzzy truth values, is essential in the type-2 fuzzy sets settings [10], [11]. Also, the need of aggregation of fuzzy sets arises in decision making problems when the alternatives are assessed by fuzzy sets. Recall that Yager's OWA operators are of special significance in solving decision making problems. This leads to growing interest of scholars to investigate OWA operators for various kinds of elements [4], e.g., for intervals [2], [15], fuzzy intervals [17], [18], gradual intervals [12], i.e., also for fuzzy sets.

In [6], [8] the concept of an ordered weighted averaging (OWA for short) operator is extended to any complete lattice endowed with a t-norm and a t-conorm. The intention of authors was to avoid the need of a linear order in environments

2010 Mathematics Subject Classification: 03E72.
Keywords: OWA operator, type-2 fuzzy set, fuzzy set, fuzzy truth value, aggregation, distributive weighting vector.
This work was supported by the Grant VEGA 1/0420/15.

in which only partial, non-linear, order is available. Our aim is to study a specific nature of ideas from [6], [8] in one particular case of complete lattice, namely that of all normal convex fuzzy truth values (fuzzy sets in $[0,1]$). It is well-known that this set is not linearly ordered. We discuss the notion of (distributive) weighting vector, formulate a sufficient and necessary condition under which given elements constitute a distributive weighting vector and study algebraic properties and linearity of the proposed OWA operator for fuzzy sets.

The paper is organized as follows. Section 2 contains basic definitions and notations that are used in the remaining parts of the paper. In Section 3, we propose an OWA operator on the set of normal convex fuzzy sets in [0,1] and study its properties. The conclusions are discussed in Section 4.

2. Preliminaries

In this section, we present some basic concepts and terminology that will be used throughout the paper.

Let X be a set. A *fuzzy set* in X is a mapping from X to $[0,1]$. Let $\mathcal{F}(X)$ denote the class of all fuzzy sets in X, and let $\mathcal{F}$ denote the class of all fuzzy sets in $[0,1]$. A *type-2 fuzzy set* in X is a fuzzy set whose membership grades are fuzzy sets in $[0,1]$. Hence, type-2 fuzzy set in X is a mapping

$$\widetilde{f}\colon X \to \mathcal{F}$$

and the elements of $\mathcal{F}$ are called *fuzzy truth values*.

A fuzzy set f in X is *normal* if there exists $x \in X$ such that $f(x) = 1$. Let X be a linear space, a fuzzy set f in X is *convex* if it satisfies $f\big(\lambda x_1 + (1-\lambda)x_2\big) \geq \min\big(f(x_1), f(x_2)\big)$ for all $\lambda \in [0,1]$, for each $x_1, x_2 \in X$. We denote by $\mathcal{F}_{NC}$ the class of all normal convex fuzzy truth values. We will use operations $\sqcup$, $\sqcap$, relations $\sqsubseteq$, $\preceq$ and special elements $\widetilde{0}$, $\widetilde{1}$ on $\mathcal{F}$ given by:

$$\begin{aligned}
(f \sqcup g)(z) &= \sup_{x\vee y=z} \big(f(x) \wedge g(y)\big), \qquad f \sqsubseteq g \quad \text{if and only if} \quad f \sqcap g = f,\\
(f \sqcap g)(z) &= \sup_{x\wedge y=z} \big(f(x) \wedge g(y)\big), \qquad f \preceq g \quad \text{if and only if} \quad f \sqcup g = g,\\
\widetilde{0}(x) &= \begin{cases} 1 & \text{if } x = 0,\\ 0, & \text{otherwise,}\end{cases} \qquad \widetilde{1}(x) = \begin{cases} 1 & \text{if } x = 1,\\ 0, & \text{otherwise.}\end{cases}
\end{aligned} \tag{1}$$

The algebra of fuzzy truth values $(\mathcal{F}, \sqcup, \sqcap, \widetilde{0}, \widetilde{1}, \sqsubseteq, \preceq)$ is closely described in [7] and [13]. In [13] it is showed that $(\mathcal{F}_{NC}, \sqcup, \sqcap, \widetilde{0}, \widetilde{1}, \sqsubseteq)$ is a bounded, distributive lattice, and in [3] the authors showed that the lattice is complete. Recall that the two orders $\sqsubseteq$ and $\preceq$ coincide on the set of normal convex fuzzy sets.

In 1988 Yager [16] introduced *OWA operator* which is one of the most widely used aggregation methods for real numbers.

Definition 2.1. Let $\mathbf{w} = (w_1, \dots, w_n) \in [0,1]^n$ be a weighting vector with $w_1 + \dots + w_n = 1$. An OWA operator associated with $\mathbf{w}$ is a mapping $OWA_{\mathbf{w}} : [0,1]^n \to [0,1]$ defined by

$$OWA_{\mathbf{w}}(x_1, \dots, x_n) = \sum_{i=1}^{n} w_i x_{(i)} \,,$$

where $x_{(i)}$ denotes the ith largest number among $x_1, \dots, x_n$.

3. OWA operators defined on the set of all normal convex fuzzy sets in $[0,1]$

In this section we apply the ideas of [6], [8] to the settings of type-2 fuzzy sets. In other words, we will study distributive weighting vectors and consequently OWA operators on the set of fuzzy truth values $\mathcal{F}$. Let us start with the notion of a t-norm and a t-conorm on $\mathcal{F}$.

