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EXPLICIT EVALUATION OF SOME QUADRATIC EULER-TYPE SUMS CONTAINING DOUBLE-INDEX HARMONIC NUMBERS

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ABSTRACT. In this paper a number of new explicit expressions for quadratic Euler-type sums containing double-index harmonic numbers H_{2n} are given. These are obtained using ordinary generating functions containing the square of the harmonic numbers H_n. As a by-product of the generating function approach used new proofs for the remarkable quadratic series of Au-Yeung

$$\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^2 = \frac{17\pi^4}{360},$$

together with its closely related alternating cousin are given. New proofs for other closely related quadratic Euler-type sums that are known in the literature are also obtained.

1. Introduction

In this paper, we find a number of explicit expressions for quadratic Euler--type sums containing double-index harmonic numbers. These are obtained using (ordinary) generating functions containing the square of the harmonic numbers. As a by-product of using a generating function approach, it provides new proofs for some classic quadratic Euler sums including the remarkable quadratic series of Au-Yeung, to be described in a moment's time.

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Various types of harmonic numbers exist which we now define. The nth single--index or unitary harmonic number is defined by

$$H_n = \sum_{k=1}^{n} \frac{1}{k},$$

while what we term the nth double-index harmonic number is defined by

$$H_{2n} = \sum_{k=1}^{2n} \frac{1}{k}.$$

No simple, direct relationship between H_n and H_{2n} exists though the Botez--Catalan identity [40, p. 338] of

$$\overline{H}_{2n} = \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} = H_{2n} - H_n,$$

comes tantalizingly close. Here $\overline{H}_n = \sum_{k=1}^{n} (-1)^{k+1}/k$ is the nth skew-harmonic number [7]. A final harmonic number is the nth generalized harmonic number of order p. It is defined by

$$H_n^{(p)} = \sum_{k=1}^{n} \frac{1}{k^p},$$

where p is a positive integer. Here $H_n^{(1)} \equiv H_n$, and by convention, $H_0 = H_0^{(p)} = \overline{H}_0 \equiv 0$.

In responding to a letter sent by Goldbach to Euler in 1742, the latter considered infinite series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^q} \sum_{k=1}^{n} \frac{1}{k^p} = \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q}, \tag{1}$$

where $q \geqslant 2$ is a positive integer. Since Euler's time these infinite series have been well studied and constitute what we now term linear Euler sums while many modifications, extensions, and generalisations along the line of the basic linear Euler sum have been considered. In this paper, we plan to consider quadratic sums of the form

$$\sum_{n=1}^{\infty} \frac{(\pm 1)^n H_{a(n)}^2}{n^q}, \tag{2}$$

for the cases where $a(n) = n$ and $a(n) = 2n$. The former gives rise to the familiar and well studied quadratic Euler sums containing single-index harmonic numbers while the latter gives rise to the far less common quadratic Euler-type sums containing double-index harmonic numbers. Variations of these series arising from such sums are also considered. We intend to do this for the cases $q = 0, 1,$ and 2 by first finding generating functions for harmonic numbers containing the term H_n^2.

The non-alternating case of (2) when $a(n) = n$ and $q = 2$ is well known in the literature. Indeed,

$$\sum_{n=1}^{\infty} \left(\frac{H_n}{n}\right)^2 = \frac{17}{4}\zeta(4) = \frac{17\pi^4}{360}. \tag{3}$$

Here ζ denotes the Riemann zeta function defined by $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, $\mathrm{Re}(s) > 1$. After its more recent rediscovery, the series in (3) is usually referred to as the quadratic series of Au-Yeung [39] (for an interesting account of its rediscovery, see [1]). Regarding its evaluation, it seems to have been first given by Martin Kneser [22] after having been initially proposed as a problem by H. F. Sandham in *The American Mathematical Monthly* [32]. There, series manipulation followed by the evaluation of a logarithmic integral was used to find its value. It is listed in Hansen's *A Table of Series and Products* [20, Entry 55.8.2, p. 366] and in the *NIST Handbook of Mathematical Functions* [29, Entry 25.16.13, p. 614]. Since first appearing, many alternative proofs for (3) have appeared in the literature [3–6,11–16,18,34,39,45]. It also appears as a problem in [31, Problem 2.6.1., p. 110], [17, Problem 3.70, p. 150], and [40, Problem 4.22, p. 292] further confirming the important role this sum plays in the theory of non-linear Euler sums. The proof we give for (3) in Section 3 using a generating function approach represents a new proof to this most classic of quadratic Euler sums.

The alternating case of (2) when $a(n) = n$ and $q = 2$, while considerably less famous than its Au-Yeung cousin, is also known [1,13,42], [40, Problem 4.52 (iii), p. 310]. Its value is

$$\begin{aligned}\sum_{n=1}^{\infty} (-1)^n \left(\frac{H_n}{n}\right)^2 = {} & 2\,\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{41}{16}\zeta(4) + \frac{7}{4}\log(2)\zeta(3) \\ & - \frac{1}{2}\log^2(2)\zeta(2) + \frac{1}{12}\log^4(2).\end{aligned} \tag{4}$$

Here Li_s denotes the polylogarithm function of order s defined by $\sum_{n=1}^{\infty} z^n/n^s$ for $|z| \leqslant 1$ provided $\mathrm{Re}(s) > 1$. Orders two, three, and four for the polylogarithm function are referred to as the dilogarithm, trilogarithm, and tetralogarithm respectively. Once again, the proof we give for (4) in Section 3 using a generating function approach represents a new proof for this slightly less famous alternating quadratic cousin of Au-Yeung.

Compared to Euler sums containing harmonic numbers with the usual unitary index, Euler sums containing harmonic numbers with non-unitary indices have been much less studied in the literature. Those which have appeared have mainly been confined to a linear term containing double-index harmonic numbers [30, 37], multi-index linear sums containing the term H_{kn} where k is a positive integer [10, 23], or in the study of so-called Jordan sums [2, 4, 8, 12, 13, 21, 35, 45], [9, pp. 189–199] a variant Euler-type sum where the harmonic number term appearing in the linear Euler sum of (1) is replaced with the term

$$\Lambda_n = 1 + \frac{1}{3} + \cdots + \frac{1}{2n-1} = \sum_{k=1}^{n} \frac{1}{2k-1} = H_{2n} - \frac{1}{2}H_n.$$

In investigating quadratic sums containing double-index harmonic numbers explicit expressions for the following six quadratic Euler-type sums will be given:

$$\begin{aligned} &\sum_{n=1}^{\infty}(-1)^n \frac{H_{2n}^2}{n}, && \sum_{n=1}^{\infty} \frac{(-1)^n H_{2n}^2}{2n+1}, \\ &\sum_{n=1}^{\infty}\left(\frac{H_{2n}}{n}\right)^2, && \sum_{n=1}^{\infty}(-1)^n\left(\frac{H_{2n}}{n}\right)^2, \\ &\sum_{n=1}^{\infty} \frac{H_{2n}^2}{(2n+1)^2}, && \sum_{n=1}^{\infty} \frac{(-1)^n H_{2n}^2}{(2n+1)^2}. \end{aligned} \tag{5}$$

These are given in Section 4. A number of these sums appear to be new results in the literature.

2. Generating functions containing the square of the harmonic numbers

In this section, we find a number of generating functions that contain the square of the harmonic numbers H_n^2. These are then used to find a number of quadratic Euler and Euler-types sums.

