DE GRUYTER
OPEN

TATRA MOUNTAINS
Mathematical Publications
DOI: 10.1515/tmmp-2016-0016
Tatra Mt. Math. Publ. **66** (2016), 13–24

CONJUNCTION AND DISJUNCTION BASED FUZZY INTERVAL ORDERS IN AGGREGATION PROCESS

Urszula Bentkowska — Anna Król — Bernard De Baets

ABSTRACT. This contribution deals with fuzzy interval orders which are defined by the use of generalized properties, the Ferrers property and total connectedness, of fuzzy relations. Definitions of the generalized properties involve binary operations including t-norms and t-conorms and, more generally, fuzzy conjunctions and disjunctions. In particular, conditions for n-argument functions to preserve fuzzy interval orders during aggregation process are presented. Moreover, some existing results on the Ferrers property are generalized and applied for fuzzy interval orders.

1. Introduction

In this contribution, we pay attention to fuzzy interval orders with definitions based on the notions of the Ferrers property and total connectedness. The Ferrers property is less demanding than transitivity property used in the most of orders, so it is worth to examine the Ferrers property and fuzzy interval orders from the application point of view. There are two ways (not equivalent) of fuzzification of the Ferrers property [7]. We will use only one of them.

In 1956 Luce [19] introduced the notion of a semiorder to model a situation of intransitive indifference with a "*threshold of discrimination or perception*". This was done for the preference structure consisting of the strict preference relation, indifference relation and incomparability relation. Later on, Fishburn [11], [12] extended the notion of semiorder to that of an interval order, whose underlying idea is to assign an "interval of evaluations" to each alternative. What makes interval orders and semi-orders so interesting is the

2010 Mathematics Subject Classification: 03E72, 68T37.
Keywords: fuzzy relation, fuzzy connective, fuzzy conjunction, fuzzy disjunction, aggregation function, Ferrers property, fuzzy interval order.
Partially supported by the Centre for Innovation and Transfer of Natural Sciences and Engineering Knowledge in Rzeszów, through Project Number RPPK.01.03.00-18-001/10.

fact that the strict preference relation is transitive, whereas the indifference relation may be intransitive. In some situations in practice, decision makers may not consider some small differences as significant. For example, let us suppose a decision maker looking for a new car and comparing two cars with prices equal to 20 000 euro for the first one and 20 500 euro for the second one. It may be no surprise if the decision maker says that he/she is indifferent to these two prices. Such an idea of an indifference threshold cannot be handled by total orders or preorders. Interval orders are used in such situations. The special case where the length of intervals is constant (hence the indifference threshold is constant) corresponds to a semiorder [21].

Among binary relations, which satisfy suitable forms of pseudo-transitivity, without necessarily being fully transitive, one of the most studied binary relations are interval orders. It is due to their capability of modeling various types of situations in a wide range of applied fields, e.g., decision aids, mathematical psychology, choice theory under risk, genetics, information storage, scheduling, mathematical programming, archaeology.

We will consider fuzzy interval orders in the context of their preservation in aggregation process (cf. [6], [8], [13], [15], [22], [24]) which is due to the possible applications, e.g., in fuzzy preference modelling, multicriteria decision making problem and solving other issues related to imprecise and uncertain information. In decision making problems, a set $X = \{x_1, \ldots, x_m\}$ represents a set of objects, where $m \in \mathbb{N}$. Additionally, a set $K = \{k_1, \ldots, k_n\}$ of criteria under which the objects are supposed to be evaluated is considered. Fuzzy relations $R_1, \ldots, R_n$ reflect judgements of decision makers. The considered aggregation process involves also an n-argument function F. With the use of given fuzzy relations $R_1, \ldots, R_n$ and the function F, we consider a new fuzzy relation $R_F = F(R_1, \ldots, R_n)$ representing a final decision on evaluated objects (after considering the involved criteria). Although we focus on aggregation functions, the aim of this paper is to give the results under the weakest assumptions on F used for the aggregation process. Therefore, we start our considerations with an arbitrary n-ary functions.

