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ON SOME COMBINATORIAL PROPERTIES OF $P(r,n)$-PELL QUATERNIONS

Dorota Bród — Anetta Szynal-Liana

Rzeszow University of Technology, Rzeszow, POLAND

ABSTRACT. In this paper we introduce a new one parameter generalization of the Pell quaternions – $P(r,n)$-Pell quaternions. We give some of their properties, among others the Binet formula, convolution identity and the generating function.

1. Introduction

Let $\mathbb{H}$ be the set of quaternions q of the form

$$q = a + bi + cj + dk,$$

where $a, b, c, d \in \mathbb{R}$ and i, j, k are complex operators such that

$$i^2 = j^2 = k^2 = ijk = -1 \tag{1}$$

and

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j. \tag{2}$$

Quaternions were introduced by W. Hamilton in 1843 as an extension of the complex numbers. The addition, the subtraction and the multiplication of quaternions were defined analogously to the complex numbers.

Let $q_1 = a_1 + b_1 i + c_1 j + d_1 k$ and $q_2 = a_2 + b_2 i + c_2 j + d_2 k$ be two quaternions. Then the addition and the subtraction of them is defined as follows

$$q_1 \pm q_2 = (a_1 \pm a_2) + (b_1 \pm b_2)i + (c_1 \pm c_2)j + (d_1 \pm d_2)k.$$

The quaternion multiplication is also defined analogously to the complex numbers multiplication using the rules (1) and (2). Unlike the multiplication of real numbers and complex numbers the multiplication of quaternions is not commutative.

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The conjugate of a quaternion is defined by

$$\overline{q} = \overline{a + bi + cj + dk} = a - bi - cj - dk.$$

The norm of a quaternion is defined by

$$N(q) = q \cdot \overline{q} = \overline{q} \cdot q = a^2 + b^2 + c^2 + d^2.$$

For the quaternion theory see [14].

Let F_n be the nth Fibonacci number defined recursively by $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with the initial terms $F_0 = 0$, $F_1 = 1$. There are many numbers defined by the linear recurrence relations and they are also called the numbers of the Fibonacci type, for example Lucas numbers, Pell numbers, Pell-Lucas numbers, Jacobsthal numbers, Jacobsthal-Lucas numbers. These numbers have many applications in distinct areas of mathematics and also in quaternions theory.

In 1963 Horadam [6] introduced Fibonacci and Lucas quaternions. For the properties of Fibonacci quaternions see [5, 9]. In 1993 Horadam [8] mentioned the possibility of introducing Pell quaternions. Interesting results concerning Pell quaternions, Pell-Lucas quaternions have been obtained quite recently (in 2016) and can be found in [4, 13]. Jacobsthal quaternions and Jacobsthal-Lucas quaternions were introduced in 2016, see [12].

Motivated by mentioned concepts, we introduce and study the $P(r, n)$-Pell quaternions in this paper.

2. The $P(r, n)$-Pell numbers

The Pell sequence $\{P_n\}$

$$0, 1, 2, 5, 12, 29, 70, 169, 408, 985, \ldots, P_n, \ldots$$

is defined recursively by $P_n = 2P_{n-1} + P_{n-2}$, for $n \geq 2$ with $P_0 = 0$, $P_1 = 1$. The nth Pell number for $n \geq 0$ is explicitly given by the Binet-type formula

$$P_n = \frac{(1 + \sqrt{2})^n - (1 - \sqrt{2})^n}{2\sqrt{2}}.$$

Moreover, the Pell numbers are defined by the following formula

$$P_n = \sum_{k=0}^{[\frac{n-1}{2}]} \binom{n}{2k+1} 2^k.$$

Some interesting properties of the Pell numbers can be found in [7].

In the literature, there are some generalizations of the Pell numbers, see [2, 3, 10, 11]. In [1], a one parameter generalization of the Pell numbers was investigated. We recall this generalization.

Let $n \geq 0$, $r \geq 1$ be integers, the *r-Pell sequence* $\{P(r,n)\}$ is defined by the following recurrence relation

$$P(r,n) = 2^r P(r,n-1) + 2^{r-1} P(r,n-2) \quad \text{for } n \geq 2 \tag{3}$$

with initial conditions $P(r,0) = 2$, $P(r,1) = 1 + 2^{r+1}$.
It is easily seen that $P(1,n) = P_{n+2}$.

