sciendo

TATRA MOUNTAINS
Mathematical Publications
DOI: 10.2478/tmmp-2020-0035
Tatra Mt. Math. Publ. **77** (2020), 99–108

ON SOME MULTINOMIAL SUMS RELATED TO THE FIBONACCI TYPE NUMBERS

Iwona Włoch[1] — Andrzej Włoch[2]

[1]Department of Discrete Mathematics, Rzeszow University of Technology, Rzeszow, POLAND

[2]Department of Mathematical Modeling, Rzeszow University of Technology, Rzeszow, POLAND

ABSTRACT. In this paper we investigate Fibonacci type sequences defined by kth order linear recurrence. Based on their companion matrix and its graph interpretation we determine multinomial and binomial formulas for these sequences. Moreover we present a graphical rule for calculating the words of these sequences from the Pascal's triangle.

1. Introduction

Fibonacci numbers are defined recursively by $F_n = F_{n-1}+F_{n-2}$ for $n \geq 2$ with initial conditions $F_0 = 0$, $F_1 = 1$. Using $F_{n-2} = F_n - F_{n-1}$, Fibonacci numbers can be extended to negative integers and $F_{-n} = (-1)^{n+1}F_n$. Consequently, the sequence $\{F_n\}$ has the form $\dots, -8, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, 8, \dots$ Many generalizations of Fibonacci numbers were studied, see for example the list given in [1]. In [9] k–Fibonacci numbers were defined by $g_n = g_{n-1} + g_{n-2} + \dots + g_{n-k}$ for $n \geq k \geq 2$ with $g_0 = g_1 = \dots = g_{k-2} = 0$ and $g_{k-1} = 1$. Some properties of the sequence $\{g_n\}$ were studied in [9] and next in [8]. More general case of sequence $\{g_n\}$ was studied by Kalman, see for details [5]. In [2] Er introduced a family of k sequences of generalized Fibonacci numbers in the following way.

Let $k \geq 2$, c_j, $j \in \{1, \dots, k\}$ be integers. Then for an integer $1 \leq i \leq k$ generalized Fibonacci numbers f_n^i are defined as

$$f_n^i = \sum_{j=1}^{k} c_j f_{n-j}^i \quad \text{for } n > 0 \tag{1}$$

2010 Mathematics Subject Classification: 11B37, 11B39, 11B65.
Keywords: Fibonacci numbers, recurrence, multinomial formulas.

with initial conditions $f_n^i = \begin{cases} 1 & \text{if } i = 1-n, \\ 0 & \text{otherwise,} \end{cases}$ for $1-k \leq n \leq 0$.

The number f_n^i is the nth term of the ith sequence. If $k = 2$ and $c_1 = c_2 = 1$, then $F_n = f_{n-1}^1 = f_n^2$ for $n \geq 0$.

Based on an approach taken by Kalman [5], Er used for these sequences a $k \times k$ matrix A of the form

$$A = \begin{bmatrix} c_1 & c_2 & \dots & c_k \\ 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and showed that

$$A^n = \begin{bmatrix} f_n^1 & f_n^2 & \dots & f_n^k \\ f_{n-1}^1 & f_{n-1}^2 & \dots & f_{n-1}^k \\ \vdots & \vdots & \ddots & \vdots \\ f_{n-k+1}^1 & f_{n-k+1}^2 & \dots & f_{n-k+1}^k \end{bmatrix}. \tag{2}$$

In this paper, we consider a generalization of Fibonacci numbers which relates to sequences (1).

Let $k \geq 2$, $c_i \geq 0$, $i \in \{1, \dots, k\}$ be integers such that there are at least two positive integers c_p, c_q where $p \neq q$ and $1 \leq p, q \leq k$. Generalized Fibonacci numbers are defined recursively by the kth order linear recurrence relation

$$f_n = c_1 f_{n-1} + c_2 f_{n-2} + \dots + c_k f_{n-k} \text{ for } n > 0 \tag{3}$$

with given nonnegative integers $f_{1-k}, \dots, f_{-1}, f_0$, and there is $1-k \leq j \leq 0$ such that $f_j > 0$.

For special values of k, c_i, and f_{1-i}, $i \in \{1, \dots, k\}$, the formula (3) gives the well-known classical sequences of the Fibonacci type, see the Table 1.

TABLE 1. Generalized Fibonacci numbers for special values k, c_i and f_i.