The notions of fuzzy relation properties, in their simplest forms, may involve functions min and max. These ones were generalized by the use of a t-norm and t-conorm, respectively [13, Chapter 2.5]. In particular, the following properties were examined: T-asymmetry, T-antisymmetry, S-connectedness, T-transitivity, negative S-transitivity, T-S-semitransitivity, and the T-S-Ferrers property of fuzzy relations, where T is a t-norm and S a t-conorm, also with regard to their preservation in aggregation process [9]. However, the assumptions put on widely used t-norms are not always necessary or desired. This is why a lot of definitions of binary operations which can play a role of weaker fuzzy connectives were introduced and studied, for example, fuzzy conjunctions: weak t-norms, overlap functions, t-seminorms (or semicopulas, or conjunctors), and pseudo-t-norms, sometimes along with their dual disjunctions.

In this paper, we consider the properties of fuzzy relations with definitions based on fuzzy conjunctions and disjunctions including t-norms and t-conorms. In order to obtain the most general results, we start with binary operations in the unit interval without any additional assumptions. As a result, we examine fuzzy interval orders which are totally B-connected and fulfil the B_1-B_2-Ferrers property, where $B, B_1, B_2 \colon [0,1]^2 \to [0,1]$ are binary operations.

In Section 2, we provide basic definitions and results concerning n-ary functions in $[0,1]$ including fuzzy connectives and dominance between functions. Next, in Section 3, we present basic information about fuzzy relations and some useful results related to preservation of fuzzy relation properties in aggregation process. Finally, in Section 4 we put the main results of this contribution connected with fuzzy interval orders in the aggregation process.

2. Basic notions

In this section, we present some facts concerning n-ary functions in $[0,1]$, aggregation functions, fuzzy connectives (conjunctions and disjunctions) useful in further considerations. Additionally, we recall the notion of dominance between operations.

Definition 2.1 ([5]). Let $n \in \mathbb{N}$. A function $A \colon [0,1]^n \to [0,1]$ which is increasing, i.e.,

$$A(x_1, \ldots, x_n) \leqslant A(y_1, \ldots, y_n) \qquad \text{for} \quad x_i, y_i \in [0,1],\ x_i \leqslant y_i,\ i = 1, \ldots, n$$

is called an aggregation function if $A(0, \ldots, 0) = 0$ and $A(1, \ldots, 1) = 1$.

Example 2.2. Some well-known families of aggregation functions are

- median
$$\operatorname{med}(x_1, \ldots, x_n) = \begin{cases} \frac{s_k + s_{k+1}}{2} & \text{for } n = 2k, \\ s_{k+1} & \text{for } n = 2k+1, \end{cases}$$
where $(s_1, \ldots, s_n)$ is an increasingly ordered sequence of the values $x_1, \ldots, x_n$, which means that $s_1 \leqslant \cdots \leqslant s_n$.
- projections
$$P_k(x_1, \ldots, x_n) = x_k \qquad \text{for} \quad k = 1, 2, \ldots n,$$
- a weighted arithmetic mean
$$A_w(x_1, \ldots, x_n) = \sum_{k=1}^{n} w_k x_k \qquad \text{for} \quad w_k > 0,\ \sum_{k=1}^{n} w_k = 1,$$

- a quasi-linear mean

$$F(x_1,\dots,x_n)=\varphi^{-1}\left(\sum_{k=1}^{n} w_k\varphi(x_k)\right) \qquad \text{for} \quad w_k>0,\ \sum_{k=1}^{n} w_k=1,$$

where $x_1,\dots,x_n \in [0,1]$, $\varphi : [0,1] \to \mathbb{R}$ is a continuous, strictly increasing function.