Definition 3.1. A mapping $T : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ is said to be a t-norm on $(\mathcal{F}, \sqsubseteq)$ if it is commutative, associative, increasing in each component and has a neutral element $\widetilde{1}$.

A mapping $S : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ is said to be a t-conorm on $(\mathcal{F}, \sqsubseteq)$ if it is commutative, associative, increasing in each component and has a neutral element $\widetilde{0}$.

The operations $\sqcap$ and $\sqcup$ given by (1) are t-norm and t-conorm on $\mathcal{F}$, respectively. The following propositions are easy to check, see [5], [9] and [14].

Proposition 3.2. *The operation $\sqcap$ given by* (1) *is a t-norm on* $(\mathcal{F}, \sqsubseteq)$.

Proposition 3.3. *The operation $\sqcup$ given by* (1) *is a t-conorm on* $(\mathcal{F}, \sqsubseteq)$.

According to the following lemma, it is possible to construct linearly ordered vector from any given vector in $\mathcal{F}^n_{NC}$.

Lemma 3.4 ([6]). *Let $(f_1, \dots, f_n) \in \mathcal{F}^n_{NC}$, and let*

$$g_1 = f_1 \sqcup \dots \sqcup f_n,$$

$$g_2 = \big((f_1 \sqcap f_2) \sqcup \dots \sqcup (f_1 \sqcap f_n)\big) \sqcup \big((f_2 \sqcap f_3) \sqcup \dots \sqcup (f_2 \sqcap f_n)\big) \sqcup \dots \sqcup \big((f_{n-1} \sqcap f_n)\big),$$

$$\vdots$$

$$g_k = \sqcup \big\{ f_{j_1} \sqcap \dots \sqcap f_{j_k} \,|\, \{j_1, \dots, j_k\} \subseteq \{1, \dots, n\} \big\},$$

$$\vdots$$

$$g_n = f_1 \sqcap \dots \sqcap f_n.$$

Then

$$g_n \sqsubseteq g_{n-1} \sqsubseteq \dots \sqsubseteq g_1.$$

Moreover, if the set $\{f_1, \dots, f_n\}$ is linearly ordered, then the vector $(g_1, \dots, g_n)$ coincides with $(f_{\sigma(1)}, \dots, f_{\sigma(n)})$ for some permutation σ of $\{1, \dots, n\}$.

We proceed with the study of distributive weighting vector in $\mathcal{F}_{NC}^n$ and some of its properties.

Definition 3.5. Let $w_1, \dots, w_n \in \mathcal{F}$. A vector $(w_1, \dots, w_n)$ is said to be a weighting vector in $(\mathcal{F}, \sqsubseteq)$ if $w_1 \sqcup \dots \sqcup w_n = \widetilde{1}$, and it is said to be a distributive weighting vector if it also satisfies

$$f \sqcap (w_1 \sqcup \dots \sqcup w_n) = (f \sqcap w_1) \sqcup \dots \sqcup (f \sqcap w_n)$$

for all $f \in \mathcal{F}$.

The following theorem gives a necessary condition under which elements $w_1, \dots, w_n$ constitute a weighting vector in $(\mathcal{F}_{NC}, \sqsubseteq)$.

Theorem 3.6. *Let $w_1, \dots, w_n \in \mathcal{F}_{NC}$. If $w_1 \sqcup \dots \sqcup w_n = \widetilde{1}$, then $w_i = \widetilde{1}$ for some $i \in \{1, \dots, n\}$.*

Proof.

1. We show that $w_i(1) = 1$ for some $i \in \{1, \dots, n\}$. From $w_1 \sqcup \dots \sqcup w_n = \widetilde{1}$ it follows $(w_1 \sqcup \dots \sqcup w_n)(1) = 1$, hence there exist $a_1, \dots a_n \in [0,1]$ such that $\max(a_1, \dots, a_n) = 1$ and $\min\big(w_1(a_1), \dots, w_n(a_n)\big) = 1$, and consequently there exist $a_1, \dots a_n \in [0,1]$ such that $\max(a_1, \dots, a_n) = 1$ and $w_1(a_1) = \dots = w_n(a_n) = 1$. It means that for some $i \in \{1, \dots, n\}$ it holds $a_i = 1$ and $w_i(a_1) = 1$, thus $w_i(1) = 1$ for some $i \in \{1, \dots, n\}$. Let us write $w_{k_0}(1) = 1$.

2. Now we are going to show that $w_{k_0}(x) = 0$ for all $x \in [0,1[$ if $w_i \neq \widetilde{1}$ for all $i \in \{1, \dots, n\} - \{k_0\}$. Let there exist $x_0 \in [0,1[$ such that $w_{k_0}(x_0) > 0$. Then there exist

$$b_1, \dots, b_{k_0-1},\, b_{k_0+1}, \dots, b_n \in [0,1[\quad \text{such that}$$
$$w_1(b_1), \dots, w_{k_0-1}(b_{k_0-1}), w_{k_0+1}(b_{k_0+1}), \dots, w_n(b_n) > 0,$$

hence

$$\big(w_1 \sqcup \dots \sqcup w_n\big)\big(\max(b_1, \dots, b_{k_0-1}, x_0, b_{k_0+1}, \dots, b_n)\big)$$
$$= \min\big(w_1(b_1), \dots, w_{k_0-1}(b_{k_0-1}), w_{k_0}(x_0), w_{k_0+1}(b_{k_0+1}), \dots, w_n(b_n)\big) > 0,$$

which contradicts our assumption $w_1 \sqcup \dots \sqcup w_n = \widetilde{1}$. □

The following corollary states a simple necessary and sufficient condition under which $(w_1, \dots, w_n) \in \mathcal{F}_{NC}^n$ is a distributive weighting vector.