We start by recalling the generating function for the sequence $\{H_n\}_{n\geqslant 1}$, namely [19, 1.513 (6), p. 52]

$$\sum_{n=1}^{\infty} H_n z^n = -\frac{\log(1-z)}{1-z}, \quad |z| < 1. \tag{6}$$

Integrating (6) from 0 to z immediately yields

$$\sum_{n=1}^{\infty} \frac{H_n z^n}{n+1} = \frac{\log^2(1-z)}{2z}. \tag{7}$$

The generating function for the sequence $\{H_n^2\}_{n\geqslant 1}$ is now given in the following lemma.

Lemma 2.1. *For $|z| < 1$ the generating function for the sequence $\{H_n^2\}_{n\geqslant 1}$ is*

$$\sum_{n=1}^{\infty} H_n^2 z^n = \frac{\mathrm{Li}_2(z) + \log^2(1-z)}{1-z}. \tag{8}$$

P r o o f. Noting that

$$H_{n+1}^2 - H_n^2 = (H_{n+1} - H_n)(H_{n+1} + H_n) = \frac{2H_n}{n+1} + \frac{1}{(n+1)^2},$$

where the recurrence relation of $H_{n+1} = H_n + \frac{1}{n+1}$ for the harmonic numbers has been used, we have

$$\sum_{n=1}^{\infty} H_{n+1}^2 z^n - \sum_{n=1}^{\infty} H_n^2 z^n = 2\sum_{n=1}^{\infty} \frac{H_n z^n}{n+1} + \sum_{n=1}^{\infty} \frac{z^n}{(n+1)^2},$$

or

$$\sum_{n=1}^{\infty} H_n^2 z^{n-1} - \sum_{n=1}^{\infty} H_n^2 z^n = 2\sum_{n=1}^{\infty} \frac{H_n z^n}{n+1} + \sum_{n=1}^{\infty} \frac{z^{n-1}}{n^2}, \tag{9}$$

after the index in the left-most sum on the left of the equality and the right-most sum on the right of the equality have been shifted by $n \mapsto n-1$. The first sum to the right of the equality in (9) is (7) while the second sum is $\mathrm{Li}_2(z)/z$. Thus

$$\left(\frac{1-z}{z}\right)\sum_{n=1}^{\infty} H_n^2 z^n = \frac{\log^2(1-z)}{z} + \frac{\mathrm{Li}_2(z)}{z},$$

from which the required result follows. □

The generating function appearing in Lemma 2.1 seems to have been first given without proof in [5]. For an alternative proof, see [26].

Lemma 2.2. *For $|z| \leqslant 1$, $z \neq 1$ the generating function for the sequence $\{H_n^2/n\}_{n\geqslant 1}$ is*

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n} z^n = \mathrm{Li}_3(z) - \mathrm{Li}_2(z)\log(1-z) - \frac{1}{3}\log^3(1-z). \tag{10}$$

P r o o f. Replacing z with t in the result given in Lemma 2.1, dividing the result by t before integrating from 0 to z yields

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n} z^n = \int_0^z \frac{\log^2(1-t)}{t(1-t)}\,\mathrm{d}t + \int_0^z \frac{\mathrm{Li}_2(t)}{t(1-t)}\,\mathrm{d}t.$$

For the first of the integrals

$$\int_0^z \frac{\log^2(1-t)}{t(1-t)}\,\mathrm{d}t = -2\,\mathrm{Li}_3(1-z) + 2\,\mathrm{Li}_2(1-z)\log(1-z) - \frac{1}{3}\log^3(1-z) + \log(z)\log^2(1-z) + 2\zeta(3).$$

This can be seen directly by differentiation. The constant of integration $-2\zeta(3)$ comes on letting $z \to 0$. For the second of the integrals

$$\int_0^z \frac{\operatorname{Li}_2(t)}{t(1-t)}\,\mathrm{d}t = 2\operatorname{Li}_3(1-z) + \operatorname{Li}_3(z) - 2\operatorname{Li}_2(1-z)\log(1-z) \\ - \operatorname{Li}_2(z)\log(1-z) - \log(z)\log^2(1-z) - 2\zeta(3).$$

This can again be seen by direct differentiation. The constant of integration $2\zeta(3)$ comes on letting $z \to 0$. Adding together the two results found delivers the desired result. □

A result closely related to the generating function given in Lemma 2.2 can be found in [27]. We now come to the main result of this section, a generating function for the sequence $\{H_n^2/n^2\}_{n\geqslant 1}$. We give this in the following theorem.

Theorem 2.3. *For $|z| \leqslant 1$ the generating function for the sequence $\{H_n^2/n^2\}_{n\geqslant 1}$ is*

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^2} z^n = \operatorname{Li}_4(z) - 2\operatorname{Li}_4(1-z) + 2\operatorname{Li}_3(1-z)\log(1-z) + \frac{1}{2}\operatorname{Li}_2^2(z) \\ - \operatorname{Li}_2(1-z)\log^2(1-z) - \frac{1}{3}\log(z)\log^3(1-z) + 2\zeta(4). \tag{11}$$

P r o o f. From the result given in Lemma 2.2, replacing z with t, dividing by t before integrating from 0 to z yields

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^2} z^n = \int_0^z \frac{\operatorname{Li}_3(t)}{t}\,\mathrm{d}t - \int_0^z \frac{\operatorname{Li}_2(t)\log(1-t)}{t}\,\mathrm{d}t - \frac{1}{3}\int_0^z \frac{\log^3(1-t)}{t}\,\mathrm{d}t.$$

The first and second integrals are elementary. Here

$$\int_0^z \frac{\operatorname{Li}_3(t)}{t}\,\mathrm{d}t = \operatorname{Li}_4(z) \quad \text{and} \quad \int_0^z \frac{\operatorname{Li}_2(t)\log(1-t)}{t}\,\mathrm{d}t = -\frac{1}{2}\operatorname{Li}_2^2(z).$$

For the third integral, we have

$$\int_0^z \frac{\log^3(1-t)}{t}\,\mathrm{d}t = 6\operatorname{Li}_4(1-z) - 6\operatorname{Li}_3(z)\log(1-z) \\ + 3\operatorname{Li}_2(1-z)\log^2(1-z) + \log(z)\log^3(1-z) - 6\zeta(4),$$

a result that can be confirmed by differentiation. The constant of integration $6\zeta(4)$ comes on letting $z \to 0$. Gathering the results for the three integrals, the desired result then follows. □

3. Some series as consequences of Lemmas 2.1 and 2.2 and Theorem 2.3

From the generating functions (8), (10), and (11) found in Lemmas 2.1 and 2.2 and Theorem 2.3, a number of quadratic Euler sums, both alternating and non-alternating, together with some closely related variants can be found. As these are going to rely on a number of special values for the polylogarithmic functions, these can be found listed in the Appendix.

