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GENERALIZATIONS OF CERTAIN REPRESENTATIONS OF REAL NUMBERS

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ABSTRACT. In the present paper, real number representations that are generalizations of classical positive and alternating representations of numbers, are introduced and investigated. The main metric relation, properties of cylinder sets are proven. The theorem on the representation of real numbers from a certain interval is formulated.

One of the peculiarities of the research presented in this paper, is introducing numeral systems with mixed bases (i.e., with bases containing positive and negative numbers). In 2016, an idea of a corresponding analytic representation of numbers was presented in [14, Serbenyuk, S.: *On some generalizations of real numbers representations*, arXiv:1602.07929v1]. These investigations were presented in [15, Serbenyuk, S.: *Generalizations of certain representations of real numbers*, arXiv:1801.10540] in January 2018.

Also, an idea of such investigations was presented by the author of this paper at the conference in 2015 (see [9, Serbenyuk, S.: *Quasi-nega-$\widetilde{Q}$-representation as a generalization of a representation of real numbers by certain sign-variable series*, `https://www.researchgate.net/publication/303255656`]).

Investigations of various numeral systems (encodings of real numbers) are useful for the development of the metric and probability theory of real numbers, for the development of the theory of coding information, for modelling and studying various mathematical objects possessing a pathological (complicated) structure such as: non-monotonic, singular, and nowhere differentiable functions, dynamical systems with chaotic trajectories, certain types of random variables, etc. (see, for example, [1], [3]–[8], [12, 13, 16, 17, 19–21]).

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Let $\mathbb{N}_B$ be a fixed subset of positive integers, $\rho_0 = 0$,

$$\rho_n = \begin{cases} 1 & \text{if } n \in \mathbb{N}_B, \\ 2 & \text{if } n \notin \mathbb{N}_B, \end{cases}$$

and

$$\tilde{Q}'_{\mathbb{N}_B} = \begin{pmatrix} (-1)^{\rho_1} q_{0,1} & (-1)^{\rho_1+\rho_2} q_{0,2} & \dots & (-1)^{\rho_{n-1}+\rho_n} q_{0,n} & \dots \\ (-1)^{\rho_1} q_{1,1} & (-1)^{\rho_1+\rho_2} q_{1,2} & \dots & (-1)^{\rho_{n-1}+\rho_n} q_{1,n} & \dots \\ \vdots & \vdots & \ddots & \vdots & \dots \\ (-1)^{\rho_1} q_{m_1-1,1} & (-1)^{\rho_1+\rho_2} q_{m_2-2,2} & \dots & (-1)^{\rho_{n-1}+\rho_n} q_{m_n,n} & \dots \\ (-1)^{\rho_1} q_{m_1,1} & (-1)^{\rho_1+\rho_2} q_{m_2-1,2} & \dots & & \dots \\ & (-1)^{\rho_1+\rho_2} q_{m_2,2} & \dots & & \dots \end{pmatrix}$$

be a fixed matrix. Here $n = 1, 2, \dots$, $m_n \in \mathbb{N} \cup \{0, +\infty\}$ ($\mathbb{N}$ is the set of all positive integers), and numbers $q_{i,n}$ ($i = \overline{0, m_n}$, i.e., $i \in \{0, 1, 2, \dots, m_n\}$) satisfy the following system of conditions:

$$\begin{cases} 1^\circ. & q_{i,n} > 0 & \text{for all} & i = \overline{0, m_n} \quad \text{and} \quad n = 1, 2, \dots, \\ 2^\circ. & \sum_{i=0}^{m_n} q_{i,n} = 1 & \text{for any} & n \in \mathbb{N}, \\ 3^\circ. & \prod_{j=1}^{\infty} q_{i_j,j} = 0 & \text{for all sequences} & (i_n). \end{cases}$$

That is, the matrix $\tilde{Q}'_{\mathbb{N}_B}$ is a matrix whose n-th column (for an arbitrary positive integer n) is finite or infinite and the column sums to $(-1)^{\rho_{n-1}+\rho_n}$, i.e., each column sums to 1 or -1. Different columns can contain different numbers of elements (i.e., there exists a finite number of elements if the condition $m_n < \infty$ holds and there exists an infinite number of elements if $m_n = \infty$ is true).