Corollary 3.7. *A vector $(w_1, \dots, w_n) \in \mathcal{F}_{NC}^n$ is a distributive weighting vector in $(\mathcal{F}_{NC}, \sqsubseteq)$ if and only if there exists $i \in \{1, \dots, n\}$ such that $w_i = \widetilde{1}$.*

Proof.

1. Necessity: Let $(w_1, \dots, w_n) \in \mathcal{F}_{NC}^n$ be a distributive weighting vector. Then, according to Definition 3.5, $w_1 \sqcup \dots \sqcup w_n = \widetilde{1}$; and from Theorem 3.6 it follows $w_i = \widetilde{1}$ for some $i \in \{1, \dots, n\}$.

2. Sufficiency: Let us first observe that $(\mathcal{F}_{NC}, \sqsubseteq)$ is a distributive lattice, thus it is sufficient to show that $(w_1, \ldots, w_n)$ is a weighting vector in $(\mathcal{F}_{NC}, \sqsubseteq)$. Let $w_i = \widetilde{1}$ for some $i \in \{1, \ldots, n\}$. The proof follows from the observation that $\widetilde{1} \sqcup f = \widetilde{1}$ for all $f \in \mathcal{F}_{NC}$. □

Now we can use the notion of distributive weighting vector and define an OWA operator on the set of normal convex fuzzy truth values $\mathcal{F}_{NC}$.

Definition 3.8. Let $\mathbf{w} = (w_1, \ldots, w_n) \in \mathcal{F}_{NC}^n$ be a distributive weighting vector in $(\mathcal{F}_{NC}, \sqsubseteq)$. The mapping $F_{\mathbf{w}} : \mathcal{F}_{NC}^n \to \mathcal{F}_{NC}$ given, for all $(f_1, \ldots, f_n) \in \mathcal{F}_{NC}^n$, by

$$F_{\mathbf{w}}(f_1, \ldots, f_n) = (w_1 \sqcap g_1) \sqcup \ldots \sqcup (w_n \sqcap g_n),$$

where $(g_1, \ldots, g_n)$ is a linearly ordered vector constructed from $(f_1, \ldots, f_n)$ according to Lemma 3.4, is called an n-ary OWA operator on $\mathcal{F}_{NC}$.

Example 3.9. Let weighting vector be $\mathbf{w} = (w_1, \widetilde{1})$ and w_1, f_1, f_2 be fuzzy truth values given by Fig. 1. Then $g_1 = f_1 \sqcup f_2$, $g_2 = f_1 \sqcap f_2$, and

$$F_{\mathbf{w}}(f_1, f_2) = (w_1 \sqcap g_1) \sqcup (\widetilde{1} \sqcap g_2) = (w_1 \sqcap g_1) \sqcup g_2.$$

The results are depicted in Fig. 1 (for simplicity, fuzzy truth values g_1 and g_2 are not depicted—they can be found in Fig. 2).

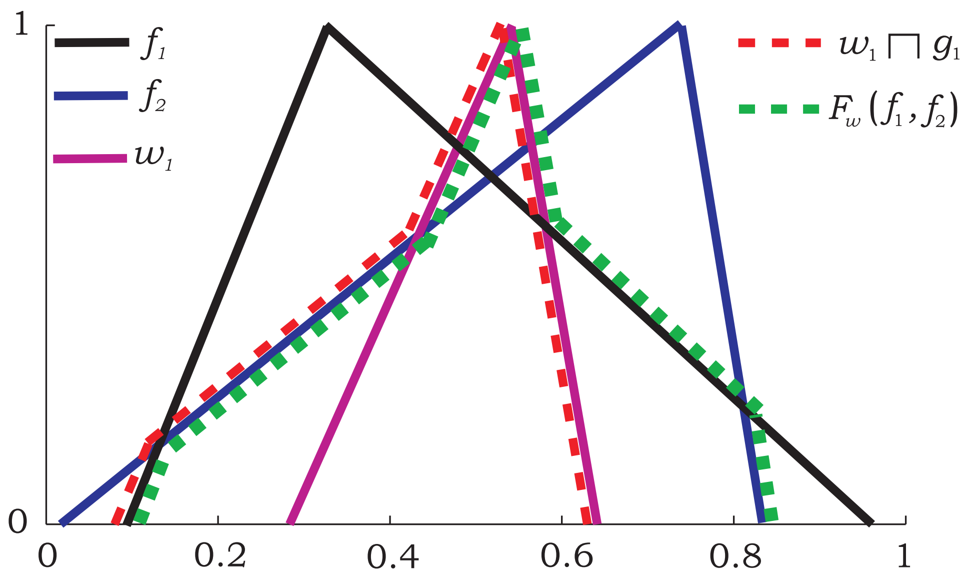

Figure 1. See Example 3.9.

The remainder of this paper will be devoted to the study of properties of the n-ary OWA operator on $\mathcal{F}_{NC}$. In the following theorem we show that the operator satisfies three basic properties of each aggregation operator, i.e., it is aggregation operator on $\mathcal{F}_{NC}$.

