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ON THE COMBINATORIAL PROPERTIES OF BIHYPERBOLIC BALANCING NUMBERS

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ABSTRACT. In this paper, we introduce bihyperbolic balancing and Lucas-balancing numbers. We give some of their properties, among others the Binet formula, Catalan, Cassini, d'Ocagne identities and the generating function.

1. Introduction

A hyperbolic number is defined as $z = x + \mathbf{h}y$, where $x, y \in \mathbb{R}$ and $\mathbf{h}$ is a unipotent (hyperbolic) imaginary unit such that $\mathbf{h}^2 = 1$ and $\mathbf{h} \neq \pm 1$. Hence the set of hyperbolic numbers is defined as

$$\mathbb{H} = \{z\colon z = x + \mathbf{h}y, x, y \in \mathbb{R}, \mathbf{h}^2 = 1\}.$$

Hyperbolic imaginary units were introduced in 1848 by James Cockle (see [4]–[7]). Let $\mathbb{H}_2$ be the set of bihyperbolic numbers defined by

$$\zeta = x_0 + j_1x_1 + j_2x_2 + j_3x_3,$$

where $x_0, x_1, x_2, x_3 \in \mathbb{R}$ and $j_1, j_2, j_3 \notin \mathbb{R}$ are operators such that

$$j_1^2 = j_2^2 = j_3^2 = 1 \tag{1}$$

and

$$j_1j_2 = j_2j_1 = j_3, \quad j_1j_3 = j_3j_1 = j_2, \quad j_2j_3 = j_3j_2 = j_1. \tag{2}$$

The addition and multiplication on $\mathbb{H}_2$ are commutative and associative. Also, the multiplication is distributive over addition. Hence $(\mathbb{H}_2, +, \cdot)$ is a commutative ring. Hyperbolic numbers and bihyperbolic numbers are well-studied in the literature, see [1, 11, 12]. In this paper we introduce bihyperbolic balancing numbers and bihyperbolic Lucas-balancing numbers.

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The sequence of balancing numbers, denoted by $\{B_n\}$, was introduced by Behera and Panda in [2]. A balancing number n with balancer r is the solution of the Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r).$$

For example, 6 is a balancing number with balancer 2, 35 is a balancing number with balancer 14. In [2] it was proved that the balancing numbers satisfy the following recurrence relation

$$B_n = 6B_{n-1} - B_{n-2} \quad \text{for } n \geq 2 \tag{3}$$

with initial conditions $B_0 = 0$, $B_1 = 1$. The sequence of balancing numbers is given by Binet formula

$$B_n = \frac{r_1^n - r_2^n}{r_1 - r_2}, \tag{4}$$

where r_1, r_2 are the roots of the characteristic equation $r^2 - 6r + 1 = 0$, associated with the recurrence relation (3), i.e.,

$$r_1 = 3 + 2\sqrt{2}, \quad r_2 = 3 - 2\sqrt{2}. \tag{5}$$

Note that

$$r_1 + r_2 = 6, \tag{6}$$

$$r_1 - r_2 = 4\sqrt{2}, \tag{7}$$

$$r_1 r_2 = 1. \tag{8}$$

It is well known that n is a balancing number if and only if n^2 is a triangular number, i.e., $8n^2 + 1$ is a perfect square, see [2]. In [8], the author introduced Lucas-balancing numbers, defined as follows: if B_n is a balancing number, the number $C_n = \sqrt{8B_n^2 + 1}$ is called a Lucas-balancing number. The sequence $\{C_n\}$ of Lucas-balancing numbers is also defined by the recurrence relation

$$C_n = 6C_{n-1} - C_{n-2} \quad \text{for } n \geq 2 \tag{9}$$

with initial terms $C_0 = 1$, $C_1 = 3$. The Binet formula for the Lucas-balancing numbers has the following form

$$C_n = \frac{1}{2}\left(r_1^n + r_2^n\right), \tag{10}$$

where r_1, r_2 are given by (5).