In the present paper, a representation of real numbers by the following series is introduced:

$$(-1)^{\rho_1} a_{i_1,1} + \sum_{n=2}^{\infty} \left((-1)^{\rho_n} a_{i_n,n} \prod_{j=1}^{n-1} q_{i_j,j} \right), \tag{1}$$

where

$$a_{i_n,n} = \begin{cases} \sum_{i=0}^{i_n-1} q_{i,n} & \text{if} \quad i_n \neq 0, \\ 0 & \text{if} \quad i_n = 0. \end{cases}$$

We say that an expansion of a number x in series (1) is *the sign-variable $\tilde{Q}'_{\mathbb{N}_B}$-expansion of* x and write

$$\Delta^{\tilde{Q}'_{\mathbb{N}_B}}_{i_1 i_2 \cdots i_n \cdots}$$

The last notation is called *the $\tilde{Q}'_{\mathbb{N}_B}$-representation of* x or *the quasi-nega-$\tilde{Q}$-representation of* x. The set $\{0, 1, \dots, m_n\}$ is an alphabet of the symbol i_n in the $\Delta^{\tilde{Q}'_{\mathbb{N}_B}}_{i_1 i_2 \dots i_n \dots}$

We can model expansion (1) in the following way. Suppose that $\rho_0 = 0$ and consider the matrix $\tilde{Q}'_{\mathbb{N}_B}$. Then

$$\sum_{i=0}^{i_1-1} \left((-1)^{\rho_1} q_{i,1}\right) + \sum_{n=2}^{\infty} \left(\left(\sum_{i=0}^{i_n-1} \left((-1)^{\rho_{n-1}+\rho_n} q_{i,n}\right) \right) \left(\prod_{j=1}^{n-1} (-1)^{\rho_{j-1}+\rho_j} q_{i_j,j} \right) \right)$$

$$= (-1)^{\rho_1} a_{i_1,1} + \sum_{n=2}^{\infty} \left(\left(\sum_{i=0}^{i_n-1} (-1)^{\rho_{n-1}+\rho_n} q_{i,n} \right) \right.$$

$$\left. \times \left((-1)^{\rho_0+\rho_1} q_{i_1,1} \cdots (-1)^{\rho_{n-3}+\rho_{n-2}} q_{i_{n-2},n-2} (-1)^{\rho_{n-2}+\rho_{n-1}} q_{i_{n-1},n-1} \right) \right)$$

$$= (-1)^{\rho_1} a_{i_1,1} + \sum_{n=2}^{\infty} \left((-1)^{2\rho_1+2\rho_2+\cdots+2\rho_{n-1}+\rho_n} a_{i_n,n} \prod_{j=1}^{n-1} q_{i_j,j} \right)$$

$$= (-1)^{\rho_1} a_{i_1,1} + \sum_{n=2}^{\infty} \left((-1)^{\rho_n} a_{i_n,n} \prod_{j=1}^{n-1} q_{i_j,j} \right).$$

It is obvious that the last series is absolutely convergent and its sum belongs to $[a', a'']$, where

$$a' = \inf \Delta^{\tilde{Q}'_{\mathbb{N}_B}}_{i_1 i_2 \ldots i_n \ldots} = (-1)^{\rho_1} \check{a}_{i_1,1} + \sum_{n=2}^{\infty} \left((-1)^{\rho_n} \check{a}_{i_n,n} \prod_{j=1}^{n-1} \check{q}_{i_j,j} \right)$$

$$= (-1)^{\rho_1} \check{a}_{i_1,1} - \sum_{1<n\in\mathbb{N}_B} \left(a_{m_n,n} \prod_{j=1}^{n-1} \check{q}_{i_j,j} \right),$$