By (3) we obtain:

$$\begin{aligned} P(r,0) &= 2, \\ P(r,1) &= 1 + 2^{r+1}, \\ P(r,2) &= 2^{r+1} + 2 \cdot 4^r, \\ P(r,3) &= 2^{r-1} + 3 \cdot 4^r + 2 \cdot 8^r, \\ P(r,4) &= \tfrac{3}{2} \cdot 4^r + 4 \cdot 8^r + 2 \cdot 16^r, \\ P(r,5) &= \tfrac{1}{4} \cdot 4^r + 3 \cdot 8^r + 5 \cdot 16^r + 2 \cdot 32^r. \end{aligned} \tag{4}$$

In [1], it was proved that the r-Pell numbers can be used for counting the independent sets of special classes of graphs.

We will recall some properties of the r-Pell numbers, which will be useful in the next section.

Theorem 2.1 ([1] Binet formula)**.** *Let $n \geq 0$, $r \geq 1$ be integers. Then*

$$P(r,n) = \left(1 + \frac{2^r + 1}{\sqrt{4^r + 2^{r+1}}}\right) r_1^n + \left(1 - \frac{2^r + 1}{\sqrt{4^r + 2^{r+1}}}\right) r_2^n,$$

where

$$r_1 = \frac{1}{2}\left(2^r + \sqrt{4^r + 2^{r+1}}\right), \qquad r_2 = \frac{1}{2}\left(2^r - \sqrt{4^r + 2^{r+1}}\right).$$

Proposition 2.2 ([1])**.** *Let $n \geq 4$, $r \geq 1$ be integers. Then*

$$P(r,n) = (8^r + 4^r) P(r,n-3) + (2^{3r-1} + 2^{2r-2}) P(r,n-4).$$

Theorem 2.3 ([1])**.** *Let n, r be positive integers. Then*

$$\sum_{l=0}^{n-1} P(r,l) = \frac{P(r,n) + 2^{r-1} P(r,n-1) - 3}{3 \cdot 2^{r-1} - 1}.$$

Theorem 2.4 ([1] Cassini identity)**.** *Let n, r be positive integers. Then*

$$P(r,n+1) P(r,n-1) - P^2(r,n) = (-1)^n 2^{(r-1)(n-1)}. \tag{5}$$

Theorem 2.5 ([1] Convolution identity)**.** *Let n, m, r be integers $m \geq 2, n \geq 1$, $r \geq 1$. Then*

$$P(r,m+n) = 2^{r-1} P(r,m-1) P(r,n) + 2^{2r-2} P(r,m-2) P(r,n-1). \tag{6}$$

Theorem 2.6 ([1])**.** *The generating function of the sequence $\{P(r,n)\}$ has the following form*

$$f(t) = \frac{2 + t}{1 - 2^r t - 2^{r-1} t^2}.$$

3. The $P(r,n)$-Pell quaternions

For $n \geq 0$, we define the nth $P(r,n)$-Pell quaternion PQ_n^r as

$$PQ_n^r = P(r,n) + iP(r,n+1) + jP(r,n+2) + kP(r,n+3), \tag{7}$$

where $P(r,n)$ is the nth r-Pell number.

By (4) and (7) we obtain:

$$\begin{aligned}
PQ_0^r &= 2 + i(1+2^{r+1}) + j(2^{r+1} + 2\cdot 4^r) + k(2^{r-1} + 3\cdot 4^r + 2\cdot 8^r),\\
PQ_1^r &= 1 + 2^{r+1} + i(2^{r+1} + 2\cdot 4^r) + j(2^{r-1} + 3\cdot 4^r + 2\cdot 8^r)\\
&\quad + k(\frac{3}{2}\cdot 4^r + 4\cdot 8^r + 2\cdot 16^r),\\
PQ_2^r &= 2^{r+1} + 2\cdot 4^r + i(2^{r-1} + 3\cdot 4^r + 2\cdot 8^r) + j(\frac{3}{2}\cdot 4^r + 4\cdot 8^r + 2\cdot 16^r)\\
&\quad + k\,(\frac{1}{4}\cdot 4^r + 3\cdot 8^r + 5\cdot 16^r + 2\cdot 32^r).
\end{aligned} \tag{8}$$

Remark 1. For $r = 1$ we obtain $PQ_n^1 = PQ_{n+2}$, where PQ_n denotes the nth Pell quaternion.