Table 1.

n	0	1	2	3	4	5	6	7
B_n	0	1	6	35	204	1189	6930	40391
C_n	1	3	17	99	577	3363	19601	114243

The Table 1 includes initial terms of the balancing and Lucas-balancing numbers for $n = 0, 1, \ldots, 7$. Many interesting properties of the balancing and Lucas-balancing numbers are presented in [2, 3, 8–10, 13]. We give some of them.

$$\begin{aligned}
B_{n-r}B_{n+r} - B_n^2 &= -B_r^2 && \text{(Catalan identity)},\\
C_{n-r}C_{n+r} - C_n^2 &= C_r^2 - 1 && \text{(Catalan identity)},\\
B_{n-1}B_{n+1} - B_n^2 &= -1 && \text{(Cassini identity)},\\
C_{n-1}C_{n+1} - C_n^2 &= 8 && \text{(Cassini identity)},\\
B_mB_{n+1} - B_{m+1}B_n &= B_{m-n} && \text{(d'Ocagne identity)},\\
C_mC_{n+1} - C_{m+1}C_n &= -8B_{m-n} && \text{(d'Ocagne identity)}.
\end{aligned}$$

In this paper we use the following identities:

$$\sum_{i=0}^{n} B_i = \frac{B_{n+1} - B_n - 1}{4}, \tag{11}$$

$$\sum_{i=0}^{n} C_i = \frac{C_{n+1} - C_n + 2}{4}, \tag{12}$$

$$3B_n - B_{n-1} = C_n\,, \tag{13}$$

$$B_{n+2} - B_{n-2} = 12C_n\,. \tag{14}$$

2. Bihyperbolic balancing numbers

In this section, we introduce bihyperbolic balancing numbers and bihyperbolic Lucas-balancing numbers.

Let $n \geq 0$ be an integer. We define the nth bihyperbolic balancing number BhB_n as

$$BhB_n = B_n + j_1B_{n+1} + j_2B_{n+2} + j_3B_{n+3}\,, \tag{15}$$

where B_n is nth balancing number.

In the same way, we define the bihyperbolic Lucas-balancing numbers BhC_n

$$BhC_n = C_n + j_1 C_{n+1} + j_2 C_{n+2} + j_3 C_{n+3}\,, \tag{16}$$

where C_n is nth Lucas-balancing number.

Using (15), (16) and Table 1, we get

$$\begin{aligned} BhB_0 &= j_1 + 6j_2 + 35j_3\,,\\ BhB_1 &= 1 + 6j_1 + 35j_2 + 204j_3\,,\\ BhB_2 &= 6 + 35j_1 + 204j_2 + 1189j_3\,,\\ BhB_3 &= 35 + 204j_1 + 1189j_2 + 6930j_3\\ &\vdots \end{aligned} \tag{17}$$

$$\begin{aligned} BhC_0 &= 1 + 3j_1 + 17j_2 + 99j_3\,,\\ BhC_1 &= 3 + 17j_1 + 99j_2 + 577j_3\,,\\ BhC_2 &= 17 + 99j_1 + 577j_2 + 3363j_3\,,\\ BhC_3 &= 99 + 577j_1 + 3363j_2 + 19601j_3\\ &\vdots \end{aligned} \tag{18}$$

By the definition of bihyperbolic balancing and Lucas-balancing numbers, we get the following recurrence relations.