$$a'' = \sup \Delta^{\tilde{Q}'_{\mathbb{N}_B}}_{i_1 i_2 \ldots i_n \ldots} = (-1)^{\rho_1} \hat{a}_{i_1,1} + \sum_{n=2}^{\infty} \left((-1)^{\rho_n} \hat{a}_{i_n,n} \prod_{j=1}^{n-1} \hat{q}_{i_j,j} \right)$$

$$= (-1)^{\rho_1} \hat{a}_{i_1,1} + \sum_{1<n\notin\mathbb{N}_B} \left(a_{m_n,n} \prod_{j=1}^{n-1} \hat{q}_{i_j,j} \right),$$

where

$$\check{a}_{i_n,n} = \begin{cases} a_{m_n,n} & \text{if } n \in \mathbb{N}_B, \\ a_{0,n} & \text{if } n \notin \mathbb{N}_B, \end{cases} \qquad \check{q}_{i_n,n} = \begin{cases} q_{m_n,n} & \text{if } n \in \mathbb{N}_B, \\ q_{0,n} & \text{if } n \notin \mathbb{N}_B, \end{cases}$$

and

$$\hat{a}_{i_n,n} = \begin{cases} a_{0,n} & \text{if } n \in \mathbb{N}_B, \\ a_{m_n,n} & \text{if } n \notin \mathbb{N}_B, \end{cases} \qquad \hat{q}_{i_n,n} = \begin{cases} q_{0,n} & \text{if } n \in \mathbb{N}_B, \\ q_{m_n,n} & \text{if } n \notin \mathbb{N}_B. \end{cases}$$

Let $s>1$ be a fixed positive integer and $Q\equiv(q_n)$ be a fixed sequence of positive integers such that $q_n>1$. Denote by $[x]$ the integer part of x. It is easy to see that the $\tilde{Q}^{'}_{\mathbb{N}_B}$-representation is:

- the nega-$\tilde{Q}$-representation [10] whenever $\mathbb{N}_B$ is the set of all odd numbers:

$$x=\Delta^{(-\tilde{Q})}_{i_1i_2...i_n...}\equiv -a_{i_1,1}+\sum^{\infty}_{n=2}\left((-1)^n a_{i_n,n}\prod^{n-1}_{j=1}q_{i_j,j}\right);$$

- the representation ([18]) by a positive series introduced by G. Cantor in [2] whenever $\mathbb{N}_B=\varnothing$ and $q_{i,n}=\frac{1}{q_n}$ for any $n\in\mathbb{N}$, $i=\overline{0,q_n-1}$:

$$x=\Delta^{Q}_{i_1i_2...i_n...}\equiv\frac{i_1}{q_1}+\frac{i_2}{q_1q_2}+\dots+\frac{i_n}{q_1q_2\dots q_n}+\dots,$$

i.e., in this case, the matrix $\tilde{Q}^{'}_{\mathbb{N}_B}$ is the following

$$\left(\begin{array}{ccccc}
\frac{1}{q_1} & \frac{1}{q_2} & \dots & \frac{1}{q_n} & \dots\\
\frac{1}{q_1} & \frac{1}{q_2} & \dots & \frac{1}{q_n} & \dots\\
\vdots & \vdots & \ddots & \vdots & \dots\\
\frac{1}{q_1} & \frac{1}{q_2} & \dots & \frac{1}{q_n} & \dots\\
\frac{1}{q_1} & \frac{1}{q_2} & \dots & & \dots\\
 & \frac{1}{q_2} & \dots & & \dots
\end{array}\right);$$