Definition 2.3. Let $n \in \mathbb{N}$. We say that a function $F\colon [0,1]^n \to [0,1]$

- has a zero element $z \in [0,1]$ if for each $k \in \{1,\dots,n\}$ and each $x_1,\dots,x_{k-1},x_{k+1},\dots,x_n \in [0,1]$

$$F(x_1,\dots,x_{k-1},z,x_{k+1},\dots,x_n)=z,$$

- is without zero divisors if it has a zero element z and

$$\underset{x_1,\dots,x_n\in[0,1]}{\forall}\left(F(x_1,\dots,x_n)=z \ \Rightarrow \left(\underset{1\leqslant k\leqslant n}{\exists}\ x_k=z\right)\right).$$

Definition 2.4 ([10])**.** An operation $C : [0,1]^2 \to [0,1]$ is called a fuzzy conjunction if it is increasing with respect to each variable and $C(1,1)=1$, $C(0,0)=C(0,1)=C(1,0)=0$. An operation $D\colon [0,1]^2 \to [0,1]$ is called a fuzzy disjunction if it is increasing with respect to each variable and $D(0,0)=0$, $D(1,1)=D(0,1)=D(1,0)=1$.

Corollary 2.5. *A fuzzy conjunction has a zero element* 0. *A fuzzy disjunction has a zero element* 1.

Definition 2.6. An operation $C\colon [0,1]^2 \to [0,1]$ is called

- an overlap function [3] if it is a commutative, continuous fuzzy conjunction without zero divisors, fulfilling condition $C(x,y)=1$ if and only if $xy=1$,
- a t-norm [25] if it is a commutative, associative, increasing operation with neutral element 1.

Definition 2.7. An operation $D\colon [0,1]^2 \to [0,1]$ is called:

- a grouping function [4] if it is a commutative, continuous fuzzy disjunction without zero divisors, fulfilling condition $D(x,y)=0$ if and only if $x=y=0$,
- a t-conorm [18] if it is a commutative, associative, increasing operation with neutral element 0,
- a strict t-conorm $S\colon [0,1]^2 \to [0,1]$ if it is a t-conorm which is continuous and strictly increasing in $[0,1)^2$.

Example 2.8 ([18]). The Łukasiewicz t-norm and t-conorm are described in the following way, $T_L(x, y) = \max(x+y-1, 0)$ and $S_L(x, y) = \min(x+y, 1)$, respectively. The product t-norm and t-conorm are given by the formulas $T_P(x, y) = xy$ and $S_P(x, y) = x + y - xy$, respectively.

Definition 2.9. A t-norm T is called nilpotent if it is continuous and each $x \in (0, 1)$ is a nilpotent element of T, i.e., for each $x \in (0, 1)$ there exists $n \in \mathbb{N}$ such that $x_T^{(n)} = 0$.

Theorem 2.10. *Any nilpotent t-norm is isomorphic to the Łukasiewicz t-norm* T_L*, i.e.,*

$$T(x, y) = \varphi^{-1}\Big(T_L\big(\varphi(x), \varphi(y)\big)\Big), \qquad x, y \in [0, 1],$$

where $\varphi\colon [0, 1] \to [0, 1]$ *is an increasing bijection.*

Definition 2.11 (cf. [16], [17])**.** A rotation invariant t-norm is a t-norm T that verifies for all $x, y, z \in [0, 1]$,

$$T(x, y) \leqslant z \Leftrightarrow T(x, 1 - z) \leqslant 1 - y.$$

Proposition 2.12 ([7])**.** *All rotation invariant t-norms T fulfil the property*

$$T_L \leqslant T \leqslant T_{nM},$$

where T_{nM} is the nilpotent minimum [14], i.e.,

$$T_{nM}(x, y) = \begin{cases} 0 & \textit{if } x + y \leqslant 1, \\ \min(x, y), & \textit{otherwise.} \end{cases}$$

Rotation invariant t-norms T satisfy the property $T(x, y) > 0 \Leftrightarrow x + y > 1$, i.e., the lower-left triangle of the unit square constitutes the zero divisors of T. We will not go into further details with this family of t-norms, however this subject was developed for example in [20].

Definition 2.13 ([5])**.** Let $F\colon [0, 1]^n \to [0, 1]$. A function F^d is called a dual function to F, if for all $x_1, \dots, x_n \in [0, 1]$,

$$F^d(x_1, \dots, x_n) = 1 - F(1 - x_1, \dots, 1 - x_n).$$

F is called a self-dual function, if it holds $F = F^d$.