Sequence	k	c_1	c_2	c_3	-2	-1	0	Recursion
Fibonacci	2	1	1			0	1	$F_n = F_{n-1} + F_{n-2}$ for $n \geq 1$
Lucas	2	1	1			2	1	$L_n = L_{n-1} + L_{n-2}$ for $n \geq 1$
Pell	2	2	1			0	1	$P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 1$
Jacobsthal	2	1	2			0	1	$J_n = J_{n-1} + 2J_{n-2}$ for $n \geq 1$
Padovan	3	0	1	1	1	1	1	$Pv(n) = Pv(n-2) + Pv(n-3)$, $n \geq 1$
Tribonacci	3	1	1	1	0	0	1	$T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for $n \geq 1$

Apart from the recurrence relations for Fibonacci type sequences, binomial formulas were determined. We recall some of them:

1. F. E. A. Lucas [7]

$$F_{n-1} = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-i-1}{i},$$

2. H. W. Gould [10]

$$L_{n-1} = n \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} \frac{1}{n-i},$$

3. A. F. Haradam [4]

$$P_{n-1} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i+1} 2^i,$$

4. S. Falcon Santana, J. L. Diaz-Barrero [3]

$$P_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} 2^{n-2i},$$

5. E. Kilic, H. Prodinger [6]

$$P_n = \sum_{0 \le i \le j \le n} \binom{n-i}{j} \binom{j}{i},$$

6. B. Cloitre [10]

$$J_{n-1} = \sum_{i=0}^{\lceil \frac{n}{2} \rceil} 2^i \binom{n-i}{i},$$

7. E. Kilic, H. Prodinger [6]

$$T_n = \sum_{0 \le j \le i \le n} \binom{n-i}{i-j} \binom{i-j}{j}.$$

Other binomial formulas for integer sequences can be found in [10]. Inspired by these results, we looked for a general binomial formula for all sequences defined by the recurrence (3).

The purpose of this paper is to investigate the connections between sequences, matrices and directed multigraphs. This leads to the discovery of a method of deriving multinomial and binomial formulas for these sequences. Obtained formulas allow us to formulate a graphical rule for calculating generalized Fibonacci numbers from the Pascal's triangle. Consequently, we present new binomial formulas for these numbers.

2. Main results

Firstly, we give dependencies between nth generalized Fibonacci number f_n and nth term of the ith sequence.

Theorem 2.1. *Let $k \geq 2$ and $n \geq 1-k$ be integers. Then $f_n = \sum_{i=1}^{k} f_{1-i} f_n^i$.*

P r o o f. (by the induction on n) If $1-k \leq n \leq 0$, then from the definition of f_n^i we obtain

$$\sum_{i=1}^{k} f_{1-i} f_n^i = f_0 f_n^1 + f_{-1} f_n^2 + \cdots + f_n f_n^{1-n} + \cdots + f_{1-k} f_n^k = f_n f_n^{1-n} = f_n \cdot 1 = f_n.$$

Assume that $n > 0$ and let $f_n = \sum_{i=1}^{k} f_{1-i} f_n^i$. We shall show that

$$f_{n+1} = \sum_{i=1}^{k} f_{1-i} f_{n+1}^i.$$

Using the equation (3) we have that

$$\begin{aligned}
\sum_{i=1}^{k} f_{1-i} f_{n+1}^i &= f_0(c_1 f_n^1 + c_2 f_{n-1}^1 + \cdots + c_k f_{n-k+1}^1) \\
&\quad + f_{-1}(c_1 f_n^2 + c_2 f_{n-1}^2 + \cdots + c_k f_{n-k+1}^2) \\
&\quad \vdots \\
&\quad + f_{1-k}(c_1 f_n^k + c_2 f_{n-1}^k + \cdots + c_k f_{n-k+1}^k) \\
&= c_1 \sum_{i=1}^{k} f_{1-i} f_n^i + c_2 \sum_{i=1}^{k} f_{1-i} f_{n-1}^i + \cdots + c_k \sum_{i=1}^{k} f_{1-i} f_{n-k+1}^i.
\end{aligned}$$