Theorem 3.10. *Let* $\mathbf{w} = (w_1, \dots, w_n)$ *be a distributive weighting vector in* $(\mathcal{F}_{NC}, \sqcap, \sqcup, \sqsubseteq, \widetilde{0}, \widetilde{1})$ *and* $F_{\mathbf{w}}$ *the corresponding OWA operator given by Definition 3.8. Then:*

(i) $F_{\mathbf{w}}(\widetilde{0}, \dots, \widetilde{0}) = \widetilde{0}$.

(ii) $F_{\mathbf{w}}(\widetilde{1}, \dots, \widetilde{1}) = \widetilde{1}$.

(iii) $f_1 \sqsubseteq f_1^*, \dots, f_n \sqsubseteq f_n^*$ *imply*

$$F_{\mathbf{w}}(f_1, \dots, f_n) \sqsubseteq F_{\mathbf{w}}(f_1^*, \dots, f_n^*) \quad \textit{for all} \quad f_1, f_1^*, \dots, f_n, f_n^* \in \mathcal{F}_{NC}.$$

Proof.

(i) $F_{\mathbf{w}}(\widetilde{0}, \dots, \widetilde{0}) = (w_1 \sqcap \widetilde{0}) \sqcup \dots \sqcup (w_n \sqcap \widetilde{0}) = \widetilde{0} \sqcup \dots \sqcup \widetilde{0} = \widetilde{0}$.

(ii) $F_{\mathbf{w}}(\widetilde{1}, \dots, \widetilde{1}) = (w_1 \sqcap \widetilde{1}) \sqcup \dots \sqcup (w_n \sqcap \widetilde{1}) = w_1 \sqcup \dots \sqcup w_n = \widetilde{1}$.

(iii) Let $f_1 \sqsubseteq f_1^*, \dots, f_n \sqsubseteq f_n^*$. Then we have

$$g_1 \sqsubseteq g_1^*, \dots, g_n \sqsubseteq g_n^*$$

and consequently

$$\begin{aligned} &F_{\mathbf{w}}(f_1, \dots, f_n) \\ &= (w_1 \sqcap g_1) \sqcup \dots \sqcup (w_n \sqcap g_n) \sqsubseteq (w_1 \sqcap g_1^*) \sqcup \dots \sqcup (w_n \sqcap g_n^*) \\ &= F_{\mathbf{w}}(f_1^*, \dots, f_n^*). \end{aligned}$$

□

Theorem 3.11. *Let* $F_{\mathbf{w}}$ *be an* n*-ary OWA operator on* $\mathcal{F}_{NC}$*. Then*

$$f_1 \sqcap \dots \sqcap f_n \sqsubseteq F_{\mathbf{w}}(f_1, \dots, f_n) \sqsubseteq f_1 \sqcup \dots \sqcup f_n \quad \textit{for all} \quad f_1, \dots, f_n \in \mathcal{F}_{NC}.$$

Proof. We prove the right inequality, the left one can be checked in a similar way.

$$\begin{aligned} F_{\mathbf{w}}(f_1, \dots, f_n) &= (w_1 \sqcap g_1) \sqcup \dots \sqcup (w_n \sqcap g_n) \sqsubseteq (w_1 \sqcap g_1) \sqcup \dots \sqcup (w_n \sqcap g_1) \\ &= (w_1 \sqcup \dots \sqcup w_n) \sqcap g_1 = \widetilde{1} \sqcap g_1 = g_1 = f_1 \sqcup \dots \sqcup f_n. \end{aligned}$$

□

The theorem says that the results of $F_{\mathbf{w}}(f_1, \dots, f_n)$ are bounded by $f_1 \sqcap \dots \sqcap f_n$ and $f_1 \sqcup \dots \sqcup f_n$. It is worth pointing out that for standard OWA operators for real numbers from $\min(x_1, \dots, x_n) \leq OWA_w(x_1, \dots, x_n) \leq \max(x_1, \dots, x_n)$ it follows that $OWA_w(x_1, \dots, x_n) \geq x_i$ for some $i \in \{1, \dots, n\}$, and $OWA_w(x_1, \dots, x_n) \leq x_j$

for some $j \in \{1, \dots, n\}$. However, the similar property does not hold for $F_{\mathbf{w}}$, i.e., it is possible that

$$F_{\mathbf{w}}(f_1, \dots, f_n) \sqsubset f_i \qquad \text{for all} \quad i \in \{1, \dots, n\}$$

or

$$F_{\mathbf{w}}(f_1, \dots, f_n) \sqsupset f_j \qquad \text{for all} \quad j \in \{1, \dots, n\}.$$

See Example 3.12 where $F_{\mathbf{w}}(f_1, f_2) \sqsupset f_1$ and $F_{\mathbf{w}}(f_1, f_2) \sqsupset f_2$.

Example 3.12. Let weighting vector be $\mathbf{w} = (\widetilde{1}, w_2)$ and w_2, f_1, f_2 be fuzzy truth values given by Fig. 2. Then $g_1 = f_1 \sqcup f_2$, $g_2 = f_1 \sqcap f_2$, and (see Lemma 3.13 for the last equality)

$$F_{\mathbf{w}}(f_1, f_2) = (\widetilde{1} \sqcap g_1) \sqcup (w_2 \sqcap g_2) = g_1 \sqcup (w_2 \sqcap g_2) = g_1.$$

It is easy to check that $F_{\mathbf{w}}(f_1, f_2) = g_1 \sqsupset f_i$, for $i = 1, 2$.