By the definition of $P(r,n)$-Pell quaternions we get the following results.

Theorem 3.1. *Let $n \geq 0$, $r \geq 1$ be integers. Then*

$$2^r PQ_{n+1}^r + 2^{r-1} PQ_n^r = PQ_{n+2}^r.$$

Proof. Using (3) and (7), we have

$$\begin{aligned}
&2^r PQ_{n+1}^r + 2^{r-1} PQ_n^r\\
&= 2^r\big(P(r,n+1) + iP(r,n+2) + jP(r,n+3) + kP(r,n+4)\big)\\
&\quad + 2^{r-1}\big(P(r,n) + iP(r,n+1) + jP(r,n+2) + kP(r,n+3)\big)\\
&= P(r,n+2) + iP(r,n+3) + jP(r,n+4) + kP(r,n+5)\\
&= PQ_{n+2}^r,
\end{aligned}$$

which ends the proof. □

Remark 2. If $r = 1$ and $n \geq 0$, then we obtain the basic equality for the Pell quaternions

$$PQ_{n+2} = 2PQ_{n+1} + PQ_n.$$

Proposition 3.2. *Let $n \geq 4$, $r \geq 0$ be integers. Then*

$$PQ_n^r = (8^r + 4^r)PQ_{n-3}^r + (2^{3r-1} + 2^{2r-2})PQ_{n-4}^r.$$

P r o o f. Using Proposition 2.2, we obtain:

$$\begin{aligned}
PQ_n^r &= P(r,n) + iP(r,n+1) + jP(r,n+2) + kP(r,n+3)\\
&= (8^r+4^r)P(r,n-3) + (2^{3r-1}+2^{2r-2})P(r,n-4)\\
&\quad + i\big((8^r+4^r)P(r,n-2) + (2^{3r-1}+2^{2r-2})P(r,n-3)\big)\\
&\quad + j\big((8^r+4^r)P(r,n-1) + (2^{3r-1}+2^{2r-2})P(r,n-2)\big)\\
&\quad + k\big((8^r+4^r)P(r,n) + (2^{3r-1}+2^{2r-2})P(r,n-1)\big)\\
&= (8^r+4^r)\big(P(r,n-3) + iP(r,n-2) + jP(r,n-1) + kP(r,n)\big)\\
&\quad + (2^{3r-1}+2^{2r-2})\\
&\quad \cdot \big(P(r,n-4) + iP(r,n-3) + jP(r,n-2) + kP(r,n-1)\big).
\end{aligned}$$

Hence we have

$$PQ_n^r = (8^r+4^r)PQ_{n-3}^r + (2^{3r-1}+2^{2r-2})PQ_{n-4}^r,$$

which ends the proof. □

Remark 3. If $r=1$ and $n \geq 4$, then we obtain the well-known equality for the Pell quaternions

$$PQ_n = 12PQ_{n-3} + 5PQ_{n-4}.$$

Theorem 3.3. *Let $n \geq 0$, $r \geq 1$ be integers. Then*

$$PQ_n^r - iPQ_{n+1}^r - jPQ_{n+2}^r - kPQ_{n+3}^r = P(r,n) + P(r,n+2) + P(r,n+4) + P(r,n+6).$$

P r o o f. Using multiplication rules (1) and (2), we obtain:

$$\begin{aligned}
&PQ_n^r - iPQ_{n+1}^r - jPQ_{n+2}^r - kPQ_{n+3}^r\\
&= P(r,n) + iP(r,n+1) + jP(r,n+2) + kP(r,n+3)\\
&\quad - i\big(P(r,n+1) + iP(r,n+2) + jP(r,n+3) + kP(r,n+4)\big)\\
&\quad - j\big(P(r,n+2) + iP(r,n+3) + jP(r,n+4) + kP(r,n+5)\big)\\
&\quad - k\big(P(r,n+3) + iP(r,n+4) + jP(r,n+5) + kP(r,n+6)\big)\\
&= P(r,n) + P(r,n+2) + P(r,n+4) + P(r,n+6)\\
&\quad - (ij+ji)P(r,n+3) - (ik+ki)P(r,n+4) - (jk+kj)P(r,n+5)\\
&= P(r,n) + P(r,n+2) + P(r,n+4) + P(r,n+6).
\end{aligned}$$

□

Theorem 3.4. *Let $n \geq 0$, $r \geq 1$ be integers. Then*

(i) $PQ_n^r + \overline{PQ_n^r} = 2P(r,n)$,
(ii) $(PQ_n^r)^2 = 2P(r,n)PQ_n^r - N(PQ_n^r)$.