We give first several quadratic Euler-type sums. Here each contains a factor of 2^n in the denominator of the summand and comes about from setting $z = \pm\frac{1}{2}$ in (8), (10), and (11). When $z = \frac{1}{2}$ one has

$$\sum_{n=1}^{\infty} \frac{H_n^2}{2^n} = \zeta(2) + \log^2(2),$$

$$\sum_{n=1}^{\infty} \frac{H_n^2}{2^n n} = \frac{7}{8}\zeta(3), \tag{12}$$

$$\sum_{n=1}^{\infty} \frac{H_n^2}{2^n n^2} = -\operatorname{Li}_4\left(\frac{1}{2}\right) + \frac{37}{16}\zeta(4) - \frac{7}{4}\zeta(3)\log(2) + \frac{1}{4}\zeta(2)\log^2(2) - \frac{1}{24}\log^4(2), \tag{13}$$

where the special values for the dilogarithm and trilogarithm given in (27) have been used. Alternative derivations for (12) and (13) can be found in [11, 43]. When $z = -\frac{1}{2}$ one has

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n^2}{2^n} = \frac{1}{3}\operatorname{Li}_2\left(\frac{1}{4}\right) - \frac{1}{3}\zeta(2) + \log^2(2) - \frac{4}{3}\log(2)\log(3) + \frac{2}{3}\log^2(3)$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_n^2}{2^n n} = \frac{1}{4}\operatorname{Li}_3\left(\frac{1}{4}\right) + \frac{1}{2}\log\left(\frac{2}{3}\right)\operatorname{Li}_2\left(\frac{1}{4}\right) - \frac{7}{8}\zeta(3) + \frac{1}{2}\zeta(2)\log(3) + \frac{2}{3}\log^3(2) - \frac{1}{3}\log^3(3) - \frac{3}{2}\log^2(2)\log(3) + \log(2)\log^2(3),$$

where the special values for the dilogarithm and trilogarithm given in (28) have been used.

Classic quadratic Euler sums are obtained from their respective generating functions by setting $z = \pm 1$ in (8), (10), and (11) whenever the resultant series converge. For the non-alternating case, setting $z = 1$ in (8) and (10) sees each series diverge while setting $z = 1$ into (11) immediately yields (3), thereby providing a new proof for this most remarkable of quadratic Euler sums.

For the alternating case, setting $z=-1$ in (8) sees the series diverge, substituting $z=-1$ into (10) is straight forward, while substituting $z=-1$ into (11) forces one to deal with imaginary quantities. In all cases where these occur, the principal branch is selected. The imaginary parts will naturally enough cancel, as they should, leaving behind a real value. The simplest of these is $\log(-1)=i\pi$. For the others, the values found in (29), (30), and (31) of the Appendix are required. Thus

$$\sum_{n=1}^{\infty}(-1)^n H_n^2 \quad \text{diverges,}$$

$$\sum_{n=1}^{\infty}(-1)^n \frac{H_n^2}{n} = -\frac{3}{4}\zeta(3)+\frac{1}{2}\zeta(2)\log(2)-\frac{1}{3}\log^3(2), \tag{14}$$

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^n \frac{H_n^2}{n^2} &= 2\operatorname{Li}_4\left(\frac{1}{2}\right)-\frac{41}{16}\zeta(4)+\frac{7}{4}\zeta(3)\log(2)-\frac{1}{2}\zeta(2)\log^2(2)\\ &\quad+\frac{1}{12}\log^4(2).\end{aligned} \tag{15}$$

Alternative proofs for (14) can be found in [12,36,41], [17, Problem 3.57, p. 207] while our proof of (15) using a generating function approach represents a new proof for this less famous Au-Yeung cousin. Other proofs for the latter result can be found in [1, 13, 42].

4. Some series containing the square of the double-index harmonic numbers

We now give evaluations for six quadratic Euler-type sums that contain double-index harmonic numbers found listed in (5). Some of these results for the sums given we believe are new to the literature. We consider first sums which are non-alternating as these are far simpler to deal with. In such cases the following Lemma concerning convergent series is very useful.

Lemma 4.1. *For absolutely convergent series*

$$\sum_{n=1}^{\infty} a_{2n} = \frac{1}{2}\sum_{n=1}^{\infty} a_n + \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n a_n.$$

Proof. A rearrangement of terms in an absolutely convergent series immediately delivers the result. $\square$

Two quadratic Euler-type sums containing double-index harmonic numbers immediately follow from the application of Lemma 4.1. The first is a quadratic Euler-type sum analogous to the Au-Yeung series where the ordinary unitary harmonic number is replaced with its double-index counterpart.

Proposition 4.2.

$$\sum_{n=1}^{\infty}\left(\frac{H_{2n}}{n}\right)^2 = 4\,\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{27}{8}\zeta(4)+\frac{7}{2}\zeta(3)\log(2)-\zeta(2)\log^2(2)+\frac{1}{6}\log^4(2).$$

P r o o f. Employing the result in Lemma 4.1 on the series to the left which converges absolutely, one obtains

$$\sum_{n=1}^{\infty}\left(\frac{H_{2n}}{n}\right)^2 = 4\sum_{n=1}^{\infty}\frac{H_{2n}^2}{(2n)^2} = 2\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^2 + 2\sum_{n=1}^{\infty}(-1)^n\left(\frac{H_n}{n}\right)^2.$$

Combining with the results given in (3) and (15), the desired result immediately follows. □

The second non-alternating sum is a quadratic Euler-type sum containing a double-index harmonic number with the denominator in the summand modified from n^2 to $(2n+1)^2$.

Proposition 4.3.

$$\sum_{n=1}^{\infty}\frac{H_{2n}^2}{(2n+1)^2} = \mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{11}{32}\zeta(4)+\frac{7}{8}\zeta(3)\log(2)-\frac{1}{4}\zeta(2)\log^2(2)+\frac{1}{24}\log^4(2).$$

P r o o f. Applying the result given in Lemma 4.1 to the series on the left which converges absolutely, we have

$$\sum_{n=1}^{\infty}\frac{H_{2n}^2}{(2n+1)^2} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n^2}{(n+1)^2}+\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{H_n^2}{(n+1)^2},$$

or

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_{2n}^2}{(2n+1)^2} &= \frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n^2}{n^2}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^nH_n^2}{n^2}-\sum_{n=1}^{\infty}\frac{H_n}{n^3}+\sum_{n=1}^{\infty}(-1)^n\frac{H_n}{n^3}\\ &\quad+\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^4}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\\ &= \frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n^2}{n^2}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^nH_n^2}{n^2}-\sum_{n=1}^{\infty}\frac{H_n}{n^3}+\sum_{n=1}^{\infty}(-1)^n\frac{H_n}{n^3}\\ &\quad+\frac{15}{16}\zeta(4).\end{aligned}\tag{16}$$

after shifting the indices by $n \mapsto n-1$ before applying the recurrence relation for the nth harmonic numbers of $H_{n-1} = H_n - \frac{1}{n}$. The first and second sums appearing in (16) are just the quadratic series of Au-Yeung and its alternating cousin, values for which were given in (3) and (15) respectively. The third and fourth series that appear are linear Euler sums with their values being well known in the literature (see, for example [14, p. 16, 32]). Here

$$\sum_{n=1}^{\infty} \frac{H_n}{n^3} = \frac{5}{4}\zeta(4), \tag{17}$$

and

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^n H_n}{n^3} = 2\,\mathrm{Li}_4\left(\frac{1}{2}\right) & \\ - \frac{11}{4}\zeta(4) + \frac{7}{4}\zeta(3)\log(2) - \frac{1}{2}\zeta(2)\log^2(2) & \\ + \frac{1}{12}\log^4(2). &\end{aligned} \tag{18}$$

For a rather simple proof of the second of these sums the interested reader is referred to [25]. Combining all values found into (16) leads to the desired result. □

Remark 1. From Propositions 4.2 and 4.3 we have an interesting relation between two quadratic Euler-type sums containing double-index harmonic numbers and $\zeta(4)$. It is

$$2\sum_{n=1}^{\infty} \frac{H_{2n}^2}{(2n)^2} - 2\sum_{n=1}^{\infty} \frac{H_{2n}^2}{(2n+1)^2} = \zeta(4).$$

The final four quadratic Euler-type series containing double-index harmonic numbers we give are of the alternating type. To find their values we make use of the various generating functions we have given in Lemmas 2.1 and 2.2, and Theorem 2.3. Before presenting these we give a useful result for convergent series and two generating functions for sequences containing terms linear in H_n.