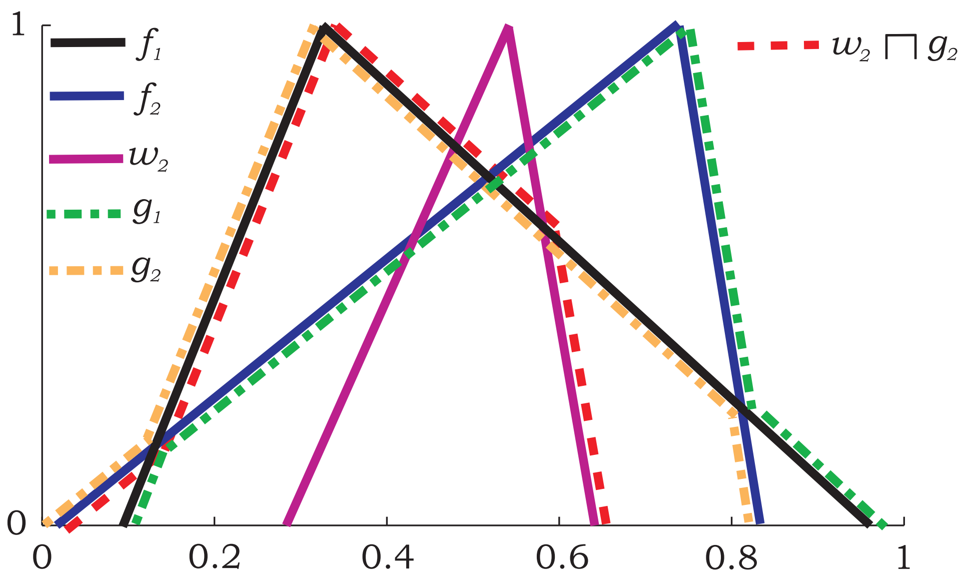

Figure 2. See Example 3.12.

The following lemma was used in previous example and will also be needed in proof of Theorem 3.14.

Lemma 3.13. *Let $f, g, h \in \mathcal{F}_{NC}$ with $f \sqsubseteq g$. Then $g \sqcup (f \sqcap h) = g$.*

Proof. Applying the distributive laws ([13, Proposition 36]) and absorption laws ([13, Proposition 37]) we have:

$$g \sqcup (f \sqcap h) = (g \sqcup f) \sqcap (g \sqcup h) = g \sqcap (g \sqcup h) = g.$$

□

Note that the property of Lemma 3.13 does not hold in $\mathcal{F}$. The reason is that the absorption laws fail if g is not convex or h is not normal.

We can now strengthen [6, Proposition 3.8] in the settings of fuzzy truth values. Our result is that if $\widetilde{1}$ is on the first position of a weighting vector $\mathbf{w}$, then our OWA operator is simply maximum, no matter what are the other weights—see item 1 of the following theorem. Note that there are much stronger assumptions for a similar assertion on minimum—see item 2 of the theorem.

Theorem 3.14. *Let $\mathbf{w}=(w_1,\dots,w_n)$ be a distributive weighting vector in $\mathcal{F}_{NC}$, and let $F_{\mathbf{w}}$ be an n-ary OWA operator on $\mathcal{F}_{NC}$.*

(i) *If $w_1=\widetilde{1}$, then $F_{\mathbf{w}}(f_1,\dots,f_n)=f_1\sqcup\dots\sqcup f_n$.*

(ii) *If $w_n=\widetilde{1}$ and $w_i\sqsubseteq f_1\sqcap\dots\sqcap f_n$ for all $i\in\{1,\dots,n-1\}$, then $F_{\mathbf{w}}(f_1,\dots,f_n)=f_1\sqcap\dots\sqcap f_n$.*

(iii) *If $w_k=\widetilde{1}$ for some $k\in\{1,\dots,n\}$ and $w_i\sqsubseteq g_k$ for all $i\in\{1,\dots,k-1\}$, then $F_{\mathbf{w}}(f_1,\dots,f_n)=g_k$.*

Proof.

(i) Let $\mathbf{w}=(\widetilde{1},w_2,\dots,w_n)$. Then

$$\begin{aligned}F_{\mathbf{w}}(f_1,\dots,f_n)&=(\widetilde{1}\sqcap g_1)\sqcup(w_2\sqcap g_2)\sqcup\dots\sqcup(w_n\sqcap g_n)\\&=g_1\sqcup(w_2\sqcap g_2)\sqcup\dots\sqcup(w_n\sqcap g_n)\end{aligned}$$

and applying Lemma 3.13 $(n-1)$ times we conclude

$$F_{\mathbf{w}}(f_1,\dots,f_n)=g_1=f_1\sqcup\dots\sqcup f_n.$$

(ii) Let $\mathbf{w}=(w_1,\dots,w_{n-1},\widetilde{1})$ with $w_i\sqsubseteq g_n$ for all $i\in\{1,\dots,n-1\}$. Then $w_i\sqsubseteq g_i$ for all $i\in\{1,\dots,n-1\}$, and we have

$$\begin{aligned}F_{\mathbf{w}}(f_1,\dots,f_n)&=(w_1\sqcap g_1)\sqcup\dots\sqcup(w_{n-1}\sqcap g_{n-1})\sqcup(\widetilde{1}\sqcap g_n)\\&=w_1\sqcup\dots\sqcup w_{n-1}\sqcup g_n=g_n=f_1\sqcap\dots\sqcap f_n.\end{aligned}$$