Fuzzy disjunctions are dual to fuzzy conjunctions, grouping functions are dual to overlap functions, t-conorms are dual functions to t-norms, in particular S_L is dual to T_L, S_P is dual to T_P, max is dual to min. Now, we recall the notion of dominance.

Definition 2.14 ([26]). Let $m, n \in \mathbb{N}$. A function $F\colon [0,1]^m \to [0,1]$ dominates function $G\colon [0,1]^n \to [0,1]$ ($F \gg G$) if for an arbitrary matrix $[a_{ik}] = A \in [0,1]^{m\times n}$ the following inequality holds

$$F\big(G(a_{11},\dots,a_{1n}),\dots,G(a_{m1},\dots,a_{mn})\big) \geqslant G\big(F(a_{11},\dots,a_{m1}),\dots,F(a_{1n},\dots,a_{mn})\big).$$

Example 2.15 ([1]). Any weighted arithmetic mean dominates t-norm T_L and any weighted arithmetic mean is dominated by S_L. Minimum dominates any fuzzy conjunction. Fuzzy disjunctions dominate maximum.

3. Fuzzy interval orders

Here we recall the notion of a fuzzy relation, some properties of fuzzy relations and their preservation in aggregation process. Moreover, we propose a notion of the B-B_1-B_2-fuzzy interval order.

Definition 3.1 ([27]). A fuzzy relation in a set $X \neq \emptyset$ is an arbitrary function $R\colon X \times X \to [0,1]$. The family of all fuzzy relations in X is denoted by $FR(X)$.

Definition 3.2 (cf. [1], [13], [23]). Let $B, B_1, B_2 : [0,1]^2 \to [0,1]$ be binary operations. Relation $R \in FR(X)$ is:

- reflexive, if $\forall_{x\in X} R(x,x) = 1$,
- totally B-connected, if $\forall_{x,y\in X} B\big(R(x,y), R(y,x)\big) = 1$,
- B_1-B_2-Ferrers, if $\forall_{x,y,z,w\in X} B_1\big(R(x,y), R(z,w)\big) \leqslant B_2\big(R(x,w), R(z,y)\big)$,
- a B-B_1-B_2-fuzzy interval order, if it is totally B-connected and B_1-B_2-Ferrers,
- a B_1-B_2-fuzzy interval order, if it is totally B_2-connected and B_1-B_2-Ferrers.

We present the notions of the given properties in the most general version, i.e., with operations $B, B_1, B_2\colon [0,1]^2 \to [0,1]$. However, the natural approach is to consider a fuzzy disjunction B in definition of the B-connectedness, a fuzzy conjunction B_1 and a fuzzy disjunction B_2 in the Ferrers property.

Let $F : [0,1]^n \to [0,1]$, $R_1,\dots,R_n \in FR(X)$. An aggregated fuzzy relation $R_F \in FR(X)$ is described by the formula $R_F(x,y) = F\big(R_1(x,y),\dots,R_n(x,y)\big)$, $x, y \in X$. A function F preserves a property of fuzzy relations if for every $R_1,\dots,R_n \in FR(X)$ having this property, R_F also has this property. Preservation of reflexivity, total B-connectedness, the B_1-B_2-Ferrers property and also other properties of this kind was considered in [1]. We will recall here only the results for the properties that will be useful in the sequel.

Theorem 3.3. *Let $R_1, \ldots, R_n \in FR(X)$ be reflexive. The relation R_F is reflexive, if and only if the function F satisfies the condition $F(1, \ldots, 1) = 1$.*

Theorem 3.4. *Let card $X \geqslant 2$, B have a zero element 1 and be without zero divisors. A function F preserves total B-connectedness (B-connectedness) if and only if it satisfies the following condition for all $x, y \in [0,1]^n$,*

$$\underset{1 \leqslant k \leqslant n}{\forall} \max(x_k, y_k) = 1 \Rightarrow \max\big(F(x), F(y)\big) = 1. \tag{1}$$