Then from the induction's hypothesis

$$\sum_{i=1}^{k} f_{1-i} f_{n+1}^i = c_1 f_n + c_2 f_{n-1} + \cdots + c_k f_{n-k+1} = f_{n+1}$$

and by the induction's rule the Theorem is proved. □

Theorem 2.2. *Let $k \geq 2$, $n \geq 1$, $1 \leq i \leq k$ be integers. Then*

$$f_n^i = \sum_{t=0}^{k-i} \sum_{\substack{\alpha_1,\alpha_2,\ldots,\alpha_k \\ \alpha_1+2\alpha_2+\cdots+k\alpha_k = n-t-i}} c_{i+t} \cdot c_1^{\alpha_1} \cdot c_2^{\alpha_2} \cdots c_k^{\alpha_k} \binom{\alpha_1 + \cdots + \alpha_k}{\alpha_1, \ldots, \alpha_k} \quad \text{for } c_i > 0, i \in \{1, \ldots, k\}, \tag{4}$$

$$f_n^i = \sum_{\substack{0 \leq t \leq k-i \\ c_{i+t} > 0}} \sum_{\substack{\alpha_1,\alpha_2,\ldots,\alpha_k \\ \alpha_1+2\alpha_2+\cdots+k\alpha_k = n-t-i}} c_{i+t} \cdot \prod_{c_i > 0} c_i^{\alpha_i} \cdot \binom{\alpha_1 + \cdots + \alpha_k}{\alpha_1, \ldots, \alpha_k} \quad \text{for } c_i \geq 0, i \in \{1, \ldots, k\} \tag{5}$$

if $n = \sum_{j=1}^{k} \alpha_j \cdot j$, where $\alpha_j = 0$ if $c_j = 0$ or $\alpha_j \geq 0$ if $c_j > 0$.

P r o o f. The companion matrix A of the sequence $\{f_n^i\}$ can be considered as an adjacency matrix of a special multidigraph. For convenience, lets consider a multidigraph D defined by A^T presented on the Fig. 1, where c_i denotes the multiplicity of the arc $v_i v_1$.

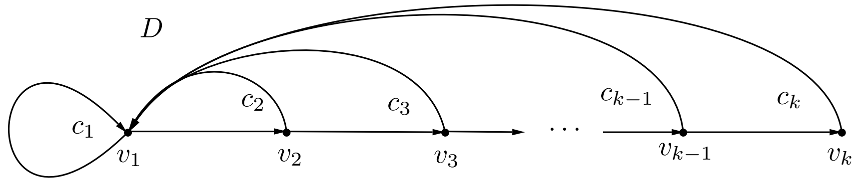

FIGURE 1. Digraph D.

Elements of the matrix $(A^n)^T = (A^T)^n$ give the number of all different paths of the length n between corresponding vertices in the digraph D. For a fixed $1 \le i \le k$, the element $a_{i,1}$ of the matrix $(A^n)^T$ is equal to the total number of different paths from v_i to v_1. By (2), it holds that $a_{i,1} = f_n^i$. Each such path consists of a path $v_i - \cdots - v_{i+t} - v_1$, $t \in \{0, \ldots k-i\}$, of the length $t+1$ and the finite sequence $\mathcal{C}$ of elementary cycles in random order. Cycles have the form $C_i = v_1 - v_2 - \cdots - v_i - v_1$ and lengths i, $i \in \{1, 2, \ldots, k\}$. Suppose that the path consists of α_i-times cycle C_i. Clearly, $\alpha_i \ge 0$, $i \in \{1, 2, \ldots, k\}$. Then the length of this path can be written as $1\alpha_1 + 2\alpha_2 + \cdots + k\alpha_k + t + 1 = n$ and by rewriting $1\alpha_1 + 2\alpha_2 + \cdots + k\alpha_k = n - t - 1$. To calculate the number of such paths observe that the arc $v_{i+t}v_1$ can be chosen in c_{i+t} ways. The remaining part of this path consists of $\alpha_1 + \alpha_2 + \cdots + \alpha_k$ cycles C_i, $i \in \{1, 2, \ldots, k\}$. The position of the cycle C_1 in the sequence $\mathcal{C}$ can be chosen in $\binom{\alpha_1+\alpha_2+\cdots+\alpha_k}{\alpha_1}$ ways, the position of the cycle C_2 in $\binom{\alpha_2+\cdots+\alpha_k}{\alpha_2}$ ways, and so on. Consequently, we have $\binom{\alpha_1+\alpha_2+\cdots+\alpha_k}{\alpha_1}\binom{\alpha_2+\cdots+\alpha_k}{\alpha_2}\cdots\binom{\alpha_k}{\alpha_k}$ possibilities that create a path of length $n-t-1$, which can be rewritten as a multinomial coefficient $\binom{\alpha_1+\cdots+\alpha_k}{\alpha_1,\ldots,\alpha_k}$.