Lemma 4.4. *If $a_n > 0$ and all series converge, then*

$$\sum_{n=1}^{\infty} (-1)^n a_{2n} = \operatorname{Re} \sum_{n=1}^{\infty} i^n a_n\,,$$

where i is the imaginary unit and Re *denotes the real part.*

P r o o f. The result can be directly seen by writing out the terms in the series appearing on the right before taking the real part. □

Lemma 4.5. *For $|z| \leqslant 1$ the generating function for the sequence $\{H_n/n^2\}_{n\geqslant 1}$ is*

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} z^n = \mathrm{Li}_3(z) - \mathrm{Li}_3(1-z) + \mathrm{Li}_2(1-z)\log(1-z) + \frac{1}{2}\log(z)\log^2(1-z) + \zeta(3).$$

P r o o f. Starting with the generating function for the sequence $\{H_n\}_{n\geqslant 1}$ given in (6), replacing z with t, dividing by t throughout before integrating from 0 to z yields

$$\sum_{n=1}^{\infty} \frac{H_n z^n}{n} = -\int_0^z \frac{\log(1-t)}{t(1-t)}\,\mathrm{d}t = -\int_0^z \frac{\log(1-t)}{t}\,\mathrm{d}t - \int_0^z \frac{\log(1-t)}{1-t}\,\mathrm{d}t,$$

following a partial fraction decomposition, or

$$\sum_{n=1}^{\infty} \frac{H_n z^n}{n} = \mathrm{Li}_2(z) + \frac{1}{2}\log^2(1-z), \tag{19}$$

where in the first integral we have recognised the integral representation for the dilogarithm. Continuing, replacing z with t in (19), dividing throughout by t before integrating from 0 to z yields

$$\sum_{n=1}^{\infty} \frac{H_n z^n}{n^2} = \int_0^z \frac{\mathrm{Li}_2(t)}{t}\,\mathrm{d}t + \frac{1}{2}\int_0^z \frac{\log^2(1-t)}{t}\,\mathrm{d}t. \tag{20}$$

Since

$$\int_0^z \frac{\mathrm{Li}_2(t)}{t}\,\mathrm{d}t = \mathrm{Li}_3(z),$$

and

$$\int_0^z \frac{\log^2(1-t)}{t}\,\mathrm{d}t = \log(z)\log^2(1-z) + 2\,\mathrm{Li}_2(1-z)\log(1-z) - 2\,\mathrm{Li}_3(1-z) + 2\zeta(3),$$

a result that can be confirmed by differentiation with the constant of integration $-2\zeta(3)$ found on letting $z \to 0$, combining the values found into (20), the result immediately follows. $\square$

Lemma 4.6. *For $|z| \leqslant 1$ the generating function for the sequence $\{H_n/n^3\}_{n\geqslant 1}$ is*

$$\sum_{n=1}^{\infty} \frac{H_n}{n^3} z^n = 2\,\mathrm{Li}_4(z) + \mathrm{Li}_4\left(\frac{z}{z-1}\right) - \mathrm{Li}_4(1-z) - \mathrm{Li}_3(z)\log(1-z)$$
$$+ \frac{1}{24}\log^4(1-z) - \frac{1}{6}\log(z)\log^3(1-z)$$
$$+ \frac{1}{2}\zeta(2)\log^2(1-z) + \zeta(3)\log(1-z) + \zeta(4).$$

P r o o f. Let $g(z) = \sum_{n=1}^{\infty} H_n z^n/n^3$. Observing that

$$\int_0^1 t^{n-1}\log^2(t)\,\mathrm{d}t = \frac{2}{n^3}, \quad n \in \mathbb{N},$$

then

$$g(z) = \frac{1}{2}\sum_{n=1}^{\infty} H_n z^n \int_0^1 t^{n-1}\log^2(t)\,\mathrm{d}t$$
$$= \frac{1}{2}\int_0^1 \frac{\log^2(t)}{t}\sum_{n=1}^{\infty} H_n (zt)^n\,\mathrm{d}t,$$

where the interchange made between the summation and the integration is permissible due to Fubini's theorem. Recognising the series as (6) with z replaced with zt, we have

$$-2g(z) = \int_0^1 \frac{\log^2(t)\log(1-zt)}{t(1-zt)}\,\mathrm{d}t,$$

or

$$\begin{aligned}-2g(z) = &\int_0^z \frac{\log^2(u)\log(1-u)}{u}\,\mathrm{d}u + \int_0^z \frac{\log^2(u)\log(1-u)}{1-u}\,\mathrm{d}u\\ &- 2\log(z)\int_0^z \frac{\log(u)\log(1-u)}{u}\,\mathrm{d}u - 2\log(z)\int_0^z \frac{\log(u)\log(1-u)}{1-u}\,\mathrm{d}u\\ &+ \log^2(z)\int_0^z \frac{\log(1-u)}{u}\,\mathrm{d}u + \log^2(z)\int_0^z \frac{\log(1-u)}{1-u}\,\mathrm{d}u,\end{aligned} \tag{21}$$

after a partial fraction decomposition and a substitution of $u = zt$ have been made. For the first, and third through to sixth integrals appearing in (21), we have

$$\int_0^z \frac{\log^2(u)\log(1-u)}{u}\,\mathrm{d}u = -2\operatorname{Li}_4(z) - \operatorname{Li}_2(z)\log^2(z) + \operatorname{Li}_3(z)\log(z) + 2\zeta(4),$$

$$\int_0^z \frac{\log(u)\log(1-u)}{u}\,\mathrm{d}u = -\operatorname{Li}_2(z)\log(z) + \operatorname{Li}_3(z),$$

$$\int_0^z \frac{\log(u)\log(1-u)}{1-u}\,\mathrm{d}u = -\operatorname{Li}_3(1-z) + \operatorname{Li}_2(1-z)\log(1-z) + \zeta(3),$$

$$\int_0^z \frac{\log(1-u)}{u}\,\mathrm{d}u = -\operatorname{Li}_2(z),$$

$$\int_0^z \frac{\log(1-u)}{1-u}\,\mathrm{d}u = -\frac{1}{2}\log^2(1-z).$$

Each of the above integrals can be readily confirmed by differentiation and the constant of integration found by letting $z \to 0$. For the remaining integral, the second one which will we call I_2, this can be found by taking advantage of the following algebraic identity

$$a^2 b = \frac{1}{3}\left(a^3 - b^3 - (a-b)^3 + 3ab^2\right).$$

Setting $a = \log(u)$ and $b = \log(1-u)$ we see that

$$\begin{aligned}\log^2(u)\log(1-u) = {} & \frac{1}{3}\log^3(u) - \frac{1}{3}\log^3(1-u) - \frac{1}{3}\log^3\left(\frac{u}{1-u}\right) \\ & + \log(u)\log^2(1-u).\end{aligned}$$

Thus

$$\begin{aligned}I_2 = {} & \frac{1}{3}\int_0^z \frac{\log^3(u)}{1-u}\,\mathrm{d}u - \frac{1}{3}\int_0^z \frac{\log^3(1-u)}{1-u}\,\mathrm{d}u \\ & - \frac{1}{3}\int_0^z \log^3\left(\frac{u}{1-u}\right)\frac{\mathrm{d}u}{1-u} \\ & + \int_0^z \frac{\log(u)\log^2(1-u)}{1-u}\,\mathrm{d}u\,.\end{aligned}$$