Example 3.5. Let B be a fuzzy disjunction without zero divisors (e.g., a strict t-conorm or a grouping function). Examples of functions fulfilling (1) for all $x, y \in [0,1]^n$ are $F = \max$, $F =$med or functions F with the zero element $z = 1$ with respect to a certain coordinate, i.e.,

$$\underset{1 \leqslant k \leqslant n}{\exists} \ \underset{i \neq k}{\forall} \ \underset{x_i \in [0,1]}{\forall} F(x_1, \ldots, x_{k-1}, 1, x_{k+1}, \ldots, x_n) = 1.$$

Theorem 3.6. *If a function $F \colon [0,1]^n \to [0,1]$, which is increasing in each of its arguments fulfils $F \gg B_1$ and $B_2 \gg F$, then it preserves the B_1-B_2-Ferrers property.*

Lemma 3.7. *Let $B \colon [0,1]^2 \to [0,1]$ and B^d be a corresponding dual operation. If $F \colon [0,1]^n \to [0,1]$ is a self-dual function, then $F \gg B$ implies $B^d \gg F$.*

The condition given in Lemma 3.7 is only the sufficient one. Let us consider projections $F = P_k$, $B = T$ being a t-norm, $S = T^d$. Then $S \gg P_k$ and $P_k \gg T$, but $F \neq F^d$.

Example 3.8. Any weighted arithmetic mean preserves the B_1-B_2-Ferrers property for t-norm $T_L = B_1$ and t-conorm $S_L = B_2$.

Corollary 3.9. *Any quasi-linear mean preserves the T-S-Ferrers property for a nilpotent t-norm T and $S = T^d$.*

Let us notice that the conditions given in Theorem 3.6 are only the sufficient ones.

Example 3.10. Let us consider $F = T_P$ and fuzzy relations represented by the matrices

$$R_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad R_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Relations R_1, R_2 are min-max-Ferrers ([13]). Moreover, $R = F(R_1, R_2)$ is min-max-Ferrers, where $R \equiv 0$. However, it is not true that $F \gg \min$ (the only t-norm that dominates minimum is minimum itself).

4. Aggregation of fuzzy interval orders

Now, we will consider fuzzy interval orders and their properties in the process of aggregation. In this section, B, B_1, B_2 denote binary operations on unit interval, i.e., $B, B_1, B_2\colon [0,1]^2 \to [0,1]$.

Theorem 4.1. *Let B_1 have an idempotent element 1. A reflexive B_1-B_2-Ferrers relation is totally B_2-connected.*

Proof. If R is a reflexive B_1-B_2-Ferrers fuzzy relation, then we get

$$1 = B_1(1,1) = B_1\big(R(x,x), R(y,y)\big) \leqslant B_2\big(R(x,y), R(y,x)\big),$$

which means that $B_2\big(R(x,y), R(y,x)\big) = 1$ and R is totally B_2-connected. □

Corollary 4.2 ([7]). *Let T be a t-norm and S a t-conorm. A reflexive T-S-Ferrers relation is totally S-connected.*

Theorem 4.3. *Let B_1 have a zero element 0, an idempotent element 1 and for each $x, y \in [0,1]$ such that $x+y > 1$ fulfil $B_1(x,y) = B_1(y,x)$ and let B_2 be dual to B_1 such that $B_1 \leqslant B_2$. The following assertions are equivalent:*

(1) *A reflexive B_1-B_2-Ferrers relation is totally S_L-connected.*
(2) *Operation $B_1\colon [0,1]^2 \to [0,1]$ fulfils $B_1(x,y) > 0$ for any pair $(x,y) \in [0,1]^2$ such that $x + y > 1$.*

Proof. (1) ⇒ (2) Let us consider an operation B_1 such that there exists a pair $(x,y) \in [0,1]^2$ fulfilling $x + y > 1$ and $B_1(x,y) = 0$. Then a reflexive relation that is B_1-B_2-Ferrers but not totally S_L-connected may be build. For example, let $X = \{x_1, x_2\}$ and $R(x_1,x_2) = 1 - x$, $R(x_2,x_1) = 1 - y$.

