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ON THE DIOPHANTINE EQUATION $a^x + (a+2)^y = z^2$ WHERE $a \equiv 5 \pmod{42}$

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ABSTRACT. In this paper, we show that the Diophantine equation

$$a^x + (a+2)^y = z^2,$$

where $a \equiv 5 \pmod{42}$ and $a \in \mathbb{N}$ has no solution in non-negative integers.

1. Introduction

In 1844, Catalan [1] conjectured that $(3,2,2,3)$ is a unique solution (a,b,x,y) for the Diophantine equation

$$a^x - b^y = 1,$$

where a, b, x and y are integers such that $\min\{a,b,x,y\} > 1$.

In 2004, The Catalan's conjecture was proved by P. Mihailescu [2].

In 2011, A. Suvarmani [3] studied the Diophantine equation

$$2^x + p^y = z^2,$$

where p is a prime number such that x, y and z are non-negative integers.

In 2013, B. Sroysang [4] showed that the Diophantine equation

$$5^x + 7^y = z^2$$

has no non-negative integer solution, where x, y and z are non-negative integers.

Later that same year, he proved [5, 6] that the Diophantine equations

$$47^x + 49^y = z^2 \quad \text{and} \quad 89^x + 91^y = z^2$$

have no non-negative integer solution, where x, y and z are non-negative integers.

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In this paper, we will prove that the Diophantine equation

$$a \equiv 5 \pmod{42} \quad \text{and} \quad a \in \mathbb{N},$$

has no solution in non-negative integers.

2. Preliminaries

Lemma 2.1. *If $5^x \equiv 5 \pmod{6}$, then x is odd.*

Proof. Since $5 \equiv -1 \pmod{6}$, thus $5^{2n-1} \equiv -1 \pmod{6}$. Hence $x = 2n-1$ is odd for all $n \in \mathbb{N}$. □

Lemma 2.2. *If $a \equiv 5 \pmod{42}$ and x is odd, then*

$$a^x \equiv 3 \pmod{7} \quad or \quad a^x \equiv 5 \pmod{7} \quad or \quad a^x \equiv 6 \pmod{7}.$$

Proof. Since $a \equiv 5 \pmod{42}$, thus $a \equiv 5 \pmod{7}$ and x is odd, we can write

$$x = 6n-5 \quad \text{or} \quad x = 6n-3 \quad \text{or} \quad x = 6n-1, \quad \text{where} \quad n \in \mathbb{N}.$$

If $x = 6n-1$ and since $a \equiv -2 \pmod{7}$, thus $a^{6n-6} \equiv 1 \pmod{7}$, hence $a^{6n-1} \equiv a^5 \equiv (-2)^5 \equiv 3 \pmod{7}$.

If $x = 6n-5$ and since $a \equiv -2 \pmod{7}$, thus $a^{6n-6} \equiv 1 \pmod{7}$, hence $a^{6n-5} \equiv a \equiv 5 \pmod{7}$.

If $x = 6n-3$ and since $a \equiv -2 \pmod{7}$, thus $a^{6n-6} \equiv 1 \pmod{7}$, hence $a^{6n-3} \equiv a^3 \equiv -8 \equiv 6 \pmod{7}$.

□

3. Main Results

Theorem 3.1. *The Diophantine equation $a^x + (a+2)^y = z^2$, where $a \equiv 5 \pmod{42}$ and $a \in \mathbb{N}$ has no non-negative integer solution such that x, y and z are non-negative integers.*

Proof. Case I:

If $y = 0$, then we have the Diophantine equation $a^x + 1 = z^2$.

If $x = 0$, then $z^2 = 2$ which is impossible.

If $x = 1$, then $a + 1 = z^2 \equiv 6 \pmod{42}$ which is impossible.

If $x > 1$, then the Diophantine equation $z^2 - a^x = 1$ has no non-negative solution by the Catalan's conjecture.

Case II:

If $x = 0$, then we have the Diophantine equation $(a+2)^y + 1 = z^2$.

If $y = 0$, then $z^2 = 2$ which is impossible.

If $y = 1$, then $a + 3 = z^2 \equiv 8 \pmod{42}$ which is impossible.

If $y > 1$, then the Diophantine equation $z^2 - (a+2)^y = 1$ has no non-negative solution by the Catalan's conjecture.

Case III:

If $x \geq 1$ and $y \geq 1$, then z is even. Thus

$$z^2 \equiv 0 \pmod{6} \quad \text{or} \quad z^2 \equiv 4 \pmod{6}.$$

Since $a+2 \equiv 1 \pmod{6}$. It follows that $a^x + (a+2)^y = z^2$ thus $a^x + 1 \equiv z^2 \pmod{6}$. Hence

$$a^x \equiv 5 \pmod{6} \quad \text{or} \quad a^x \equiv 3 \pmod{6}.$$

Note that $a \equiv 5 \pmod{6}$ thus $a^x \equiv 5^x \pmod{6}$ but $5 \equiv -1 \pmod{6}$, then $5^x \equiv \pm 1 \pmod{6}$. So,

$$5^x \equiv 1 \pmod{6} \quad \text{or} \quad 5^x \equiv 5 \pmod{6}.$$

This implies that

$$a^x \equiv 5^x \equiv 5 \pmod{6}.$$

Thus by Lemma 2.1 x is odd. Hence by Lemma 2.2 $a^x \equiv 3, 5, 6 \pmod{7}$. Since $(a+2)^y \equiv 0 \pmod{7}$ it follows that

$$z^2 \equiv \quad 3, 5, 6 \pmod{7}.$$

This is a contradiction since

$$z^2 \equiv \; 0, 1, 2, 4 \pmod{7}.$$

□

Corollary 3.2. *The Diophantine equation*

$$a^x + (a+2)^y = w^4, \quad \textit{where} \quad a \equiv 5 \pmod{42} \quad \textit{and} \quad a \in \mathbb{N}$$

has no non-negative integer solution such that x, y and w are non-negative integers.

Proof. Let $z = w^2$, then $a^x + (a+2)^y = z^2$. By theorem 3.1 the Diophantine equation $a^x + (a+2)^y = z^2$ has no non-negative integer solution. This implies that the Diophantine equation $a^x + (a+2)^y = w^4$ has no non-negative integer solution. □

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