For the first, second, and fourth integrals we have

$$\int_0^z \frac{\log^3(u)}{1-u}\,\mathrm{d}u = -6\,\mathrm{Li}_4(z) - \mathrm{Li}_2(z)\log^2(z) + 6\,\mathrm{Li}_3(z)\log(z),$$
$$- \log^3(z)\log(1-z) + 6\zeta(4),$$
$$\int_0^z \frac{\log^3(1-u)}{1-u}\,\mathrm{d}u = -\frac{1}{4}\log^4(1-z),$$
$$\int_0^z \frac{\log(u)\log^2(1-u)}{1-u}\,\mathrm{d}u = 2\,\mathrm{Li}_4(1-z) + \mathrm{Li}_2(1-z)\log^2(1-z)$$
$$- 2\,\mathrm{Li}_3(1-z)\log(1-z) - 2\zeta(4).$$

Again each of the above integrals can be confirmed by differentiation with the constant of integration found by letting $z \to 0$. For the third integral appearing in I_2, making a substitution of $t = u/(1-u)$ leads to

$$\begin{aligned}\int_0^z \log^3\left(\frac{u}{1-u}\right)\frac{\mathrm{d}u}{1-u} &= \int_0^{\frac{z}{1-z}} \frac{\log^3 t}{1+t}\,\mathrm{d}t \\ &= \Big[6\,\mathrm{Li}_4(-t) - 6\,\mathrm{Li}_3(-t)\log(t) + \\ &\qquad 3\,\mathrm{Li}_2(-t)\log^2(t) + \log(1+t)\log^3(t)\Big]_0^{\frac{z}{1-z}} \\ &= 6\,\mathrm{Li}_4\left(\frac{z}{z-1}\right) - 6\,\mathrm{Li}_3\left(\frac{z}{z-1}\right)\log\left(\frac{z}{1-z}\right) \\ &\quad + 3\,\mathrm{Li}_2\left(\frac{z}{z-1}\right)\log^2\left(\frac{z}{1-z}\right) - \log^3\left(\frac{z}{1-z}\right)\log(1-z).\end{aligned}$$

Thus

$$\begin{aligned}I_2 = &-2\,\mathrm{Li}_4(z) + 2\,\mathrm{Li}_4(1-z) - 2\,\mathrm{Li}_4\left(\frac{z}{z-1}\right) + 2\,\mathrm{Li}_3(z)\log(z) \\ &- 2\,\mathrm{Li}_3(1-z)\log(1-z) + 2\,\mathrm{Li}_3\left(\frac{z}{z-1}\right)\log\left(\frac{z}{1-z}\right) \\ &+ \mathrm{Li}_2(1-z)\log^2(1-z) - \mathrm{Li}_2(z)\log^2(z) - \mathrm{Li}_2\left(\frac{z}{z-1}\right)\log^2\left(\frac{z}{1-z}\right) \\ &+ \frac{1}{3}\log^3\left(\frac{z}{1-z}\right)\log(1-z) + \frac{1}{3}\log^3(z)\log(1-z) \\ &+ \frac{1}{12}\log^4(1-z) - 2\zeta(4).\end{aligned}$$

Piecing all the results together we find

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n z^n}{n^3} &= 2\operatorname{Li}_4(z)-\operatorname{Li}_4(1-z)+\operatorname{Li}_4\left(\frac{z}{z-1}\right)-\operatorname{Li}_3(1-z)\log(z)\\
&\quad-\operatorname{Li}_3(z)\log(z)+\operatorname{Li}_3(1-z)\log(1-z)-\operatorname{Li}_3\left(\frac{z}{z-1}\right)\log\left(\frac{z}{1-z}\right)\\
&\quad+\operatorname{Li}_2(1-z)\log(z)\log(1-z)-\frac{1}{2}\operatorname{Li}_2(1-z)\log^2(1-z)\\
&\quad+\frac{1}{2}\operatorname{Li}_2(z)\log^2(z)+\frac{1}{2}\operatorname{Li}_2\left(\frac{z}{z-1}\right)\log^2\left(\frac{z}{1-z}\right)\\
&\quad+\frac{1}{4}\log^2(z)\log^2(1-z)+\zeta(3)\log(z)-\frac{1}{6}\log^3\left(\frac{z}{1-z}\right)\log(1-z)\\
&\quad+\frac{1}{6}\log^3(z)\log(1-z)-\frac{1}{24}\log^4(1-z)+\zeta(4).\end{aligned}\tag{22}$$

Further simplification leading to the desired result is possible by expanding each of the logarithmic terms appearing in (22) containing $z/(1-z)$ in its argument followed by applying the following three polylogarithmic identities to the expression that results:

(i) Euler's reflexion formula for the dilogarithm, namely (41);

(ii) Landen's dilogarithm identity of [24, (1.12), p. 5]

$$\operatorname{Li}_2(z)+\operatorname{Li}_2\left(\frac{z}{z-1}\right)=-\frac{1}{2}\log^2(1-z),$$

and;

(iii) Landen's trilogarithm identity of [24, (6.10), p. 155]

$$\begin{aligned}\operatorname{Li}_3(z)+\operatorname{Li}_3(1-z)+\operatorname{Li}_3\left(\frac{z}{z-1}\right)&=\frac{1}{6}\log^3(1-z)-\frac{1}{2}\log(z)\log^2(1-z)\\
&\quad+\zeta(2)\log(1-z)+\zeta(3).\end{aligned}$$

□

Remark 2. Substituting $z=-1$ and allowing $z\to 1^-$ in the generating function given in Lemma 4.6 immediately provides one with alternate derivations for the results quoted in (18) and (17) respectively.

Proposition 4.7.

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_{2n}^2}{n}=-\frac{3}{16}\zeta(3)+\frac{1}{2}\zeta(2)\log(2)-\frac{1}{2}\pi\mathbf{G}-\frac{1}{12}\log^3(2),$$

where $\mathbf{G}$ *is Catalan's constant* $\sum_{n=0}^{\infty}(-1)^n/(2n+1)^2$.

P r o o f. Applying the result given in Lemma 4.4 on the series to the left we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_{2n}^2}{n} = 2 \sum_{n=1}^{\infty} \frac{(-1)^n H_{2n}^2}{(2n)} = 2 \operatorname{Re} \sum_{n=1}^{\infty} i^n \frac{H_n^2}{n} .$$

From the generating function given in Lemma 2.2, substituting $z = i$ into (10) gives

$$\sum_{n=1}^{\infty} i^n \frac{H_n^2}{n} = \operatorname{Li}_3(i) - \operatorname{Li}_2(i) \log(1-i) - \frac{1}{3} \log^3(1-i),$$

or

$$\begin{aligned} \sum_{n=1}^{\infty} i^n \frac{H_n^2}{n} &= -\frac{3}{32}\zeta(3) + \frac{1}{4}\zeta(2)\log(2) \\ &\quad - \frac{1}{4}\pi\mathbf{G} - \frac{1}{24}\log^3(2) + \frac{i\pi}{8}\zeta(2) \\ &\quad - \frac{i}{2}\mathbf{G}\log(2) + \frac{i\pi}{16}\log^2(2), \end{aligned} \tag{23}$$

where values quoted in (33), (35), and (37) have been used. Taking the real part of (23) before multiplying throughout by a factor of 2 delivers the desired result. □

A recent alternate derivation for this result can be found in [38, Eq. (23)].

A double-index harmonic number series analogous to Au-Yeung's alternating quadratic series cousin is given in the next Proposition.