(2) ⇒ (1) Let $R \in FR(X)$, $x, y \in X$ and R be reflexive and B_1-B_2-Ferrers. Applying these assumptions we obtain

$$\begin{aligned} 1 &= B_1\big(R(x,x), R(y,y)\big) \\ &\leqslant B_2\big(R(x,y), R(y,x)\big) \\ &= 1 - B_1\big(1 - R(x,y), 1 - R(y,x)\big), \end{aligned}$$

which implies that $B_1\big(1-R(x,y), 1-R(y,x)\big) = 0$. As a result from (2) it follows that $1-R(x,y)+1-R(y,x) \leqslant 1$, which means that R is totally S_L-connected. □

Corollary 4.4 ([7]). *Let us consider a t-norm T and its dual t-conorm S. The following statements are equivalent:*

(1) *A reflexive T-S-Ferrers relation is totally S_L-connected.*
(2) *T fulfils $T(x,y) > 0$ for any pair $(x,y) \in [0,1]^2$ such that $x + y > 1$.*

In particular, the above corollary can be applied to all rotation invariant t-norms ([7]). The next results concern total max-connectedness. Let us observe that this notion is also named as strong completeness (cf. [13]).

Theorem 4.5. *Let B have a zero element 1 and have no zero divisors. Then total B-connectedness is equivalent to total* max*-connectedness.*

Proof. Let $R \in FR(X)$, B have a zero element 1 and have no zero divisors. Total B-connectedness is equivalent to

$$B\big(R(x,y), R(y,x)\big) = 1 \Leftrightarrow R(x,y) = 1 \vee R(y,x) = 1 \Leftrightarrow \max\big(R(x,y), R(y,x)\big) = 1,$$

which is equivalent to the fact that R is totally max-connected. □

Corollary 4.6 ([2])**.** *Let S be a t-conorm without zero divisors. Then total S-connectedness is equivalent to total* max*-connectedness.*

Theorem 4.7. *Let a commutative operation B_1 have a zero element 0, an idempotent element 1 and let B_2 be dual to B_1 such that $B_1 \leqslant B_2$. The following assertions are equivalent:*

1. *A reflexive B_1-B_2-Ferrers relation is totally* max*-connected.*
2. *B_1 has no zero divisors.*

Proof. (1) $\Rightarrow$ (2) Let us suppose, to the contrary, B_1 is not without zero divisors. Then there exist $x, y \in (0,1]$ such that $B_1(x,y) = 0$. Let us now consider a relation $R \in FR(X)$, where $X = \{x_1, x_2\}$ and $R(x_1,x_2) = 1 - x$, $R(x_2,x_1) = 1 - y$. R is the B_1-B_2-Ferrers relation but it is not totally max-connected which contradicts to (1).

(2) $\Rightarrow$ (1) Let R be a reflexive B_1-B_2-Ferrers relation. By Theorem 4.1 it is also totally B_2-connected. From (2) and the assumption that B_2 is dual to B_1 it follows that B_2 has a zero element 1 and has no zero divisors. By Theorem 4.5 we obtain that R is totally max-connected. □

Corollary 4.8 ([7])**.** *Let T be a t-norm and S its dual t-conorm. Then the following conditions are equivalent:*

1. *A reflexive T-S-Ferrers relation is totally* max*-connected.*
2. *T has no zero divisors.*

The above-mentioned results from Section 4 simplify the considerations on aggregation of fuzzy interval orders (condition on F for preservation of reflexivity is much easier than the one for total connectedness). Applying these results and results of Section 3 for some special cases of operations B, B_1, B_2, F, we get, for example, the following statements.

Theorem 4.9. *Let T be a rotation invariant t-norm, $R_1, \ldots, R_n \in FR(X)$ be reflexive and T-S_L-Ferrers. If a function $F : [0,1]^n \to [0,1]$, which is increasing in each of its arguments, fulfils $F(1, \ldots, 1) = 1$, $F \gg T$ and $S_L \gg F$, then $R_F = F(R_1, \ldots, R_n)$ is a T-S_L fuzzy interval order.*

Since T_L is an example of a rotation invariant t-norm, in particular we get the following results.