Theorem 2.1. *Let $n \geq 2$ be an integer. Then*

(i) $BhB_n = 6BhB_{n-1} - BhB_{n-2}\,,$

(ii) $BhC_n = 6BhC_{n-1} - BhC_{n-2}\,,$

where BhB_0, BhB_1, BhC_0, BhC_1 are given by (17), (18), *respectively.*

Proof. (i) Using (15) and (3), we have

$$\begin{aligned} 6BhB_{n-1} - BhB_{n-2} &= 6(B_{n-1} + j_1 B_n + j_2 B_{n+1} + j_3 B_{n+2})\\ &\quad -(B_{n-2} + j_1 B_{n-1} + j_2 B_n + j_3 B_{n+1})\\ &= 6B_{n-1} - B_{n-2} + j_1(6B_n - B_{n-1})\\ &\quad +j_2(6B_{n+1} - B_n) + j_3(6B_{n+2} - B_{n+1})\\ &= B_n + j_1 B_{n+1} + j_2 B_{n+2} + j_3 B_{n+3} = BhB_n\,. \end{aligned}$$

We omit the proof of (ii). □

Theorem 2.2. *Let $n \geq 1$ be an integer. Then*

$$BhC_n = 3BhB_n - BhB_{n-1}\,.$$

P r o o f. By formulas (15) and (13), we have

$$\begin{aligned}
3BhB_n - BhB_{n-1} &= 3(B_n + j_1 B_{n+1} + j_2 B_{n+2} + j_3 B_{n+3}) \\
&\quad - B_{n-1} - j_1 B_n - j_2 B_{n+1} - j_3 B_{n+2} \\
&= 3B_n - B_{n-1} + j_1(3B_{n+1} - B_n) \\
&\quad + j_2(3B_{n+2} - B_{n+1}) \\
&\quad + j_3(3B_{n+3} - B_{n+2}) \\
&= C_n + j_1 C_{n+1} + j_2 C_{n+2} + j_3 C_{n+3} \\
&= BhC_n.
\end{aligned}$$

□

Theorem 2.3. *Let $n \geq 2$ be an integer. Then*

$$BhB_{n+2} - BhB_{n-2} = 12BhC_n\,.$$

P r o o f. Using (15), (14) and (16), we obtain

$$\begin{aligned}
BhB_{n+2} - BhB_{n-2} &= B_{n+2} + j_1 B_{n+3} + j_2 B_{n+4} + j_3 B_{n+5} \\
&\quad - B_{n-2} - j_1 B_{n-1} - j_2 B_n - j_3 B_{n+1} \\
&= B_{n+2} - B_{n-2} + j_1(B_{n+3} - B_{n-1}) \\
&\quad + j_2(B_{n+4} - B_n) + j_3(B_{n+5} - B_{n+1}) \\
&= 12(C_n + j_1 C_{n+1} + j_2 C_{n+2} + j_3 C_{n+3}) \\
&= 12BhC_n.
\end{aligned}$$

□

Theorem 2.4. *Let $n \geq 0$, $r \geq 1$ be integers. Then*

$$\begin{aligned}
&BhB_n - j_1 BhB_{n+1} - j_2 BhB_{n+2} - j_3 BhB_{n+3} = \\
&\qquad B_n - B_{n+2} - B_{n+4} + B_{n+6} - 2j_3 BhB_{n+3}\,.
\end{aligned}$$

P r o o f. Using multiplication rules (1) and (2), we obtain

$$\begin{aligned}
&BhB_n - j_1 BhB_{n+1} - j_2 BhB_{n+2} - j_3 BhB_{n+3} \\
&= B_n + j_1 B_{n+1} + j_2 B_{n+2} + j_3 B_{n+3} \\
&\quad - j_1(B_{n+1} + j_1 B_{n+2} + j_2 B_{n+3} + j_3 B_{n+4}) \\
&\quad - j_2(B_{n+2} + j_1 B_{n+3} + j_2 B_{n+4} + j_3 B_{n+5}) \\
&\quad - j_3(B_{n+3} + j_1 B_{n+4} + j_2 B_{n+5} + j_3 B_{n+6}) \\
&= B_n + j_1 B_{n+1} + j_2 B_{n+2} + j_3 B_{n+3} \\
&\quad - j_1 B_{n+1} - B_{n+2} - j_3 B_{n+3} - j_2 B_{n+4} \\
&\quad - j_2 B_{n+2} - j_3 B_{n+3} - B_{n+4} - j_1 B_{n+5} \\
&\quad - j_3 B_{n+3} - j_2 B_{n+4} - j_1 B_{n+5} - B_{n+6} \\
&= B_n - B_{n+2} - B_{n+4} - B_{n+6} \\
&\quad - 2\big(j_1 B_{n+5} + j_2 B_{n+4} + j_3 B_{n+3}\big) \\
&= B_n - B_{n+2} - B_{n+4} + B_{n+6} \\
&\quad - 2j_3(B_{n+3} + j_1 B_{n+4} + j_2 B_{n+5} + j_3 B_{n+6}) \\
&= B_n - B_{n+2} - B_{n+4} + B_{n+6} - 2j_3 BhB_{n+3}\,,
\end{aligned}$$