Proposition 4.8.

$$\begin{aligned} \sum_{n=1}^{\infty} (-1)^n \left(\frac{H_{2n}}{n} \right)^2 &= \frac{5}{2} \operatorname{Li}_4\left(\frac{1}{2}\right) + \frac{21}{8}\zeta(4) \\ &\quad - 2\mathbf{G}^2 + \frac{1}{8}\zeta(2)\log^2(2) + \frac{5}{48}\log^4(2) \\ &\quad + \pi\mathbf{G}\log(2) + \frac{35}{16}\zeta(3)\log(2) + 2\pi \operatorname{Im} \operatorname{Li}_3(1-i). \end{aligned}$$

Here Im *denotes the imaginary part.*

P r o o f. Applying the result given in Lemma 4.4 on the series to the left we have

$$\sum_{n=1}^{\infty} (-1)^n \left(\frac{H_{2n}}{n} \right)^2 = 4 \sum_{n=1}^{\infty} (-1)^n \frac{H_{2n}^2}{(2n)^2} = 4 \operatorname{Re} \sum_{n=1}^{\infty} i^n \frac{H_n^2}{n^2} .$$

From the generating function given in Theorem 2.3, substituting $z = i$ into (11) gives

$$\begin{aligned}\sum_{n=1}^{\infty} i^n \frac{H_n^2}{n^2} &= \operatorname{Li}_4(i) + \frac{1}{2}\operatorname{Li}_2^2(i) - 2\operatorname{Li}_4(1-i) - \operatorname{Li}_2(1-i)\log^2(1-i)\\ &\quad + 2\operatorname{Li}_3(1-i)\log(1-i)\\ &\quad - \frac{1}{3}\log(i)\log^3(1-i) + 2\zeta(4),\end{aligned}$$

or

$$\begin{aligned}\sum_{n=1}^{\infty} i^n \frac{H_n^2}{n^2} &= \frac{653}{256}\zeta(4) - \frac{1}{2}\mathbf{G}^2 - 2\operatorname{Li}_4(1-i) + \frac{3}{32}\zeta(2)\log^2(2) + \frac{1}{4}\pi\mathbf{G}\log(2)\\ &\quad + 2\operatorname{Li}_3(1-i)\log(1-i) + i\beta(4) - \frac{i}{2}\mathbf{G}\zeta(2) + \frac{i}{4}\mathbf{G}\log^2(2) \qquad (24)\\ &\quad + \frac{i\pi}{24}\log^3(2) + \frac{3\pi i}{32}\zeta(2)\log(2),\end{aligned}$$

after substituting for the values (33), (34), (36), (37), and (42) given in the Appendix. Taking the real part of (24), the value for $\operatorname{Re}\operatorname{Li}_4(1-i)$ given in (46) is needed and we are left with the term $\operatorname{Re}[\operatorname{Li}_3(1-i)\log(1-i)]$ to deal with. As

$$\begin{aligned}\operatorname{Re}[\operatorname{Li}_3(1-i)\log(1-i)] &= \operatorname{Re}\operatorname{Li}_3(1-i)\cdot\operatorname{Re}\log(1-i)\\ &\quad - \operatorname{Im}\operatorname{Li}_3(1-i)\cdot\operatorname{Im}\log(1-i)\\ &= \frac{35}{128}\zeta(3)\log(2) + \frac{3}{32}\zeta(2)\log^2(2)\\ &\quad + \frac{\pi}{4}\operatorname{Im}\operatorname{Li}_3(1-i),\end{aligned}$$

where the values (35) and (44) quoted in the Appendix have been used, one finds

$$\begin{aligned}\operatorname{Re}\sum_{n=1}^{\infty} i^n \frac{H_n^2}{n^2} &= \frac{5}{8}\operatorname{Li}_4\left(\frac{1}{2}\right) + \frac{21}{32}\zeta(4) - \frac{1}{2}\mathbf{G}^2 + \frac{1}{32}\zeta(2)\log^2(2)\\ &\quad + \frac{5}{192}\log^4(2) + \frac{\pi}{4}\mathbf{G}\log(2) + \frac{35}{64}\zeta(3)\log(2)\\ &\quad + \frac{\pi}{2}\operatorname{Im}\operatorname{Li}_3(1-i).\end{aligned}$$

The desired result for the sum follows on multiplication throughout by a factor of 4. □

Remark 3. The constant $\operatorname{Im}\operatorname{Li}_3(1-i)$ needed here and the constant $\operatorname{Im}\operatorname{Li}_4(1-i)$ that we will have a need for shortly, involving the imaginary part of the trilogarithm and tetralogarithm, are believed to be in the simplest form. While currently not known, it seems highly unlikely either of these constants is reducible to more fundamental constants such as those in terms of π, $\log(2)$, $\zeta(3)$, Catalan's constant, or in the case of the latter constant, $\operatorname{Li}_4\left(\frac{1}{2}\right)$.

Proposition 4.9.

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_{2n}^2}{2n+1} = \frac{5}{16}\pi\zeta(2) + \frac{1}{2}\mathbf{G}\log(2) + \frac{3}{16}\pi\log^2(2) + 2\,\mathrm{Im}\,\mathrm{Li}_3(1-i).$$

P r o o f. Applying the result given in Lemma 4.1 to the series on the left we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_{2n}^2}{2n+1} = \mathrm{Re} \sum_{n=1}^{\infty} i^n \frac{H_n^2}{n+1},$$

or

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_{2n}^2}{2n+1} = \mathrm{Im} \sum_{n=2}^{\infty} i^n \frac{H_{n-1}^2}{n},$$

after a shift in the index of $n \mapsto n-1$ has been made. Applying the nth harmonic number recurrence relation of $H_{n-1} = H_n - \frac{1}{n}$, one obtains

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_{2n}^2}{2n+1} = \mathrm{Im}\left[\sum_{n=1}^{\infty} i^n \frac{H_n^2}{n} - 2\sum_{n=1}^{\infty} i^n \frac{H_n}{n^2} + \sum_{n=1}^{\infty} \frac{i^n}{n^3}\right]. \tag{25}$$

The first of the sums appearing in (25) is given in (23) while the third corresponds to $\mathrm{Li}_3(i)$ whose value is given by (33). The second of the sums can be found by substituting $z = i$ into the generating function given in Lemma 4.5. Doing so yields

$$\begin{aligned}\sum_{n=1}^{\infty} i^n \frac{H_n}{n^2} &= \mathrm{Li}_3(i) - \mathrm{Li}_3(1-i) + \mathrm{Li}_2(1-i)\log(1-i)\\ &\quad + \frac{1}{2}\log(i)\log^2(1-i) + \zeta(3),\end{aligned}$$

or

$$\begin{aligned}\sum_{n=1}^{\infty} i^n \frac{H_n}{n^2} &= \frac{29}{32}\zeta(3) + \frac{3}{16}\zeta(2)\log(2) - \frac{1}{4}\pi\mathbf{G} - \mathrm{Li}_3(1-i)\\ &\quad - \frac{i\pi}{16}\log^2(2) - \frac{i}{2}\mathbf{G}\log(2),\end{aligned}$$

after having substituted for the values given in (33), (35), (36), and (42). Combining the three results into (25), after taking their imaginary part, the desired result follows. □

A slightly different approach used to find this sum to the one given above can be found in [33].

Proposition 4.10.

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_{2n}^2}{(2n+1)^2} = \frac{35}{128}\pi\zeta(3) - 2\beta(4) - \frac{1}{2}\mathbf{G}\zeta(2) + \frac{1}{4}\mathbf{G}\log^2(2) + \frac{1}{24}\pi\log^3(2) + \log(2)\operatorname{Im}\mathrm{Li}_3(1-i) - 2\operatorname{Im}\mathrm{Li}_4(1-i),$$

where $\beta(s)$ is the Dirichlet beta function $\sum_{n=0}^{\infty}(-1)^n/(2n+1)^s$ of order s, $\operatorname{Re}(s) \geqslant 1$.