- the representation by alternating [11] Cantor series whenever $\mathbb{N}_B$ is the set of all odd numbers and $q_{i,n}=\frac{1}{q_n}$ for any $n\in\mathbb{N}$, $i=\overline{0,q_n-1}$:

$$x=\Delta^{-Q}_{i_1i_2...i_n...}\equiv\frac{i_1}{-q_1}+\frac{i_2}{(-q_1)(-q_2)}+\dots+\frac{i_n}{(-q_1)(-q_2)\dots(-q_n)}+\dots,$$

where

$$\tilde{Q}^{'}_{\mathbb{N}_B}=\left(\begin{array}{ccccc}
-\frac{1}{q_1} & -\frac{1}{q_2} & \dots & -\frac{1}{q_n} & \dots\\
-\frac{1}{q_1} & -\frac{1}{q_2} & \dots & -\frac{1}{q_n} & \dots\\
\vdots & \vdots & \ddots & \vdots & \dots\\
-\frac{1}{q_1} & -\frac{1}{q_2} & \dots & -\frac{1}{q_n} & \dots\\
-\frac{1}{q_1} & -\frac{1}{q_2} & \dots & & \dots\\
 & -\frac{1}{q_2} & \dots & & \dots
\end{array}\right);$$

- the s-adic representation (see [8]) whenever $\mathbb{N}_B=\varnothing$ and $q_{i,n}=\frac{1}{s}$ for all $n\in\mathbb{N}$, $i=\overline{0,s-1}$:

$$x=\Delta^{s}_{i_1i_2...i_n...}\equiv\sum^{\infty}_{n=1}\frac{i_n}{s^n},$$

i.e., in this case, our matrix is the following

$$\tilde{Q}'_{\mathbb{N}_B} = \begin{pmatrix} \frac{1}{s} & \frac{1}{s} & \dots & \frac{1}{s} & \dots \\ \frac{1}{s} & \frac{1}{s} & \dots & \frac{1}{s} & \dots \\ \vdots & \vdots & \ddots & \vdots & \dots \\ \frac{1}{s} & \frac{1}{s} & \dots & \frac{1}{s} & \dots \end{pmatrix};$$

- the *nega−s−adic−representation* ([6]) whenever $\mathbb{N}_B$ is the set of all odd numbers and $q_{i,n} = \frac{1}{s}$ for all $n \in \mathbb{N}$, $i = \overline{0, s-1}$:

$$x = \Delta^{-s}_{i_1 i_2 \dots i_n \dots} \equiv \sum_{n=1}^{\infty} \frac{(-1)^n i_n}{s^n},$$

where

$$\tilde{Q}'_{\mathbb{N}_B} = \begin{pmatrix} -\frac{1}{s} & -\frac{1}{s} & \dots & -\frac{1}{s} & \dots \\ -\frac{1}{s} & -\frac{1}{s} & \dots & -\frac{1}{s} & \dots \\ \vdots & \vdots & \ddots & \vdots & \dots \\ -\frac{1}{s} & -\frac{1}{s} & \dots & -\frac{1}{s} & \dots \end{pmatrix}.$$

Suppose that

$$q_{i,n} = \frac{1}{s} \quad \text{for all } n \in \mathbb{N}, \quad i = \overline{0, s-1}, \quad \text{and} \quad s > 1 \quad \text{is a fixed positive integer.}$$

Then we get the following expansion of real numbers

$$x = \Delta^{(\pm s, \mathbb{N}_B)}_{\alpha_1 \alpha_2 \dots \alpha_n \dots} \equiv \sum_{n=1}^{\infty} \frac{(-1)^{\rho_n} \alpha_n}{s^n}.$$

The last representation described by the author of the present paper in [14], is a generalization of the classical s-adic and nega-s-adic representations.

Remark 1. The term "nega" is used in this paper since certain expansions of real numbers are numeral systems with a negative base.

Consider other examples of the matrix $\tilde{Q}'_{\mathbb{N}_B}$.