Proof.
(i) By the definition of the conjugate of a quaternion we get the result.
(ii) By formula (7) we get:

$$\begin{aligned}(PQ_n^r)^2 &= P^2(r,n) - P^2(r,n+1) - P^2(r,n+2) - P^2(r,n+3)\\
&\quad + 2\big(iP(r,n)P(r,n+1) + jP(r,n)P(r,n+2) + kP(r,n)P(r,n+3)\big)\\
&\quad + (ij+ji)P(r,n+1)P(r,n+2) + (ik+ki)P(r,n+1)P(r,n+3)\\
&\quad + (jk+kj)P(r,n+2)P(r,n+3)\\
&= 2\big(iP(r,n)P(r,n+1) + jP(r,n)P(r,n+2) + kP(r,n)P(r,n+3)\big)\\
&\quad + P^2(r,n) - P^2(r,n+1) - P^2(r,n+2) - P^2(r,n+3)\\
&= 2P(r,n)\big(P(r,n) + iP(r,n+1) + jP(r,n+2) + kP(r,n+3)\big)\\
&\quad - \big(P^2(r,n) + P^2(r,n+1) + P^2(r,n+2) + P^2(r,n+3)\big)\\
&= 2P(r,n)PQ_n^r - N(PQ_n^r).\end{aligned}$$ □

Now, we will give the Binet formula for the $P(r,n)$-Pell quaternions.

Theorem 3.5 (Binet formula)**.** *Let $n \geq 0$, $r \geq 1$ be integers. Then*

$$PQ_n^r = C_1 r_1^n \left(1 + ir_1 + jr_1^2 + kr_1^3\right) + C_2 r_2^n \left(1 + ir_2 + jr_2^2 + kr_2^3\right),$$

where

$$r_1 = \frac{2^r + \sqrt{4^r + 2^{r+1}}}{2}, \qquad r_2 = \frac{2^r - \sqrt{4^r + 2^{r+1}}}{2},$$

$$C_1 = 1 + \frac{2^r + 1}{\sqrt{4^r + 2^{r+1}}}, \qquad C_2 = 1 - \frac{2^r + 1}{\sqrt{4^r + 2^{r+1}}}.$$

Proof. By Theorem 2.1 we get

$$\begin{aligned}PQ_n^r &= P(r,n) + iP(r,n+1) + jP(r,n+2) + kP(r,n+3)\\
&= C_1 r_1^n + C_2 r_2^n + i\left(C_1 r_1^{n+1} + C_2 r_2^{n+1}\right)\\
&\quad + j\left(C_1 r_1^{n+2} + C_2 r_2^{n+2}\right) + k\left(C_1 r_1^{n+3} + C_2 r_2^{n+3}\right)\\
&= C_1 r_1^n \left(1 + ir_1 + jr_1^2 + kr_1^3\right) + C_2 r_2^n \left(1 + ir_2 + jr_2^2 + kr_2^3\right).\end{aligned}$$ □

Remark 4. For $r=1$ we obtain the Binet formula for the Pell quaternions (see [4])

$$PQ_n = \frac{(1+\sqrt{2})^n A - (1-\sqrt{2})^n B}{2\sqrt{2}},$$

where

$$\begin{aligned} A &= 1 + i(1+\sqrt{2}) + j(1+\sqrt{2})^2 + k(1+\sqrt{2})^3, \\ B &= 1 + i(1-\sqrt{2}) + j(1-\sqrt{2})^2 + k(1-\sqrt{2})^3. \end{aligned}$$

The next theorem presents a summation formula for the $P(r,n)$-Pell quaternions.