which ends the proof. □

The next theorem gives the Binet formulas for the bihyperbolic balancing and Lucas-balancing numbers.

Theorem 2.5. *Let $n \geq 0$ be an integer. Then*

$$BhB_n = \frac{\hat{r_1} r_1^n - \hat{r_2} r_2^n}{r_1 - r_2}, \tag{19}$$

$$BhC_n = \frac{\hat{r_1} r_1^n + \hat{r_2} r_2^n}{2}, \tag{20}$$

where r_1 and r_2 are given by (5) *and*

$$\hat{r_1} = 1 + j_1 r_1 + j_2 r_1^2 + j_3 r_1^3\,, \tag{21}$$

$$\hat{r_2} = 1 + j_1 r_2 + j_2 r_2^2 + j_3 r_2^3\,. \tag{22}$$

P r o o f. By formula (4), we get

$$\begin{aligned} BhB_n &= B_n + j_1 B_{n+1} + j_2 B_{n+2} + j_3 B_{n+3} \\ &= \tfrac{1}{r_1 - r_2}[r_1^n - r_2^n + j_1(r_1^{n+1} - r_2^{n+1}) \\ &\quad + j_2(r_1^{n+2} - r_2^{n+2}) + j_3(r_1^{n+3} - r_2^{n+3})] \\ &= \tfrac{1}{r_1 - r_2}[r_1^n\left(1 + j_1 r_1 + j_2 r_1^2 + j_3 r_1^3\right) \\ &\quad - r_2^n\left(1 + j_1 r_2 + j_2 r_2^2 + j_3 r_2^3\right)] \\ &= \tfrac{\hat{r_1} r_1^n - \hat{r_2} r_2^n}{r_1 - r_2}. \end{aligned}$$

We omit the proof of (20). □

By simple calculations, we obtain

$$\begin{aligned} \hat{r_1}\hat{r_2} = \hat{r_2}\hat{r_1} = 1 &+ r_1 r_2 + (r_1 r_2)^2 \\ &+ (r_1 r_2)^3 + j_1(r_1 + r_2)\big(1 + (r_1 r_2)^2\big) \\ &+ j_2(r_1^2 + r_2^2)(1 + r_1 r_2) \\ &+ j_3\big(r_1^3 + r_2^3 + r_1 r_2(r_1 + r_2)\big). \end{aligned}$$

Using formulas (6)–(8) we get

$$r_1^2 + r_2^2 = (r_1 + r_2)^2 - 2r_1 r_2 = 34\,,$$

$$r_1^3 + r_2^3 = (r_1 + r_2)^3 - 3r_1 r_2(r_1 + r_2) = 198\,.$$

Thus

$$\begin{aligned} \hat{r_1}\hat{r_2} = \hat{r_2}\hat{r_1} &= 4 + 12j_1 + 68j_2 + 204j_3 \\ &= 4BhC_0 - 192j_3. \end{aligned} \tag{23}$$

3. Some identities for the bihyperbolic balancing and Lucas-balancing numbers

In this section, we give some identities such as Catalan, Cassini and d'Ocagne identities for the bihyperbolic balancing and Lucas-balancing numbers. These identities are easily proved using the Binet formulas (19) and (20).