Theorem 4.10. *Let $R_1, \ldots, R_n \in FR(X)$ be reflexive and T_L-S_L-Ferrers. If a function $F : [0,1]^n \to [0,1]$, which is increasing in each of its arguments fulfils $F(1, \ldots, 1) = 1$, $F \gg T_L$ and $S_L \gg F$, then $R_F = F(R_1, \ldots, R_n)$ is a T_L-S_L fuzzy interval order.*

Corollary 4.11. *Let $R_1, \ldots, R_n \in FR(X)$ be reflexive and T_L-S_L-Ferrers. Then fuzzy relation $R_F = F(R_1, \ldots, R_n)$ is a T_L-S_L fuzzy interval order, where F is a weighted arithmetic mean. Moreover, fuzzy relation R_F is a T-S fuzzy interval order, where F is a quasi-linear mean and T is a nilpotent t-norm, $S = T^d$.*

Theorem 4.12. *Let $R_1, \ldots, R_n \in FR(X)$ be reflexive and* min-max*-Ferrers. If a function $F : [0,1]^n \to [0,1]$, which is increasing in each of its arguments fulfils $F(1, \ldots, 1) = 1$, $F \gg \min$ and $\max \gg F$, then $R_F = F(R_1, \ldots, R_n)$ is a* min-max *fuzzy interval order.*

Examples of increasing functions which dominate minimum and are dominated by maximum are projections ([1]), so they fulfil assumptions on F in the above-mentioned theorem.

5. Conclusion

In this paper, fuzzy interval orders were considered in the context of aggregation process. In particular, fuzzy interval orders based in definitions on T as a nilpotent t-norm and $S = T^d$ and their aggregation by quasi-linear means were considered. Note that quasi-linear means are one of the most often appearing aggregation functions in decision making real-life situations [28]. In future work it would be interesting to consider other orders, in particular total preorder, total order, strict total order, partial preorder, partial order, strict partial order, or semiorder and their preservation in aggregation process.

REFERENCES

[1] BENTKOWSKA, U.—KRÓL, A.: *Preservation of fuzzy relation properties based on fuzzy conjunctions and disjunctions during aggregation process*, Fuzzy Sets Syst. **291** (2016), 98–113.