Theorem 3.6. *Let $n \geq 0$, $r \geq 1$ be integers. Then*

$$\begin{aligned} \sum_{l=0}^{n} PQ_l^r &= \frac{PQ_{n+1}^r + 2^{r-1} PQ_n^r - 3(1+i+j+k)}{3 \cdot 2^{r-1} - 1} \\ &\quad - \left(2i + (3+2^{r+1})j + (3 + 2\cdot 4^r + 2^{r+2})k\right). \end{aligned}$$

Proof. By the definition of the $P(r,n)$-Pell quaternions we have

$$\begin{aligned} \sum_{l=0}^{n} PQ_l^r &= PQ_0^r + PQ_1^r + \cdots + PQ_n^r \\ &= P(r,0) + iP(r,1) + jP(r,2) + kP(r,3) \\ &\quad + P(r,1) + iP(r,2) + jP(r,3) + kP(r,4) + \cdots \\ &\quad + P(r,n) + iP(r,n+1) + jP(r,n+2) + kP(r,n+3) \\ &= P(r,0) + P(r,1) + \cdots + P(r,n) \\ &\quad + i\big(P(r,1) + P(r,2) + \cdots + P(r,n+1) + P(r,0) - P(r,0)\big) \\ &\quad + j\big(P(r,2) + P(r,3) + \cdots + P(r,n+2) + P(r,0) + P(r,1) \\ &\quad - P(r,0) - P(r,1)\big) \\ &\quad + k\big(P(r,3) + P(r,4) + \cdots + P(r,n+3) + P(r,0) + P(r,1) + P(r,2) \\ &\quad - P(r,0) - P(r,1) - P(r,2)\big). \end{aligned}$$

Using Theorem 2.3, we obtain

$$\begin{aligned} \sum_{l=0}^{n} PQ_l^r &= \frac{1}{3\cdot 2^{r-1} - 1}\big[P(r,n+1) + 2^{r-1}P(r,n) - 3 \\ &\quad + i\big(P(r,n+2) + 2^{r-1}P(r,n+1) - 3\big) \\ &\quad + j\big(P(r,n+3) + 2^{r-1}P(r,n+2) - 3\big) \\ &\quad + k\big(P(r,n+4) + 2^{r-1}P(r,n+3) - 3\big)\big] \\ &\quad - P(r,0)i - \big(P(r,0) + P(r,1)\big)j \\ &\quad - \big(P(r,0) + P(r,1) + P(r,2)\big)k. \end{aligned}$$

Hence we have

$$\begin{aligned}\sum_{l=0}^{n} PQ_l^r \;=\;& \frac{1}{3\cdot 2^{r-1}-1}\big[P(r,n+1)+iP(r,n+2)+jP(r,n+3)+kP(r,n+4)\\ &+\; 2^{r-1}\big(P(r,n)+iP(r,n+1)+jP(r,n+2)+kP(r,n+3)\big)\\ &-\; 3(1+i+j+k)\big]\\ &-\; 2i-(3+2^{r+1})j-(3+2\cdot 4^r+2^{r+2})k\\ =\;& \frac{PQ_{n+1}^r+2^{r-1}PQ_n^r-3(1+i+j+k)}{3\cdot 2^{r-1}-1}\\ &-\; \big(2i+(3+2^{r+1})j+(3+2\cdot 4^r+2^{r+2})k\big).\end{aligned}$$

□

Theorem 3.7. *If $r=1$ and $n\geq 0$, then we obtain the summation formula for the Pell quaternions (see [4])*

$$\sum_{l=0}^{n} PQ_l = \frac{PQ_{n+1}+PQ_n-PQ_1+PQ_0}{2}.$$

P r o o f. For $r=1$ we have

$$\begin{aligned}\sum_{l=0}^{n} PQ_l^1 \;&=\; \frac{PQ_{n+1}^1+PQ_n^1-3(1+i+j+k)}{2}-(2i+7j+19k)\\ &=\; \frac{PQ_{n+1}^1+PQ_n^1-(1+2i+5j+12k)-(2+5i+12j+29k)}{2}\\ &=\; \frac{PQ_{n+1}^1+PQ_n^1-PQ_1-PQ_2}{2}.\end{aligned}$$