Theorem 3.1 (Catalan identities). *Let $n \geq 0, r \geq 0$ be integers such that $n \geq r$. Then*

(i) $BhB_{n-r}BhB_{n+r} - (BhB_n)^2 = -\hat{r_1}\hat{r_2}\big(\frac{r_1^r - r_2^r}{r_1 - r_2}\big)^2$,

(ii) $BhC_{n-r}BhC_{n+r} - (BhC_n)^2 = \frac{1}{4}\hat{r_1}\hat{r_2}(r_2^r - r_1^r)^2$.

P r o o f.
(i) By formula (19), we get

$$\begin{aligned}&BhB_{n-r}BhB_{n+r}-(BhB_n)^2\\&=\tfrac{(\hat{r_1}r_1^{n-r}-\hat{r_2}r_2^{n-r})(\hat{r_1}r_1^{n+r}-\hat{r_2}r_2^{n+r})-(\hat{r_1}r_1^{n}-\hat{r_2}r_2^{n})^2}{(r_1-r_2)^2}\\&=\tfrac{1}{(r_1-r_2)^2}\left[\hat{r_1}\hat{r_2}(r_1^nr_2^n)\big(1-(\tfrac{r_2}{r_1})^r\big)+\hat{r_2}\hat{r_1}(r_1^nr_2^n)\big(1-(\tfrac{r_1}{r_2})^r\big)\right].\end{aligned}$$

Using the fact that $r_1r_2=1$, we have

$$\begin{aligned}&BhB_{n-r}BhB_{n+r}-(BhB_n)^2\\&=\tfrac{1}{(r_1-r_2)^2}\big(\hat{r_1}\hat{r_2}\tfrac{r_1^r-r_2^r}{r_1^r}+\hat{r_2}\hat{r_1}\tfrac{r_2^r-r_1^r}{r_2^r}\big)=\tfrac{\hat{r_1}\hat{r_2}(r_1^r-r_2^r)r_2^r-\hat{r_2}\hat{r_1}(r_1^r-r_2^r)r_1^r}{(r_1-r_2)^2}\\&=\tfrac{(r_1^r-r_2^r)(\hat{r_1}\hat{r_2}r_2^r-\hat{r_2}\hat{r_1}r_1^r)}{(r_1-r_2)^2}=-\hat{r_1}\hat{r_2}\big(\tfrac{r_1^r-r_2^r}{r_1-r_2}\big)^2.\end{aligned}$$

(ii) In the same way, using formula (20), one can easily prove the result. □

By Theorem 3.1, for $r=1$ we get the Cassini identities for the bihyperbolic balancing and Lucas-balancing numbers.

Corollary 3.2. *For $n\geq 1$*

(i) $BhB_{n-1}BhB_{n+1}-(BhB_n)^2=-\hat{r_1}\hat{r_2}$,

(ii) $BhC_{n-1}BhC_{n+1}-(BhC_n)^2=8\hat{r_1}\hat{r_2}$.

Theorem 3.3 (d'Ocagne identities)**.** *Let $m\geq 0, n\geq 0$ be integers such that $m\geq n$. Then*

(i) $BhB_mBhB_{n+1}-BhB_{m+1}BhB_n=\frac{\hat{r_1}\hat{r_2}(r_1^{m-n}-r_2^{m-n})}{r_1-r_2}$,

(ii) $BhC_mBhC_{n+1}-BhC_{m+1}BhC_n=\frac{1}{4}\hat{r_1}\hat{r_2}(r_1^{m-n}-r_2^{m-n})(r_2-r_1)$.