- Suppose that the condition $m_n = \infty$ holds for all $n \in \mathbb{N}$ and the set $\mathbb{N}_B$ is the set

$$\{n : n = 4k+1, n = 4k+2\}, \quad \text{where} \quad k = 1, 2, 3, \dots$$

Let us consider the matrix

$$
\tilde{Q}'_{\mathbb{N}_B}=\begin{pmatrix}
-\frac{1}{2} & \frac{2}{3} & -\frac{3}{4} & \cdots & (-1)^n\frac{n}{n+1} & \cdots\\
-\frac{1}{4} & \frac{2}{9} & -\frac{3}{16} & \cdots & (-1)^n\frac{n}{(n+1)^2} & \cdots\\
-\frac{1}{8} & \frac{2}{27} & -\frac{3}{64} & \cdots & (-1)^n\frac{n}{(n+1)^3} & \cdots\\
\vdots & \vdots & \ddots & \vdots & \cdots & \cdots\\
-\frac{1}{2^i} & \frac{2}{3^i} & -\frac{3}{4^i} & \cdots & (-1)^n\frac{n}{(n+1)^i} & \cdots\\
\vdots & \vdots & \ddots & \vdots & \cdots & \cdots
\end{pmatrix}.
$$

- Suppose the condition $\mathbb{N}_B=\varnothing$ holds and the following conditions are true

$$
q_{i,n}=\begin{cases}
\frac{1}{2} & \text{whenever} \quad n=1, \quad \text{and} \quad m_1=1,\\
\frac{1}{n} & \text{whenever} \quad n \text{ is odd, } n>1, \quad \text{and} \quad m_n=n-1,\\
\frac{2^{i-1}(n+1)}{(n+3)^i} & \text{whenever} \quad n \text{ is even, } m_n=\infty, \quad \text{and} \quad i=1,2,\dots
\end{cases}
$$

Then we obtain the following matrix

$$
\tilde{Q}'_{\mathbb{N}_B}=\begin{pmatrix}
\frac{1}{2} & \frac{3}{5} & \frac{1}{3} & \frac{5}{7} & \cdots\\
\frac{1}{2} & \frac{6}{25} & \frac{1}{3} & \frac{10}{49} & \cdots\\
 & \frac{12}{125} & \frac{1}{3} & \frac{20}{7^3} & \cdots\\
 & \vdots & & \vdots & \vdots\\
 & \frac{3\cdot 2^{i-1}}{5^i} & & \frac{5\cdot 2^{i-1}}{7^i} & \cdots\\
 & \vdots & & \vdots & \vdots
\end{pmatrix}.
$$

Let us consider the $\tilde{Q}'_{\mathbb{N}_B}$-expansion.

Theorem 0.1. *For an arbitrary number $x\in\left[a'_0,a''_0\right]$ there exists a sequence (i_n), $i_n\in N^0_{m_n}\equiv\{0,1,\dots,m_n\}$, such that*

$$
x=(-1)^{\rho_1}a_{i_1,1}+\sum_{n=2}^{\infty}\left((-1)^{\rho_n}a_{i_n,n}\prod_{j=1}^{n-1}q_{i_j,j}\right)
$$

whenever for all $n\in\mathbb{N}$ the following system of conditions holds:

$$\begin{cases} q_{i_n,n}\left(1-\sum\limits_{n<t\in\mathbb{N}_B}\left(a_{m_t,t}\prod\limits_{r=n+1}^{t-1}\check{q}_{i_r,r}\right)\right)\le q_{i_n+1,n}\left(\sum\limits_{n<t\notin\mathbb{N}_B}\left(a_{m_t,t}\prod\limits_{r=n+1}^{t-1}\hat{q}_{i_r,r}\right)\right) \\ \hfill \textit{for } n\in\mathbb{N}_B, \\ q_{i_n,n}\left(1-\sum\limits_{n<t\notin\mathbb{N}_B}\left(a_{m_t,t}\prod\limits_{r=n+1}^{t-1}\hat{q}_{i_r,r}\right)\right)\le q_{i_n+1,n}\left(\sum\limits_{n<t\in\mathbb{N}_B}\left(a_{m_t,t}\prod\limits_{r=n+1}^{t-1}\check{q}_{i_r,r}\right)\right) \\ \hfill \textit{for } n\notin\mathbb{N}_B. \end{cases}$$