On the other hand, using $PQ_n^1=PQ_{n+2}$, we obtain

$$\begin{aligned}\sum_{l=0}^{n} PQ_l \;&=\; PQ_0+PQ_1+\cdots+PQ_n = PQ_0+PQ_1+\sum_{l=0}^{n-2} PQ_l^1\\ &=\; PQ_0+PQ_1+\frac{PQ_{n-1}^1+PQ_{n-2}^1-PQ_1-PQ_2}{2}\\ &=\; \frac{PQ_{n+1}+PQ_n-PQ_1-PQ_2+2PQ_0+2PQ_1}{2}\\ &=\; \frac{PQ_{n+1}+PQ_n-PQ_1-PQ_2+(PQ_0+2PQ_1)+PQ_0}{2}\\ &=\; \frac{PQ_{n+1}+PQ_n-PQ_1+PQ_0}{2}.\end{aligned}$$

□

Theorem 3.8 (Convolution identity)**.** *Let $m\geq 2, n\geq 1$, $r\geq 1$. Then*

$$\begin{aligned}2PQ_{m+n}^r \;&=\; 2^{r-1}PQ_{m-1}^rPQ_n^r+2^{2r-2}PQ_{m-2}^rPQ_{n-1}^r+P(r,m+n)\\ &\quad+P(r,m+n+2)+P(r,m+n+4)+P(r,m+n+6).\end{aligned}$$

P r o o f. Using (1) and (2), we have

$$\begin{aligned}
&2^{r-1}PQ^r_{m-1}PQ^r_n + 2^{2r-2}PQ^r_{m-2}PQ^r_{n-1}\\
&= 2^{r-1}\big(P(r,m-1)+iP(r,m)+jP(r,m+1)+kP(r,m+2)\big)\\
&\quad\cdot\big(P(r,n)+iP(r,n+1)+jP(r,n+2)+kP(r,n+3)\big)\\
&\quad+ 2^{2r-2}\big(P(r,m-2)+iP(r,m-1)+jP(r,m)+kP(r,m+1)\big)\\
&\quad\cdot\big(P(r,n-1)+iP(r,n)+jP(r,n+1)+kP(r,n+2)\big)\\
&= 2^{r-1}\big(P(r,m-1)P(r,n)+iP(r,m-1)P(r,n+1)\\
&\quad+ jP(r,m-1)P(r,n+2)+kP(r,m-1)P(r,n+3)\\
&\quad+ iP(r,m)P(r,n)-P(r,m)P(r,n+1)+kP(r,m)P(r,n+2)\\
&\quad- jP(r,m)P(r,n+3)+jP(r,m+1)P(r,n)-kP(r,m+1)P(r,n+1)\\
&\quad+ P(r,m+1)P(r,n+2)+iP(r,m+1)P(r,n+3)\\
&\quad+ kP(r,m+2)P(r,n)+jP(r,m+2)P(r,n+1)\\
&\quad- iP(r,m+2)P(r,n+2)-P(r,m+2)P(r,n+3)\big)\\
&\quad+ 2^{2r-2}\big(P(r,m-2)P(r,n-1)+iP(r,m-2)P(r,n)\\
&\quad+ jP(r,m-2)P(r,n+1)+kP(r,m-2)P(r,n+2)\\
&\quad+ iP(r,m-1)P(r,n-1)-P(r,m-1)P(r,n)\\
&\quad+ kP(r,m-1)P(r,n+1)-jP(r,m-1)P(r,n+2)\\
&\quad+ jP(r,m)P(r,n-1)-kP(r,m)P(r,n)-P(r,m)P(r,n+1)\\
&\quad+ iP(r,m)P(r,n+2)+kP(r,m+1)P(r,n-1)\\
&\quad+ jP(r,m+1)P(r,n)-iP(r,m+1)P(r,n+1)\\
&\quad- P(r,m+1)P(r,n+2)\big).
\end{aligned}$$