P r o o f.
(i) By formulas (19) and (4), we get

$$\begin{aligned}&BhB_mBhB_{n+1}-BhB_{m+1}BhB_n\\&=\tfrac{(\hat{r_1}r_1^{m}-\hat{r_2}r_2^{m})(\hat{r_1}r_1^{n+1}-\hat{r_2}r_2^{n+1})-(\hat{r_1}r_1^{m+1}-\hat{r_2}r_2^{m+1})(\hat{r_1}r_1^{n}-\hat{r_2}r_2^{n})}{(r_1-r_2)^2}\\&=\tfrac{1}{(r_1-r_2)^2}[\hat{r_1}\hat{r_2}(r_1^{m+1}r_2^n-r_1^mr_2^{n+1})+\hat{r_2}\hat{r_1}(r_1^nr_2^{m+1}-r_1^{n+1}r_2^m)]\\&=\tfrac{1}{(r_1-r_2)^2}(r_1r_2)^n[\hat{r_1}\hat{r_2}(r_1-r_2)r_1^{m-n}+\hat{r_2}\hat{r_1}(r_2-r_1)r_2^{m-n}]\\&=\tfrac{\hat{r_1}\hat{r_2}r_1^{m-n}-\hat{r_2}\hat{r_1}r_2^{m-n}}{r_1-r_2}=\tfrac{\hat{r_1}\hat{r_2}(r_1^{m-n}-r_2^{m-n})}{r_1-r_2}.\end{aligned}$$

(ii) By formula (20), we have

$$\begin{aligned}&BhC_mBhC_{n+1}-BhC_{m+1}BhC_n\\&=\tfrac{1}{4}(r_1r_2)^n\big(\hat{r_1}\hat{r_2}(r_2-r_1)r_1^{m-n}+\hat{r_2}\hat{r_1}(r_1-r_2)r_2^{m-n}\big)\\&=\tfrac{1}{4}\hat{r_1}\hat{r_2}(r_1^{m-n}-r_2^{m-n})(r_2-r_1).\end{aligned}$$

□

Theorem 3.4. *Let $m \geq 0, n \geq 0$ be integers. Then*

(i) $BhB_m BhC_n - BhC_m BhB_n = \frac{\hat{r_1}\hat{r_2}(r_1^{m-n} - r_2^{m-n})}{r_1 - r_2}$,

(ii) $BhB_m BhC_n + BhC_m BhB_n = \frac{(\hat{r_1})^2 r_1^{m+n} - (\hat{r_2})^2 r_2^{m+n}}{r_1 - r_2}$.

Proof.

(i)

$$\begin{aligned}
&BhB_m BhC_n - BhC_m BhB_n \\
&= \tfrac{1}{2(r_1-r_2)}[(\hat{r_1}r_1^m - \hat{r_2}r_2^m)(\hat{r_1}r_1^n + \hat{r_2}r_2^n) - (\hat{r_1}r_1^m + \hat{r_2}r_2^m)(\hat{r_1}r_1^n - \hat{r_2}r_2^n)] \\
&= \tfrac{1}{2(r_1-r_2)}[2\hat{r_1}\hat{r_2}r_1^m r_2^n - 2\hat{r_2}\hat{r_1}r_1^n r_2^m] \\
&= \tfrac{1}{r_1-r_2}[(r_1 r_2)^n(\hat{r_1}\hat{r_2}r_1^{m-n} - \hat{r_2}\hat{r_1}r_2^{m-n})] \\
&= \tfrac{\hat{r_1}\hat{r_2}(r_1^{m-n} - r_2^{m-n})}{r_1-r_2}.
\end{aligned}$$

(ii)

$$\begin{aligned}
&BhB_m BhC_n + BhC_m BhB_n \\
&= \tfrac{1}{2(r_1-r_2)}[(\hat{r_1}r_1^m - \hat{r_2}r_2^m)(\hat{r_1}r_1^n + \hat{r_2}r_2^n) + (\hat{r_1}r_1^m + \hat{r_2}r_2^m)(\hat{r_1}r_1^n - \hat{r_2}r_2^n)] \\
&= \tfrac{1}{2(r_1-r_2)}[2(\hat{r_1})^2 r_1^{m+n} - 2(\hat{r_2})^2 r_2^{m+n}] \\
&= \tfrac{(\hat{r_1})^2 r_1^{m+n} - (\hat{r_2})^2 r_2^{m+n}}{r_1-r_2}.
\end{aligned}$$