(iii) Let $\mathbf{w}=(\widetilde{0},\dots,\widetilde{0},w_k=\widetilde{1},\widetilde{0}\dots,\widetilde{0})$. Then

$$\begin{aligned}&F_{\mathbf{w}}(f_1,\dots,f_n)\\&=(w_1\sqcap g_1)\sqcup\dots\sqcup(w_{k-1}\sqcap g_{k-1})\sqcup(\widetilde{1}\sqcap g_k)\sqcup(w_{k+1}\sqcap g_{k+1})\dots\sqcup(w_n\sqcap g_n)\\&=w_1\sqcup\dots\sqcup w_{k-1}\sqcup g_k\sqcup(w_{k+1}\sqcap g_{k+1})\dots\sqcup(w_n\sqcap g_n)=g_k.\end{aligned}$$

□

Theorem 3.15. *Let $\mathbf{w}=(w_1,\dots,w_n)$ be a distributive weighting vector in $\mathcal{F}_{NC}$, and let $F_{\mathbf{w}}$ be an n-ary OWA operator on $\mathcal{F}_{NC}$. Then:*

(i) *$F_{\mathbf{w}}$ is a symmetric operator, i.e.,*

$$F_{\mathbf{w}}(f_1,\dots,f_n)=F_{\mathbf{w}}\big(f_{\sigma(1)},\dots,f_{\sigma(n)}\big)$$

for all $f_1,\dots,f_n\in\mathcal{F}_{NC}$, for each permutation σ of $\{1,\dots,n\}$.

(ii) $F_{\mathbf{w}}$ *is an idempotent operator, i.e.,*

$$F_{\mathbf{w}}(f, \dots, f) = f \qquad \text{for all} \quad f \in \mathcal{F}_{NC}.$$

Proof.

(i) Immediately follows from Definition 3.8.

(ii) According to Definition 3.5 we have

$$\begin{aligned} F_{\mathbf{w}}(f, \dots, f) &= (w_1 \sqcap f) \sqcup \dots \sqcup (w_n \sqcap f) \\ &= (w_1 \sqcup \dots \sqcup w_n) \sqcap f = \widetilde{1} \sqcap f = f. \end{aligned}$$

□

We extend definitions of homogeneity and shift-invariance of aggregation functions from $[0, 1]$, see, e.g., [1], to fuzzy sets in $[0, 1]$.

Definition 3.16. An aggregation function $M : \mathcal{F}_{NC}^n \to \mathcal{F}_{NC}$ is *homogeneous* if for all $\lambda \in]0, \infty[$ and for all $(f_1, \dots, f_n) \in \mathcal{F}_{NC}^n$ the following holds:

$$M(\lambda f_1, \dots, \lambda f_n) = \lambda M(f_1, \dots, f_n)$$

whenever $(\lambda f_1, \dots, \lambda f_n) \in \mathcal{F}_{NC}^n$.

Definition 3.17. An aggregation function $M : \mathcal{F}_{NC}^n \to \mathcal{F}_{NC}$ is *shift-invariant* (or *stable for translations*) if for all $\lambda \in [0, 1]$ and for all $(f_1, \dots, f_n) \in \mathcal{F}_{NC}^n$ the following holds:

$$M(f_1 + \lambda, \dots, f_n + \lambda) = M(f_1, \dots, f_n) + \lambda$$

whenever $(f_1 + \lambda, \dots, f_n + \lambda) \in \mathcal{F}_{NC}^n$.

Recall that for $f \in \mathcal{F}$ and appropriate λ we have

$$(\lambda f)(x) = f\left(\frac{x}{\lambda}\right) \quad \text{and} \quad (f + \lambda)(x) = f(x - \lambda),$$

moreover, λf and $f + \lambda$ are normal convex fuzzy sets whenever f is normal convex fuzzy set.

Theorem 3.18. *OWA operator* $F_{\mathbf{w}} : \mathcal{F}_{NC}^n \to \mathcal{F}_{NC}$ *given by Definition 3.8 is not homogeneous.*

Proof.

We give a counterexample. Let $\lambda = 2$, $f_1 = (0.3, 0.4, 0.5)$, $f_2 = (0.1, 0.2, 0.3)$, $w_1 = (0.2, 0.3, 0.4)$ be triangular fuzzy sets in $[0, 1]$, $w_2 = \widetilde{1}$ and $\mathbf{w} = (w_1, w_2)$. Then $\lambda f_1 = (0.6, 0.8, 1)$, $\lambda f_2 = (0.2, 0.4, 0.6)$ and

$$F_{\mathbf{w}}(f_1, f_2) = (w_1 \sqcap f_1) \sqcup (w_2 \sqcap f_2) = w_1 \sqcup f_2 = w_1,$$

$$F_{\mathbf{w}}(\lambda f_1, \lambda f_2) = (w_1 \sqcap \lambda f_1) \sqcup (w_2 \sqcap \lambda f_2) = w_1 \sqcup \lambda f_2 = \lambda f_2,$$

hence $F_{\mathbf{w}}(\lambda f_1, \lambda f_2) \neq \lambda F_{\mathbf{w}}(f_1, f_2)$. □

Although $F_{\mathbf{w}}$ is not homogeneous, the following properties of order relation $\sqsubseteq$ are satisfied.