By simple calculations and formula (6) we get

$$\begin{aligned}
&2^{r-1}PQ^r_{m-1}PQ^r_n + 2^{2r-2}PQ^r_{m-2}PQ^r_{n-1}\\
&= 2^{r-1}P(r,m-1)P(r,n)+2^{2r-2}(P(r,m-2)P(r,n-1)\\
&\quad+ i\big(2^{r-1}P(r,m-1)P(r,n+1)+2^{2r-2}P(r,m-2)P(r,n)\big)\\
&\quad+ j\big(2^{r-1}P(r,m-1)P(r,n+2)+2^{2r-2}P(r,m-2)P(r,n+1)\big)\\
&\quad+ k\big(2^{r-1}P(r,m-1)P(r,n+3)+2^{2r-2}P(r,m-2)P(r,n+2)\big)\\
&\quad+ i\big(2^{r-1}P(r,m)P(r,n)+2^{2r-2}P(r,m-1)P(r,n-1)\big)\\
&\quad+ j\big(2^{r-1}P(r,m+1)P(r,n)+2^{2r-2}P(r,m)P(r,n-1)\big)\\
&\quad+ k\big(2^{r-1}P(r,m)P(r,n+2)+2^{2r-2}P(r,m-1)P(r,n+1)\big)\\
&\quad- 2^{r-1}P(r,m)P(r,n+1)-2^{2r-2}P(r,m-1)P(r,n)\\
&\quad- 2^{r-1}P(r,m+1)P(r,n+2)-2^{2r-2}P(r,m)P(r,n+1)\\
&\quad- 2^{r-1}P(r,m+2)P(r,n+3)-2^{2r-2}P(r,m+1)P(r,n+2).
\end{aligned}$$

Using formula (6) again, we obtain

$$\begin{aligned}
&2^{r-1}PQ^r_{m-1}PQ^r_n + 2^{2r-2}PQ^r_{m-2}PQ^r_{n-1}\\
&= 2\big(P(r,m+n) + iP(r,m+n+1) + jP(r,m+n+2) + kP(r,m+n+3)\big)\\
&\quad - \big(P(r,m+n) + P(r,m+n+2) + P(r,m+n+4) + P(r,m+n+6)\big)\\
&= 2PQ^r_{m+n} - P(r,m+n) - P(r,m+n+2) - P(r,m+n+4) - P(r,m+n+6).
\end{aligned}$$

Hence we get the result. □

The next theorem presents the ordinary generating functions for the $P(r,n)$--Pell quaternions.

Theorem 3.9. *The generating function for the $P(r,n)$-Pell quaternion sequence $\{PQ^r_n\}$ is*

$$g(t) = \frac{PQ^r_0 + (PQ^r_1 - 2^r PQ^r_0)t}{1 - 2^r t - 2^{r-1}t^2}.$$

P r o o f. Let $g(t) = \sum_{n=0}^{\infty} PQ^r_n t^n$. Then, by (3.1), we get

$$\begin{aligned}
(1 - 2^r t - 2^{r-1}t^2)g(t) &= (1 - 2^r t - 2^{r-1}t^2)\cdot(PQ^r_0 + PQ^r_1 t + PQ^r_2 t^2 + \cdots)\\
&= PQ^r_0 + PQ^r_1 t + PQ^r_2 t^2 + \cdots\\
&\quad - 2^r PQ^r_0 t - 2^r PQ^r_1 t^2 - 2^r PQ^r_2 t^3 - \cdots\\
&\quad - 2^{r-1}PQ^r_0 t^2 - 2^{r-1}PQ^r_1 t^3 - 2^{r-1}PQ^r_2 t^4 - \cdots
\end{aligned}$$

Because the coefficients of t^n for $n \geq 2$ are equal to zero, we have

$$(1 - 2^r t - 2^{r-1}t^2)g(t) = PQ^r_0 + (PQ^r_1 - 2^r JP^r_0)t.$$

Hence

$$g(t) = \frac{PQ^r_0 + (PQ^r_1 - 2^r PQ^r_0)t}{1 - 2^r t - 2^{r-1}t^2}.$$

By (3) we obtain

$$\begin{aligned}
PQ^r_0 &= 2 + i(1 + 2^{r+1}) + j(2^{r+1} + 2\cdot 4^r)\\
&\quad + k(2^{r-1} + 3\cdot 4^r + 2\cdot 8^r),
\end{aligned}$$

$$PQ^r_1 - 2^r PSQ^r_0 = 1 + i2^r + j(4^r + 2^{r-1}) + k(8^r + 4^r).$$

□

Remark 5. The generating function for the Pell quaternion sequence $\{PQ_n\}$ is

$$g(t) = \frac{PQ_0 + (PQ_1 - 2PQ_0)t}{1 - 2t - t^2}.$$

At the end, we give the matrix representation of the $P(r,n)$-Pell quaternions.