Since the cycle C_i, $i \in \{1, 2, \ldots, k\}$, which appears α_i times in the sequence $\mathcal{C}$, can go through one of c_i multiple arcs $v_i v_1$, by the multiplicity of it the number of sequences has to be multiplied by $c_1^{\alpha_1} \cdot c_2^{\alpha_2} \cdots c_k^{\alpha_k}$.

Summing over all possible collections $\alpha_1, \alpha_2, \ldots, \alpha_k$ satisfying the equality $1\alpha_1 + 2\alpha_2 + \cdots + k\alpha_k = n - t - 1$, we obtain that the number of sequences $\mathcal{C}$ is equal to

$$\sum_{t=0}^{k-i} \sum_{\substack{\alpha_1,\alpha_2,\ldots,\alpha_k \\ \alpha_1+2\alpha_2+\cdots+k\alpha_k=n-t-1}} c_{i+t} \cdot c_1^{\alpha_1} \cdot c_2^{\alpha_2} \cdots c_k^{\alpha_k} \binom{\alpha_1+\cdots+\alpha_k}{\alpha_1,\ldots,\alpha_k} = f_n^i.$$

To prove (5), observe that if there is $1 \le i \le k$ such that $c_i = 0$ in (3), then the cycle C_i does not exist in the path from v_i to v_1 in the digraph D. Consequently $\alpha_i = 0$ and we put $c_i^{\alpha_i} = 1$ in the formula (4). Moreover we can omit terms which are equal to zero. Then the formula (5) follows. □

Using Theorem 2.1 and Theorem 2.2 we obtain

Corollary 2.3. *Let $k \geq 2$, $n \geq 1$, $1 \leq i \leq k$ be integers. Then*

$$f_n = \sum_{i=1}^{k} \sum_{t=0}^{k-i} \sum_{\substack{\alpha_1,\alpha_2,\dots,\alpha_k \\ \alpha_1+2\alpha_2+\dots+k\alpha_k=n-t-1}} f_i \cdot c_{i+t} \cdot c_1^{\alpha_1} \cdot c_2^{\alpha_2} \cdots c_k^{\alpha_k} \binom{\alpha_1+\dots+\alpha_k}{\alpha_1,\dots,\alpha_k}$$

$$\text{for } c_i > 0, i \in \{1,\dots,k\}, \quad (6)$$

$$f_n = \sum_{\substack{1\leq i\leq k \\ f_i>0}} \sum_{\substack{0\leq t\leq k-i \\ c_{i+t}>0}} \sum_{\substack{\alpha_1,\alpha_2,\dots,\alpha_k \\ \alpha_1+2\alpha_2+\dots+k\alpha_k=n-t-1}} f_i \cdot c_{i+t} \cdot \prod_{c_i>0} \left[c_i^{\alpha_i} \binom{\alpha_i+\dots+\alpha_k}{\alpha_i} \right]$$

$$\text{for } c_i \geq 0, i \in \{1,\dots,k\} \quad (7)$$

$$\text{if } n = \sum_{j=1}^{k} \alpha_j \cdot j, \text{ where } \alpha_j = 0 \text{ if } c_j = 0 \text{ or } \alpha_j \geq 0 \text{ if } c_j > 0.$$

It is well known that classical Fibonacci numbers can be calculated from the Pascal's triangle as a sum of binomials on special diagonals and this rule is described by the formula 1. The same sums can be obtained from the Pascal's triangle using staircase method, see Fig. 2. The step has a height 1 and length 1.

1	0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0
1	2	1	0	0	0	0	0	0	0	0
1	3	3	1	0	0	0	0	0	0	0
1	4	6	4	1	0	0	0	0	0	0
1	5	10	10	5	1	0	0	0	0	0
1	6	15	20	15	6	1	0	0	0	0
1	7	21	35	35	21	7	1	0	0	0
1	8	28	56	70	56	28	8	1	0	0
1	9	36	84	126	126	84	36	9	1	0
1	10	45	120	210	252	210	120	45	10	1

Figure 2. Stairs method of calculating Fibonacci numbers from the Pascal's triangle.