Lemma 3.19. *Let $f_1, \ldots, f_n \in \mathcal{F}(R)$ and $\lambda \in]0, \infty[$. Then*

(i) $\lambda f_1 \sqcup \ldots \sqcup \lambda f_n = \lambda(f_1 \sqcup \ldots \sqcup f_n)$,

(ii) $\lambda f_1 \sqcap \ldots \sqcap \lambda f_n = \lambda(f_1 \sqcap \ldots \sqcap f_n)$.

Proof.

$$\text{(i)}\ (\lambda \mathrm{f}_1 \sqcup \ldots \sqcup \lambda \mathrm{f}_n)(\mathrm{z}) = \sup_{x_1 \vee \ldots \vee x_n = z} \big((\lambda f_1)(x_1) \wedge \ldots \wedge (\lambda f_n)(x_n)\big)$$
$$= \sup_{\frac{x_1}{\lambda} \vee \ldots \vee \frac{x_n}{\lambda} = \frac{z}{\lambda}} \big((f_1)(x_1/\lambda) \wedge \ldots \wedge (f_n)(x_n/\lambda)\big)$$
$$= (f_1 \sqcup \ldots \sqcup f_n)(z/\lambda) = \big(\lambda(f_1 \sqcup \ldots \sqcup f_n)\big)(z).$$

(ii) The proof is analogous. □

Corollary 3.20. *Let $\lambda \in]0, \infty[$. If we take $\lambda f_1, \ldots, \lambda f_n$ instead of $f_1, \ldots, f_n \in \mathcal{F}(R)$ in Lemma (3.4), then we get chain $\lambda g_n \sqsubseteq \ldots \sqsubseteq \lambda g_1$ instead of $g_n \sqsubseteq \ldots \sqsubseteq g_1$.*

Proof. Immediately follows from Lemma (3.4) and Lemma 3.19. □

The similar results hold for shift-invariance of OWA operator $F_{\mathbf{w}}$.

Theorem 3.21. *OWA operator $F_{\mathbf{w}} : \mathcal{F}_{NC}^n \to \mathcal{F}_{NC}$ given by Definition 3.8 is not shift-invariant.*

Proof.

We give a counterexample. Let $\lambda = 0.4$, $f_1 = (0.3, 0.4, 0.5)$, $f_2 = (0.1, 0.2, 0.3)$, $w_1 = (0.2, 0.3, 0.4)$ be triangular fuzzy sets in $[0, 1]$, $w_2 = \widetilde{1}$ and $\mathbf{w} = (w_1, w_2)$. Then $f_1 + \lambda = (0.7, 0.8, 0.9)$, $f_2 + \lambda = (0.5, 0.6, 0.7)$ and

$$F_{\mathbf{w}}(f_1, f_2) = (w_1 \sqcap f_1) \sqcup (w_2 \sqcap f_2) = w_1 \sqcup f_2 = w_1,$$
$$F_{\mathbf{w}}(f_1 + \lambda, f_2 + \lambda) = \big(w_1 \sqcap (f_1 + \lambda)\big) \sqcup \big(w_2 \sqcap (f_2 + \lambda)\big)$$
$$= w_1 \sqcup (f_2 + \lambda) = f_2 + \lambda,$$

hence $F_{\mathbf{w}}(f_1 + \lambda, f_2 + \lambda) \neq F_{\mathbf{w}}(f_1, f_2) + \lambda$. □

Although $F_{\mathbf{w}}$ is not shift-invariant, the following properties of order relation $\sqsubseteq$ are satisfied.

Lemma 3.22. *Let $f_1, \ldots, f_n \in \mathcal{F}(R)$ and $\lambda \in R$. Then*

(i) $(f_1 + \lambda) \sqcup \ldots \sqcup (f_n + \lambda) = (f_1 \sqcup \ldots \sqcup f_n) + \lambda$,

(ii) $(f_1 + \lambda) \sqcap \ldots \sqcap (f_n + \lambda) = (f_1 \sqcap \ldots \sqcap f_n) + \lambda$.

P r o o f.

(i)
$$\begin{aligned}&\big((f_1+\lambda)\sqcup\ldots\sqcup(f_n+\lambda)\big)(z)\\&=\sup_{x_1\vee\ldots\vee x_n=z}\big((f_1+\lambda)(x_1)\wedge\ldots\wedge(f_n+\lambda)(x_n)\big)\\&=\sup_{(x_1-\lambda)\vee\ldots\vee(x_n-\lambda)=z-\lambda}\big((f_1)(x_1-\lambda)\wedge\ldots\wedge(f_n)(x_n-\lambda)\big)\\&=(f_1\sqcup\ldots\sqcup f_n)(z-\lambda)=\big((f_1\sqcup\ldots\sqcup f_n)+\lambda\big)(z).\end{aligned}$$

(ii) The proof is analogous. □

Corollary 3.23. *Let $\lambda\in R$. If we take $f_1+\lambda,\ldots,f_n+\lambda$ instead of $f_1,\ldots,f_n\in\mathcal{F}(R)$ in Lemma (3.4), then we get chain $g_n+\lambda\sqsubseteq\ldots\sqsubseteq g_1+\lambda$ instead of $g_n\sqsubseteq\ldots\sqsubseteq g_1$.*

P r o o f. Immediately follows from Lemma (3.4) and Lemma 3.22. □

We show that OWA operator $F_{\mathbf{w}}$ given by Definition 3.8 is an extension of Yager's OWA operator on $[0,1]$, moreover, $F_{\mathbf{w}}$ is closed on the set of all closed subintervals of $[0,1]$. From now on we denote a singleton fuzzy set in X given by some $a\in X$ as follows

$$\widetilde{a}(x)=\begin{cases}1 & \text{if } x=a,\\ 0, & \text{otherwise.}\end{cases}$$