Theorem 3.10. *Let $n \geq 1$, $r \geq 1$ be integers. Then*

$$\begin{bmatrix} PQ^r_{n+1} & PQ^r_n \\ PQ^r_n & PQ^r_{n-1} \end{bmatrix} = \begin{bmatrix} PQ^r_2 & PQ^r_1 \\ PQ^r_1 & PQ^r_0 \end{bmatrix} \cdot \begin{bmatrix} 2^r & 1 \\ 2^{r-1} & 0 \end{bmatrix}^{n-1}. \tag{9}$$

Proof. (by induction on n) For $n = 1$ the result is obvious. Assume that the formula (9) is true for $n \geq 1$. We shall show that

$$\begin{bmatrix} PQ^r_{n+2} & PQ^r_{n+1} \\ PQ^r_{n+1} & PQ^r_n \end{bmatrix} = \begin{bmatrix} PQ^r_2 & PQ^r_1 \\ PQ^r_1 & PQ^r_0 \end{bmatrix} \cdot \begin{bmatrix} 2^r & 1 \\ 2^{r-1} & 0 \end{bmatrix}^{n}.$$

Using induction's hypothesis and formula (3.1), we have

$$\begin{aligned}
&\begin{bmatrix} PQ^r_2 & PQ^r_1 \\ PQ^r_1 & PQ^r_0 \end{bmatrix} \cdot \begin{bmatrix} 2^r & 1 \\ 2^{r-1} & 0 \end{bmatrix}^{n-1} \cdot \begin{bmatrix} 2^r & 1 \\ 2^{r-1} & 0 \end{bmatrix} \\
&= \begin{bmatrix} PQ^r_{n+1} & PQ^r_n \\ PQ^r_n & PQ^r_{n-1} \end{bmatrix} \cdot \begin{bmatrix} 2^r & 1 \\ 2^{r-1} & 0 \end{bmatrix} \\
&= \begin{bmatrix} 2^r PQ^r_{n+1} + 2^{r-1} PQ^r_n & PQ^r_{n+1} \\ 2^r PQ^r_n + 2^{r-1} PQ^r_{n-1} & PQ^r_n \end{bmatrix} \\
&= \begin{bmatrix} PQ^r_{n+2} & PQ^r_{n+1} \\ PQ^r_{n+1} & PQ^r_n \end{bmatrix},
\end{aligned}$$

which ends the proof. □

In the same way one can easily give the matrix representation of the r-Pell numbers.

Theorem 3.11. *Let $n \geq 1$, $r \geq 1$ be integers. Then*

$$\begin{bmatrix} P(r,n+1) & P(r,n) \\ P(r,n) & P(r,n-1) \end{bmatrix} = \begin{bmatrix} P(r,2) & P(r,1) \\ P(r,1) & P(r,0) \end{bmatrix} \cdot \begin{bmatrix} 2^r & 1 \\ 2^{r-1} & 0 \end{bmatrix}^{n-1}.$$

Calculating the above determinants, we obtain Cassini's identity (5) for the r-Pell numbers. We have

$$\det \begin{bmatrix} P(r,n+1) & P(r,n) \\ P(r,n) & P(r,n-1) \end{bmatrix} = P(r,n+1)P(r,n-1) - P^2(r,n),$$

$$\det \begin{bmatrix} P(r,2) & P(r,1) \\ P(r,1) & P(r,0) \end{bmatrix} = \det \begin{bmatrix} 2^{r+1} + 2 \cdot 4^r & 1 + 2^{r+1} \\ 1 + 2^{r+1} & 2 \end{bmatrix} = -1,$$

$$\det \begin{bmatrix} 2^r & 1 \\ 2^{r-1} & 0 \end{bmatrix}^{n-1} = \left(-2^{r-1}\right)^{n-1}.$$

Hence,

$$P(r,n+1)P(r,n-1) - P^2(r,n) = (-1)^n 2^{(r-1)(n-1)}.$$

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Dorota Bród
Anetta Szynal-Liana
Department of Discrete Mathematics
Faculty of Mathematics and Applied Physics
Rzeszow University of Technology
Powstańców Warszawy 12
35-959 Rzeszów
POLAND
E-mail: dorotab@prz.edu.pl
aszynal@prz.edu.pl