The Corollary 2.3 gives direct formula for an arbitrary sequence defined by kth order linear recurrence. We use this formula as a tool for determining other binomial formulas for these sequences. It also gives a possibility to discover graphical rules (called staircase method) for calculating these numbers from the Pascal's triangle.

For explanation, consider as an example Fibonacci numbers and $n = 11$. Then from (7) we have that

$$F_{11} = \binom{10}{0}+\binom{9}{0}+\binom{9}{1}+\binom{8}{1}+\binom{8}{2}+\binom{7}{2}+\binom{7}{3}+\binom{6}{3}+\binom{6}{4}+\binom{5}{4}+\binom{5}{5}.$$

These binomials marked on the Pascal's triangle give simple and elegant rule (stairs) of calculating F_{11}, see Fig. 3. We can generalize this rule by extending

1	0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0
1	2	1	0	0	0	0	0	0	0	0
1	3	3	1	0	0	0	0	0	0	0
1	4	6	4	1	0	0	0	0	0	0
1	5	10	10	5	1	0	0	0	0	0
1	6	15	20	15	6	1	0	0	0	0
1	7	21	35	35	21	7	1	0	0	0
1	8	28	56	70	56	28	8	1	0	0
1	9	36	84	126	126	84	36	9	1	0
1	10	45	120	210	252	210	120	45	10	1

FIGURE 3. Stairs method of calculating F_{11} from the Pascal's triangle.

staircases in two directions. Moving such infinite staircases downwards, we can calculate the next Fibonacci number.

Based on such graphical rule, we can give a new formula for Fibonacci numbers

$$F_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{1} \binom{n-1-i-j}{i} \tag{8}$$

We have two binomials adjacent in a row on each step of a staircase. Using the basic formula

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}. \tag{9}$$

we immediately obtain that

$$F_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}.$$

Let us consider now Padovan numbers defined by

$$Pv(n) = Pv(n-2) + Pv(n-3) \quad \text{for } n \geq 0 \quad \text{with } Pv(-2) = Pv(-1) = Pv(0) = 1.$$

According to Theorem 2.1 we have that

$$Pv(n) = \sum_{i=1}^{3} Pv(1-i)Pv^i(n) = Pv^1(n) + Pv^2(n) + Pv^3(n).$$

For example, let $n = 11$. By (7) we have that

$$\begin{aligned} Pv^1(11) &= \tbinom{4}{0} + \tbinom{4}{1} + \tbinom{3}{2} + \tbinom{3}{3} && \text{(solid line on the Fig. 4)},\\ Pv^2(11) &= \tbinom{5}{0} + \tbinom{4}{1} + \tbinom{4}{2} + \tbinom{3}{3} && \text{(dashed line)},\\ Pv^3(11) &= \tbinom{4}{0} + \tbinom{3}{1} + \tbinom{3}{2} && \text{(dotted line)}. \end{aligned}$$

These binomials marked in the Pascal's triangle form stairs. By extending them we obtain a rule for calculating $Pv^1(n)$, $Pv^2(n)$, $Pv^3(n)$, see the Fig. 4.

1	0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0
1	2	1	0	0	0	0	0	0	0	0
1	3	3	1	0	0	0	0	0	0	0
1	4	6	4	1	0	0	0	0	0	0
1	5	10	10	5	1	0	0	0	0	0
1	6	15	20	15	6	1	0	0	0	0
1	7	21	35	35	21	7	1	0	0	0
1	8	28	56	70	56	28	8	1	0	0
1	9	36	84	126	126	84	36	9	1	0
1	10	45	120	210	252	210	120	45	10	1

FIGURE 4. Calculating $Pv^1(11)$, $Pv^2(11)$, $Pv^3(11)$ by (5).