Theorem 3.24. *Let $v_1,\ldots,v_n,a_1,\ldots,a_n\in[0,1]$, $\mathbf{v}=(v_1,\ldots,v_n)$ and $\mathbf{w}=(\widetilde{v_1},\ldots,\widetilde{v_n})$. Let $F_{\mathbf{w}}:\mathcal{F}^n_{NC}\to\mathcal{F}_{NC}$ be the corresponding OWA operator given by Definition 3.8 and $OWA_{\mathbf{v}}:[0,1]^n\to[0,1]$ be the standard Yager's OWA operator for real numbers given by Definition 2.1. Then*

$$F_{\mathbf{w}}(\widetilde{a_1},\ldots,\widetilde{a_n})=\widetilde{c},\qquad\textit{where}\qquad c=OWA_{\mathbf{v}}(a_1,\ldots,a_n),$$

for all $v_1,\ldots,v_n$ such that $F_{\mathbf{w}}$ and $OWA_{\mathbf{v}}$ are defined.

P r o o f. Operators $F_{\mathbf{w}}$ and $OWA_{\mathbf{v}}$ are defined simultaneously if and only if the weighting vector $(\widetilde{v_1},\ldots,\widetilde{v_n})$ is distributive and $v_1+\cdots+v_n=1$, i.e., according to Corollary 3.7, $v_k=1$ for some $k\in\{1,\ldots,n\}$ and $v_i=0$ for all $i\in\{1,\ldots,n\}-\{k\}$. Then

$$\begin{aligned}&F_{\mathbf{w}}(\widetilde{a_1},\ldots,\widetilde{a_n})\\&=(\widetilde{0}\sqcap g_1)\sqcup\ldots\sqcup(\widetilde{0}\sqcap g_{k-1})\sqcup(\widetilde{1}\sqcap g_k)\sqcup(\widetilde{0}\sqcap g_{k+1})\sqcup\ldots\sqcup(\widetilde{0}\sqcap g_n)=g_k=\widetilde{a_{\sigma(k)}}\end{aligned}$$

and

$$\begin{aligned}&OWA_{\mathbf{v}}(a_1,\ldots,a_n)\\&=0\cdot a_{\sigma(1)}+\cdots+0\times a_{\sigma(k-1)}+1\times a_{\sigma(k)}+0\times a_{\sigma(k+1)}+\cdots+0\times a_{\sigma(n)}=a_{\sigma(k)}.\end{aligned}$$

Recall that $g_i=\widetilde{a_{\sigma(i)}}$, for all $i\in\{1,\ldots,n\}$. □

From now on, $I([0,1])$ denotes the set of all closed subintervals of $[0,1]$, i.e.,

$$I([0,1]) = \big\{[a,b] \mid 0 \leq a \leq b \leq 1\big\},$$

$\widetilde{[a,b]}$ stands for the characteristic function of $[a,b]$, i.e.,

$$\widetilde{[a,b]}(x) = \begin{cases} 1, & \text{if } x \in [a,b], \\ 0, & \text{otherwise} \end{cases}$$

and K stands for the set of all characteristic functions of the closed subintervals of $[0,1]$. Recall that $(K, \sqcap, \sqcup, \sqsubseteq, \widetilde{0}, \widetilde{1})$ is a subalgebra of $(\mathcal{F}_{NC}, \sqcap, \sqcup, \sqsubseteq, \widetilde{0}, \widetilde{1})$, see [13].

Theorem 3.25. *Let* $\mathbf{w} = (w_1, \dots, w_n)$ *be a distributive weighting vector in* $(K, \sqcap, \sqcup, \sqsubseteq, \widetilde{0}, \widetilde{1})$ *and* $F_{\mathbf{w}}$ *the corresponding OWA operator given by Definition 3.8. If* $f_1, \dots, f_n \in K$, *then* $F_{\mathbf{w}}(f_1, \dots, f_n) \in K$.

Proof. The proof is straightforward. □

4. Conclusion

In [6], [8] an OWA operator on any complete lattice endowed with a t-norm and a t-conorm was introduced. In this paper we focused on OWA operators on one particular case of complete lattice, namely that of all normal convex fuzzy sets in $[0,1]$. We showed that the proposed OWA operator for fuzzy sets is an extension of standard OWA operator for real numbers and it possesses similar algebraic properties as standard one, such as boundary conditions, monotonicity, symmetry, idempotency, boundary from above by maximum and from below by minimum. However, the proposed operator is neither homogeneous nor shift-invariant, i.e., it is not linear in contrast to the standard OWA operator.

We have restricted our attention on operations $\sqcap$ and $\sqcup$. Our next intention is to apply some other t-norms and t-conorms on the set of fuzzy truth values and study properties of corresponding OWA operators. Another line of our investigation is a relationship of the proposed OWA operators to existing operators, namely type-1 OWA operators [17], [18] and OWA operators for gradual intervals [12].

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Received July 15, 2016

Institute of Information Engineering
Automation and Mathematics
Faculty of Chemical an Food Technology
Slovak University of Technology in Bratislava
Radlinského 9
SK–812-37 Bratislava
SLOVAKIA
E-mail: zdenko.takac@stuba.sk