Proof. Applying the result given in Lemma 4.4 to the series on the left we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_{2n}^2}{(2n+1)^2} = \operatorname{Re}\sum_{n=1}^{\infty} i^n \frac{H_n^2}{(n+1)^2},$$

or

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_{2n}^2}{(2n+1)^2} = \operatorname{Im}\sum_{n=2}^{\infty} i^n \frac{H_{n-1}^2}{n^2},$$

after a shift in the index of $n \mapsto n-1$ has been made. Applying the nth harmonic number recurrence relation of $H_{n-1} = H_n - \frac{1}{n}$, one obtains

$$\sum_{n=1}^{\infty} \frac{(-1)^n H_{2n}^2}{(2n+1)^2} = \operatorname{Im}\left[\sum_{n=1}^{\infty} i^n \frac{H_n^2}{n^2} - 2\sum_{n=1}^{\infty} i^n \frac{H_n}{n^3} + \sum_{n=1}^{\infty} \frac{i^n}{n^4}\right]. \tag{26}$$

The third of the sums appearing in (26) corresponds to $\mathrm{Li}_4(i)$ whose value is given by (34). It imaginary part is just $\beta(4)$.

The first of the sums appearing in (26) is given in (24). After taking its imaginary part the term $\operatorname{Im}[\mathrm{Li}_3(1-i)\log(1-i)]$ remains. In dealing with this term we have

$$\begin{aligned}\operatorname{Im}[\mathrm{Li}_3(1-i)\log(1-i)] &= \operatorname{Re}\mathrm{Li}_3(1-i)\cdot\operatorname{Im}\log(1-i) \\ &\quad + \operatorname{Im}\mathrm{Li}_3(1-i)\cdot\operatorname{Re}\log(1-i) \\ &= -\frac{35}{256}\pi\zeta(3) - \frac{3}{64}\pi\zeta(2)\log(2) \\ &\quad + \frac{1}{2}\log(2)\operatorname{Im}\mathrm{Li}_3(1-i),\end{aligned}$$

where the values (35) and (44) found in the Appendix have been used. Thus

$$\begin{aligned}\operatorname{Im}\sum_{n=1}^{\infty} i^n \frac{H_n^2}{n^2} &= \log(2)\operatorname{Im}\mathrm{Li}_3(1-i) - 2\operatorname{Im}\mathrm{Li}_4(1-i) + \beta(4) \\ &\quad - \frac{35}{128}\pi\zeta(3) - \frac{1}{2}\mathbf{G}\zeta(2) + \frac{1}{4}\mathbf{G}\log^2(2) + \frac{1}{24}\pi\log^3(2).\end{aligned}$$

The second of the sums appearing in (26) can be found by substituting $z = i$ into the generating function given in Lemma 4.6. Doing so we have

$$\begin{aligned}\sum_{n=1}^{\infty} i^n \frac{H_n}{n^3} &= 2\,\mathrm{Li}_4(i) + \mathrm{Li}_4\left(\frac{1-i}{2}\right) - \mathrm{Li}_4(1-i) - \mathrm{Li}_3(i)\log(1-i)\\ &\quad + \frac{1}{24}\log^4(1-i) - \frac{1}{6}\log(i)\log^3(1-i) + \frac{1}{2}\zeta(2)\log^2(1-i)\\ &\quad + \zeta(3)\log(1-i) + \zeta(4),\end{aligned}$$

or

$$\begin{aligned}\sum_{n=1}^{\infty} i^n \frac{H_n}{n^3} &= \frac{5}{8}\,\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{195}{256}\zeta(4) - \frac{5}{32}\zeta(2)\log^2(2) + \frac{35}{64}\zeta(3)\log(2)\\ &\quad + \frac{5}{192}\log^4(2) + 2i\beta(4) - \frac{35i\pi}{128}\zeta(3),\end{aligned}$$

after substituting in the values found in (33), (34), (35), (36), (37), (38), and (45). Further reduction here has been possible since the sum between the two tetralogarithmic terms that appeared, namely

$$\mathrm{Li}_4(1-i) + \mathrm{Li}_4(1+i),$$

reduces to $2\,\mathrm{Re}\,\mathrm{Li}_4(1-i)$, a value which is known (see (46)). Its imaginary part is therefore simply $2\beta(4) - \frac{35}{128}\pi\zeta(3)$. Combining the three results found for the sums into (26), the desired result follows. □

Remark 4. Two quadratic Euler-type sums containing quadruple-index harmonic numbers immediately follow from Propositions 4.2, 4.3, 4.8, and 4.10. The first is

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{H_{4n}}{n}\right)^2 &= 13\,\mathrm{Li}_4\left(\frac{1}{2}\right) + 12\zeta(4) + \frac{91}{8}\zeta(3)\log(2) - \frac{7}{4}\zeta(2)\log^2(2)\\ &\quad + \frac{13}{24}\log^4(2) - 4\mathbf{G}^2 + 2\pi\mathbf{G}\log(2) + 4\pi\,\mathrm{Im}\,\mathrm{Li}_3(1-i),\end{aligned}$$

and is obtained on application of Lemma 4.1 to the given quadruple-index harmonic sum together with the results established in Propositions 4.2 and 4.8. The second is

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{H_{4n}}{4n+1}\right)^2 &= \frac{1}{2}\,\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{11}{64}\zeta(4) + \frac{7}{16}\zeta(3)\log(2) - \frac{1}{8}\zeta(2)\log^2(2)\\ &\quad + \frac{1}{48}\log^4(2) + \frac{35}{256}\pi\zeta(3) - \beta(4) - \frac{1}{4}\mathbf{G}\zeta(2) + \frac{1}{8}\mathbf{G}\log^2(2)\\ &\quad + \frac{1}{48}\pi\log^3(2) + \frac{1}{2}\log(2)\,\mathrm{Im}\,\mathrm{Li}_3(1-i) - \mathrm{Im}\,\mathrm{Li}_4(1-i),\end{aligned}$$

and is obtained on application of Lemma 4.1 to the given quadruple-index harmonic sum together with the results established in Propositions 4.3 and 4.10.

5. Appendix

In this Appendix, we list a number of special values that will be needed for the logarithmic function at complex arguments and the polylogarithmic functions of orders two, three, and four at both real and complex values. In all cases where complex values occur for these functions, the principal value is taken. For those containing polylogarithms we briefly indicate how these can be found.

Recall for $\operatorname{Re}(s) > 1$, $\operatorname{Li}_s(1) = \zeta(s)$. Also $\operatorname{Li}_s(-1) = (1-2^{1-s})\zeta(s)$, $\operatorname{Re}(s) > 1$, giving us

$$\operatorname{Li}_2(-1) = -\frac{1}{2}\zeta(2), \qquad \operatorname{Li}_3(-1) = -\frac{3}{4}\zeta(3), \qquad \operatorname{Li}_4(-1) = -\frac{7}{8}\zeta(4).$$

Special values for the dilogarithm and trilogarithm at $z = \frac{1}{2}$ are known [24, (1.16), p. 6; (6.12), p. 155]. They are

$$\begin{aligned} \operatorname{Li}_2\left(\frac{1}{2}\right) &= \frac{1}{2}\zeta(2) - \frac{1}{2}\log^2(2), \\ \operatorname{Li}_3\left(\frac{1}{2}\right) &= \frac{1}{6}\log^3(2) - \frac{1}{2}\zeta(2)\log(2) + \frac{7}{8}\zeta(3). \end{aligned} \tag{27}$$

From the polylogarithmic identity [24, (7.42), p. 197]

$$\operatorname{Li}_s(-z) + \operatorname{Li}_s(z) = \frac{1}{2^{s-1}}\operatorname{Li}_s(z^2),$$

we have

$$\begin{aligned} \operatorname{Li}_2\left(-\frac{1}{2}\right) &= \frac{1}{2}\operatorname{Li}_2\left(\frac{1}{4}\right) - \operatorname{Li}_2\left(\frac{1}{2}\right), \\ \operatorname{Li}_3\left(-\frac{1}{2}\right) &= \frac{1}{4}\operatorname{Li}_3\left(\frac{1}{4}\right) - \operatorname{Li}_3\left(\frac{1}{2}\right). \end{aligned} \tag{28}$$

Both these results can be further simplified using the values given in (27).