[2] BODENHOFER, U.—KLAWONN, F.: *A formal study of linearity axioms for fuzzy orderings*, Fuzzy Sets Syst. **145** (2004), 323–354.
[3] BUSTINCE, H.—FERNANDEZ, J.—MESIAR, R.—MONTERO, J.—ORDUNA, R.: *Overlap functions*, Nonlinear Anal. **72** (2010), 1488–1499.
[4] BUSTINCE, H.—PAGOLA, M.—MESIAR, R.—HÜLLERMEIER, E.—HERRERA, F.: *Grouping, overlap functions and generalized bi-entropic functions for fuzzy modeling of pairwise comparison*, IEEE Trans. Fuzzy Syst. **20** (2012), 405–415.
[5] CALVO, T.—KOLESÁROVÁ, A.—KOMORNÍKOVÁ, M.—MESIAR, R.: *Aggregation operators: properties, classes and construction methods*, in: Aggregation operators, New trends and applications (T. Calvo et al., eds.), Stud. Fuzziness Soft Comput., Vol. 97, Physica-Verlag, Heidelberg, 2002, pp. 3–104.
[6] DE BAETS, B.—MESIAR, R.: *T–partitions*, Fuzzy Sets and Systems **97** (1998), 211–223.
[7] DÍAZ, S.—DE BAETS, B.—MONTES, S.: *On the Ferrers property of valued interval orders*, Top **19** (2011), 421–447.
[8] DREWNIAK, J.—DUDZIAK, U.: *Preservation of properties of fuzzy relations during aggregation processes*, Kybernetika (Prague) **43** (2007), 115–132.
[9] DUDZIAK, U.: *Preservation of t-norm and t-conorm based properties of fuzzy relations during aggregation process*, in: Proc. of the 8th Conf. EUSFLAT '13 (G. Pasi et al., eds.), Milano, Italy, 2013, Atlantis Press, Amsterdam, 2013, pp. 376–383.
[10] DREWNIAK, J.—KRÓL, A.: *A survey of weak connectives and the preservation of their properties by aggregations*, Fuzzy Sets Syst. **161** (2010), 202–215.
[11] FISHBURN, P. C.: *Intransitive indifference with unequal indifference intervals,* J. Math. Psych. **7** (1970), 144–149.
[12] FISHBURN, P. C.: *Interval representations for interval orders and semiorders,* J. Math. Psych. **10** (1973), 91–105.
[13] FODOR, J.—ROUBENS, M.: *Fuzzy Preference Modelling and Multicriteria Decision Support*, Kluwer Acad. Publ., Dordrecht, 1994.
[14] FODOR, J.: *Contrapositive symmetry of fuzzy implications*, Fuzzy Sets Syst. **69** (1995), 141–156.
[15] GRIGORENKO, O.—LEBEDINSKA, J.: *On another view of aggregation of fuzzy relations*, in: Proc. of 7th Conf. EUSFLAT '11 and LFA '11 (S. Galichet et al., eds.), Aix-les-Bains, France, 2011, Atlantis Pres, Amsterdam, 2011, pp. 21–27.
[16] JENEI, S.: *Geometry of left-continuous t-norms with strong induced negations*, Belg. J. Oper. Res. Statist. Comput. Sci. **38** (1998), 5–16.
[17] JENEI, S.: *New family of triangular norms via contrapositive symmetrization of residuated implications*, Fuzzy Sets Syst. **110** (2000), 157–174.
[18] KLEMENT, E. P.—MESIAR, R.—PAP, E.: *Triangular Norms*, Kluwer Acad. Publ., Dordrecht, 2000.
[19] LUCE, R. D.: *Semiorders and a theory of utility discrimination*, Econometrica **24** (1956), 178–191.
[20] MAES, K. C.—DE BAETS, B.: *Rotation-invariantt-norms: Where triple rotation and rotation-annihilation meet*, Fuzzy Sets Syst. **160** (2009), 1998–2016.
[21] ÖZTÜRK, M.—TSOUKIÀS, A.: *A valued Ferrers relation for interval comparison*, Fuzzy Sets Syst. **266** (2015), 47–66.
[22] PENEVA, V.—POPCHEV, I.: *Properties of the aggregation operators related with fuzzy relations*, Fuzzy Set Syst. **139** (2003), 615–633.
[23] PERNY, P.—ROUBENS, M.: *Fuzzy preference modeling*, in: Fuzzy sets in decision analysis, operations, research and statistics (R. Słowiński, ed.), Kluwer Acad. Publ., Boston, 1998, pp. 3–30.

[24] SAMINGER, S.—MESIAR, R.—BODENHOFER, U.: *Domination of aggregation operators and preservation of transitivity*, Internat. J. Uncertain., Fuzziness, Knowl.-Based Syst. **10** (2002), 11–35.

[25] SCHWEIZER, B.—SKLAR, A.: *Associative functions and statistical triangle inequality*, Publ. Math. Debrecen **8** (1961), 169–186.

[26] SCHWEIZER, B.—SKLAR, A.: *Probabilistic Metric Spaces*, North Holland, New York, 1983.

[27] ZADEH, L. A.: *Fuzzy sets*, Inform. Control **8** (1965), 338–353.

[28] ZIMMERMANN, H. J.—ZYSNO, P.: *Decisions and evaluations by hierarchical aggregation of information*, Fuzzy Sets Syst. **10** (1983), 243–260.

Received July 15, 2016

Urszula Bentkowska
Anna Król
Interdisciplinary Centre for
Computational Modelling
University of Rzeszów
al. Rejtana 16c
PL-35–959 Rzeszów
POLAND
E-mail: ududziak@ur.edu.pl
annakrol@ur.edu.pl

Bernard De Baets
KERMIT,
Research Unit Knowledge-based Systems
Ghent University
Coupure links 653
B–9000 Gent
BELGIUM
E-mail: bernard.debaets@ugent.be