□

Theorem 3.5. *Let $n \geq 0$, $r \geq 0$, $s \geq 0$ be integers. Then*

$$BhB_{n+r} BhC_{n+s} - BhB_{n+s} BhC_{n+r} = \frac{\hat{r_1}\hat{r_2}(r_1^r r_2^s - r_1^s r_2^r)}{r_1 - r_2}.$$

Proof. Using formulas (19) and (20), we have

$$\begin{aligned}
&BhB_{n+r} BhC_{n+s} - BhB_{n+s} BhC_{n+r} \\
&= \tfrac{1}{2(r_1-r_2)}[(\hat{r_1}r_1^{n+r} - \hat{r_2}r_2^{n+r})(\hat{r_1}r_1^{n+s} + \hat{r_2}r_2^{n+s}) \\
&\quad - (\hat{r_1}r_1^{n+s} - \hat{r_2}r_2^{n+s})(\hat{r_1}r_1^{n+r} + \hat{r_2}r_2^{n+r})] \\
&= \tfrac{1}{2(r_1-r_2)}[\hat{r_1}\hat{r_2}r_1^{n+r}r_2^{n+s} - \hat{r_1}\hat{r_2}r_1^{n+s}r_2^{n+r} \\
&\quad + \hat{r_2}\hat{r_1}r_1^{n+r}r_2^{n+s} - \hat{r_2}\hat{r_1}r_1^{n+s}r_2^{n+r}] \\
&= \tfrac{1}{2(r_1-r_2)}[(r_1 r_2)^n(\hat{r_1}\hat{r_2} + \hat{r_2}\hat{r_1})(r_1^r r_2^s - r_1^s r_2^r)] \\
&= \tfrac{\hat{r_1}\hat{r_2}(r_1^r r_2^s - r_1^s r_2^r)}{r_1-r_2}.
\end{aligned}$$

□

Theorem 3.6. *Let $n \geq 0$ be an integer. Then*

$$(BhC_n)^2 - 8(BhB_n)^2 = \hat{r_1}\hat{r_2}\,.$$

P r o o f. By (20) and (19) we get

$$\begin{aligned}(BhC_n)^2 - 8(BhB_n)^2 &= \left(\tfrac{\hat{r_1}r_1^n + \hat{r_2}r_2^n}{2}\right)^2 - 8\left(\tfrac{\hat{r_1}r_1^n - \hat{r_2}r_2^n}{4\sqrt{2}}\right)^2 \\ &= \tfrac{1}{4}[(r_1r_2)^n 2(\hat{r_1}\hat{r_2} + \hat{r_2}\hat{r_1})] = \hat{r_1}\hat{r_2}\,.\end{aligned}$$

□

The following theorem gives a summation formula for the bihyperbolic balancing numbers.

Theorem 3.7. *Let $n \geq 0$ be an integer. Then*

$$\sum_{i=0}^{n} BhB_i = \frac{1}{4}\big(BhB_{n+1} - BhB_n - (1 + j_1 + j_2 + j_3)\big) - (j_2 + 7j_3)\,.$$