Values for $\operatorname{Li}_n(2)$ where $n = 2, 3, 4$ can be found by reference to formulae found in Lewin's book [24]. From (1.10) on page 5, (6.7) on page 154, and (7.81) on page 209, we have

$$\operatorname{Li}_2(2) = \frac{3}{2}\zeta(2) - i\pi\log(2), \tag{29}$$

$$\operatorname{Li}_3(2) = \frac{7}{8}\zeta(3) + \frac{3}{2}\zeta(2)\log(2) - \frac{1}{2}i\pi\log^2(2), \tag{30}$$

$$\operatorname{Li}_4(2) = -\operatorname{Li}_4\left(\frac{1}{2}\right) + 2\zeta(4) + \zeta(2)\log^2(2) - \frac{1}{24}\log^4(2) - \frac{1}{6}i\pi\log^3(2). \tag{31}$$

Here we have made use of the special values for the dilogarithm and trilogarithm at arguments of one-half given in (27).

For an argument equal to the imaginary unit, the polylogarithm of order s reduces as follows

$$\begin{aligned}\operatorname{Li}_s(i) &= \sum_{n=1}^{\infty} \frac{\cos\left(\frac{n\pi}{2}\right)}{n^s} + i\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right)}{n^s} = 2^{-s}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} + i\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}\\ &= -2^{-s}\eta(s) + i\beta(s).\end{aligned}$$

Here $\eta(s) = (1-2^{1-s})\zeta(s)$ is the Dirichlet eta function while $\beta(s)$ is the Dirichlet beta function. Both expressions given for the Dirichlet functions are valid for $\operatorname{Re}(s) > 1$. Special values for the Dirichlet beta function at positive integer arguments are also known [28, (3:7:1), p. 33]. Those that will be needed are

$$\beta(2) = \mathbf{G}, \quad \beta(3) = \frac{\pi^3}{32} = \frac{3\pi}{16}\zeta(2). \tag{32}$$

Here $\mathbf{G}$ denotes Catalan's constant $\sum_{n=0}^{\infty}(-1)^n/(2n+1)^2$. Thus

$$\operatorname{Li}_2(i) = -\frac{1}{8}\zeta(2) + i\mathbf{G}, \quad \operatorname{Li}_3(i) = -\frac{3}{32}\zeta(3) + \frac{3i\pi}{16}\zeta(2), \tag{33}$$

where further simplification has been achieved using (32), and

$$\operatorname{Li}_4(i) = -\frac{7}{128}\zeta(4) + i\beta(4). \tag{34}$$

Next, we give some functional values for the logarithm with complex arguments.

$$\log(i) = \frac{i\pi}{2}, \quad \log(1-i) = \frac{1}{2}\log(2) - \frac{i\pi}{4}, \tag{35}$$

allowing us to readily find

$$\log^2(1-i) = -\frac{3}{8}\zeta(2) + \frac{1}{4}\log^2(2) - \frac{i\pi}{4}\log(2), \tag{36}$$

$$\log^3(1-i) = \frac{1}{8}\log^3(2) - \frac{9}{16}\zeta(2)\log(2) + \frac{3i\pi}{32}\zeta(2) - \frac{3i\pi}{16}\log^2(2), \tag{37}$$

$$\begin{aligned}\log^4(1-i) = \frac{45}{128}\zeta(4) - \frac{9}{16}\zeta(2)\log^2(2)\frac{1}{16}\log^4(2)\\ + \frac{3i\pi}{16}\zeta(2)\log(2) - \frac{i\pi}{8}\log^3(2).\end{aligned} \tag{38}$$

Also there is a need for the value

$$\log(-1-i) = \frac{1}{2}\log(2) - \frac{3i\pi}{4},$$

allowing us to readily find

$$\log^2(-1-i) = -\frac{27}{8}\zeta(2) + \frac{1}{4}\log^2(2) - \frac{3i\pi}{4}\log(2), \tag{39}$$

$$\log^4(-1-i) = \frac{3645}{128}\zeta(4) + \frac{1}{16}\log^4(2) - \frac{81}{16}\zeta(2)\log^2(2) - \frac{3i\pi}{8}\log^3(2) + \frac{81i\pi}{16}\zeta(2)\log(2). \tag{40}$$

Finally, some functional values for the polylogarithms at complex arguments are needed. We briefly indicate how these can be found. From Euler's reflexion formula for the dilogarithm [24, (1.11), p. 5], namely

$$\operatorname{Li}_2(z) + \operatorname{Li}_2(1-z) = \zeta(2) - \log(z)\log(1-z), \tag{41}$$

setting $z = i$ leads to

$$\operatorname{Li}_2(1-i) = \frac{3}{8}\zeta(2) - i\mathbf{G} - \frac{i\pi}{4}\log(2), \tag{42}$$

and from the following known identity for the trilogarithm [24, (6.53), p. 164],

$$\operatorname{Li}_3(z) + \operatorname{Li}_3(1-z) + \operatorname{Li}_3\left(1 - \frac{1}{z}\right) = \zeta(3) + \zeta(2)\log(z) + \frac{1}{6}\log^3(z) - \frac{1}{2}\log^2(z)\log(1-z), \tag{43}$$

setting $z = i$ in (43) yields

$$\operatorname{Re}\operatorname{Li}_3(1-i) = \frac{35}{64}\zeta(3) + \frac{3}{16}\zeta(2)\log(2), \tag{44}$$

where Re denotes the real part. From the following inversion identity for the tetralogarithm [24, (7.81), p. 209]

$$\operatorname{Li}_4(-z) + \operatorname{Li}_4\left(-\frac{1}{z}\right) = -\frac{7}{4}\zeta(4) - \frac{1}{24}\log^4\left(\frac{1}{z}\right) - \frac{1}{2}\zeta(2)\log^2\left(\frac{1}{z}\right),$$

setting $z = (i-1)/2$, as $1/z = -1-i$ from the above tetralogarithm identity together with the values listed in (39) and (40) we find

$$\operatorname{Li}_4\left(\frac{1-i}{2}\right) = -\operatorname{Li}_4(1+i) + \frac{1313}{1024}\zeta(4) - \frac{1}{384}\log^4(2) + \frac{11}{128}\zeta(2)\log^2(2) + \frac{i\pi}{64}\log^3(2) + \frac{21i\pi}{128}\zeta(2)\log(2). \tag{45}$$

A more difficult value for the real part of the tetralogarithm is (see [44, p. 70])

$$\operatorname{Re}\operatorname{Li}_4(1-i) = -\frac{5}{16}\operatorname{Li}_4\left(\frac{1}{2}\right) + \frac{485}{512}\zeta(4) + \frac{1}{8}\zeta(2)\log^2(2) - \frac{5}{384}\log^4(2). \tag{46}$$

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