Consequently,

$$Pv^1(n) = \sum_{i=0}^{\lfloor \frac{n-4}{2} \rfloor} \binom{\lfloor \frac{n-2-i}{2} \rfloor}{i}.$$

Moreover,

$$Pv^2(n) = Pv^1(n+1) \quad \text{and} \quad Pv^3(n) = Pv^1(n-1).$$

Using (9) in each step for adjacent binomials we obtain a new pattern for calculating these numbers, see Fig. 5. This leads to

$$Pv^1(n) = \sum_{i=0}^{\lceil \frac{\lfloor \frac{n}{2} \rfloor - 1}{3} \rceil} \binom{\lfloor \frac{n}{2} \rfloor - i}{n \pmod 2 + 2i}. \tag{10}$$

Analogously,

$$Pv^2(n) = Pv^1(n+1) \quad \text{and} \quad Pv^3(n) = Pv^1(n-1).$$

1	0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0
1	2	1	0	0	0	0	0	0	0	0
1	3	3	1	0	0	0	0	0	0	0
1	4	6	4	1	0	0	0	0	0	0
1	5	10	10	5	1	0	0	0	0	0
1	6	15	20	15	6	1	0	0	0	0
1	7	21	35	35	21	7	1	0	0	0
1	8	28	56	70	56	28	8	1	0	0
1	9	36	84	126	126	84	36	9	1	0
1	10	45	120	210	252	210	120	45	10	1

FIGURE 5. Calculating $Pv^1(11)$, $Pv^2(11)$, $Pv^3(11)$ by(5) and (9).

Using twice the formula (9) we obtain

$$Pv(n) = \sum_{i=0}^{\left\lceil \frac{\lfloor \frac{n+4}{2} \rfloor - 1}{3} \right\rceil} \binom{\lfloor \frac{n+4}{2} \rfloor - i}{n \pmod 2 + 2i} \tag{11}$$

and the corresponding stairs are presented on the Fig. 6.

1	0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0
1	2	1	0	0	0	0	0	0	0	0
1	3	3	1	0	0	0	0	0	0	0
1	4	6	4	1	0	0	0	0	0	0
1	5	10	10	5	1	0	0	0	0	0
1	6	15	20	15	6	1	0	0	0	0
1	7	21	35	35	21	7	1	0	0	0
1	8	28	56	70	56	28	8	1	0	0
1	9	36	84	126	126	84	36	9	1	0
1	10	45	120	210	252	210	120	45	10	1

FIGURE 6. Calculating $Pv(11)$ by (11).

REFERENCES

[1] BEDNARZ, U.—WŁOCH, I.—WOŁOWIEC-MUSIAŁ, M.: *Total graph interpretation of the number of the Fibonacci type*, J. Appl. Math. (2015), Article ID 837917, 7 pp.

[2] ER, M. C.: *Sums of Fibonacci numbers by matrix methods*, Fibonacci Quart. **22** (1984), no. 3, 204–207.

[3] FALCÓN SANTANA, S.—DÍAZ-BARRERO, J. L.: *Some properties of sums involving Pell numbers.* Missouri J. Math. Sci. **18** (2006), no. 1, 33–40.

[4] HORADAM, A. F.: *Pell identities*, Fibonacci Quart. **9** (1971), no. 3, 245–252, 263.

[5] D. KALMAN: *Generalized Fibonacci numbers by matrix methods*, Fibonacci Quart. **20** (1982), no. 1, 73–76.

[6] KILIÇ, E.—PRODINGER, H.: *Some double binomial sums related to Fibonacci, Pell and generalized order-k Fibonacci numbers*, Rocky Mountain J. Math. **43** (2013), no. 3, 975–987. https://projecteuclid.org/euclid.rmjm/1375361983

[7] KOSHY, T.: *Fibonacci, Lucas, and Pell numbers, and Pascal's triangle*, Appl. Probab. Trust, (2011), 125–132.

[8] LEE, G.-Y.—LEE, S.-G.—KIM, J.-S.—SHIN, H. K.: *The Binet formula and representations of k-generalized Fibonacci numbers*, Fibonacci Quart. **39** (2001), no. (2) 158–164.

[9] MILES, E. P. JR.: *Generalized Fibonacci numbers and associated matrices*, Amer. Math. Monthly, **67** (1960), 745–752.

[10] SLOANE, N. J. A.: (an editor). *The On-Line Encyclopedia of Integer Sequences*, https://oeis.org (published electronically).

Received March 31, 2020

Iwona Włoch
Department of Discrete Mathematics
The Faculty of Mathematics and Applied Physics
Rzeszow University of Technology
al. Powstancow Warszawy 8
35-959 Rzeszow
POLAND
E-mail: iwloch@prz.edu.pl

Andrzej Włoch
Department of Mathematical Modeling
The Faculty of Mathematics and Applied Physics
Rzeszow University of Technology
al. Powstancow Warszawy 8
35-959 Rzeszow
POLAND
E-mail: awloch@prz.edu.pl