P r o o f. By formula (11), we have

$$\begin{aligned}\sum_{i=0}^{n} BhB_i &= \sum_{i=0}^{n}(B_i + j_1B_{i+1} + j_2B_{i+2} + j_3B_{i+3}) \\ &= \sum_{i=0}^{n} B_i + j_1\sum_{i=0}^{n} B_{i+1} + j_2\sum_{i=0}^{n} B_{i+2} + j_3\sum_{i=0}^{n} B_{i+3} \\ &= \tfrac{1}{4}(B_{n+1} - B_n - 1) + j_1\big(\tfrac{1}{4}(B_{n+2} - B_{n+1} - 1) - B_0\big) \\ &\quad + j_2\big(\tfrac{1}{4}(B_{n+3} - B_{n+2} - 1) - B_0 - B_1\big) \\ &\quad + j_3\big(\tfrac{1}{4}(B_{n+4} - B_{n+3} - 1) - B_0 - B_1 - B_2\big) \\ &= \tfrac{1}{4}\big(B_{n+1} + j_1B_{n+2} + j_2B_{n+3} + j_3B_{n+4} \\ &\quad - (B_n + j_1B_{n+1} + j_2B_{n+2} + j_3B_{n+3}) - (1 + j_1 + j_2 + j_3)\big) \\ &\quad - j_1B_0 - j_2(B_0 + B_1) - j_3(B_0 + B_1 + B_2)\,.\end{aligned}$$

Hence

$$\sum_{i=0}^{n} BhB_i = \frac{1}{4}\big(BhB_{n+1} - BhB_n - (1 + j_1 + j_2 + j_3)\big) - (j_2 + 7j_3).$$

□

In the same way, using formula (12), we can prove the next theorem.

Theorem 3.8. *Let $n \geq 0$ be an integer. Then*

$$\sum_{i=0}^{n} BhC_i = \frac{1}{4}\big(BhC_{n+1} - BhC_n + 2(1 + j_1 + j_2 + j_3)\big) - (j_1 + 4j_2 + 21j_3)\,.$$

4. Generating functions

In [2] and [14], the following theorems were proved.

Theorem 4.1. [2] *The generating function of the balancing sequence* $\{B_n\}$ *has the following form*

$$G(B_n;x) = \frac{x}{1-6x+x^2}\,.$$

Theorem 4.2. [14] *The generating function of the Lucas-balancing sequence* $\{C_n\}$ *has the following form*

$$G(C_n;x) = \frac{1-3x}{1-6x+x^2}\,.$$

Now we will give the generating functions for the bihyperbolic balancing and the Lucas-balancing numbers.

Theorem 4.3. *The generating function of the bihyperbolic balancing sequence has the following form*

$$g(x) = \frac{x+j_1+(6-x)j_2+(35-6x)j_3}{1-6x+x^2}\,.$$

P r o o f. Let

$$g(x) = BhB_0 + BhB_1x + BhB_2x^2 + \cdots + BhB_nx^n + \cdots$$

be the generating function of the bihyperbolic balancing sequence. Hence we have

$$\begin{aligned} 6xg(x) &= 6BhB_0x + 6BhB_1x^2 + 6BhB_2x^3 + \cdots + 6BhB_{n-1}x^n + \cdots\,, \\ x^2g(x) &= BhB_0x^2 + BhB_1x^3 + BhB_2x^4 + \cdots + BhB_{n-2}x^n + \cdots \end{aligned}$$

Using the recurrence $BhB_n = 6BhB_{n-1} - BhB_{n-2}$ and fact that the coefficients of x^n for $n \geq 2$ are equal to zero, we get

$$g(x) - 6xg(x) + x^2g(x) = BhB_0 + (BhB_1 - 6BhB_0)x\,.$$

Thus

$$g(x) = \frac{BhB_0 + (BhB_1 - 6BhB_0)x}{1-6x+x^2}\,.$$

Using (17), we obtain

$$g(x) = \frac{j_1+6j_2+35j_3+(1-j_2-6j_3)x}{1-6x+x^2} = \frac{x+j_1+(6-x)j_2+(35-6x)j_3}{1-6x+x^2}\,.$$

□

In the same way we can prove the next theorem.

Theorem 4.4. *The generating function of the bihyperbolic Lucas-balancing sequence has the following form*

$$f(x) = \frac{1-3x+(3-x)j_1+(17-3x)j_2+(99-17x)j_3}{1-6x+x^2}\,.$$

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