P r o o f. Suppose that

$$\tilde{a}_{m_n,n}=\begin{cases} a_{m_n,n} & \text{if } n\in\mathbb{N}_B,\\ a_{0,n} & \text{if } n\notin\mathbb{N}_B,\end{cases}\qquad \tilde{q}_{m_n,n}=\begin{cases} q_{m_n,n} & \text{if } n\in\mathbb{N}_B,\\ q_{0,n} & \text{if } n\notin\mathbb{N}_B,\end{cases}$$

$$\tilde{a}_{0,n}=\begin{cases} a_{0,n} & \text{if } n\in\mathbb{N}_B,\\ a_{m_n,n} & \text{if } n\notin\mathbb{N}_B,\end{cases}\qquad \tilde{q}_{0,n}=\begin{cases} q_{0,n} & \text{if } n\in\mathbb{N}_B,\\ q_{m_n,n} & \text{if } n\notin\mathbb{N}_B.\end{cases}$$

Definition 0.2. *A cylinder* $\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2\dots c_n}$ of rank n with base $c_1\dots c_n$ is a set $\{x : x=\Delta^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1\dots c_n i_{n+1}i_{n+2}\dots}\}$, where $c_1,c_2,\dots,c_n$ are fixed.

The following result is an auxiliary statement for proving the last-mentioned theorem.

Lemma 0.3. *A cylinder* $\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2\dots c_n}$ *is a closed interval.*

P r o o f. Let us prove this lemma. Let

$$n\notin\mathbb{N}_B \quad\text{and}\quad x\in\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2\dots c_n},$$

i.e.,

$$x=(-1)^{\rho_1}a_{c_1,1}+\sum_{k=2}^{n}\left((-1)^{\rho_k}a_{c_k,k}\prod_{j=1}^{k-1}q_{c_j,j}\right)+ \left(\prod_{j=1}^{n}q_{c_j,j}\right)\left((-1)^{\rho_{n+1}}a_{i_{n+1},n+1}+\sum_{t=n+2}^{\infty}\left((-1)^{\rho_t}a_{i_t,t}\prod_{r=n+1}^{t-1}q_{i_r,r}\right)\right).$$

Then

$$\begin{aligned} x^{'} = &(-1)^{\rho_1}a_{c_1,1}+\sum_{k=2}^{n}\left((-1)^{\rho_k}a_{c_k,k}\prod_{j=1}^{k-1}q_{c_j,j}\right)\\ &-\left(\prod_{j=1}^{n}q_{c_j,j}\right)\left(\tilde{a}_{m_{n+1},n+1}+\sum_{t=n+2}^{\infty}\left(\tilde{a}_{m_t,t}\prod_{r=n+1}^{t-1}\tilde{q}_{m_r,r}\right)\right)\\ \le\ x\ \le &(-1)^{\rho_1}a_{c_1,1}+\sum_{k=2}^{n}\left((-1)^{\rho_k}a_{c_k,k}\prod_{j=1}^{k-1}q_{c_j,j}\right)\\ &+\left(\prod_{j=1}^{n}q_{c_j,j}\right)\left(\tilde{a}_{0,n+1}+\sum_{t=n+2}^{\infty}\left(\tilde{a}_{0,t}\prod_{r=n+1}^{t-1}\tilde{q}_{0,r}\right)\right)\\ =\ x^{''}. \end{aligned}$$

Hence

$$x\in\left[x^{'},x^{''}\right]\quad\text{and}\quad\left[x^{'},x^{''}\right]\supseteq\Lambda^{\tilde{Q}^{'}_{\mathbb{N}_B}}_{c_1c_2\ldots c_n}.$$

Since the equalities

$$\begin{aligned} x^{'} =& (-1)^{\rho_1}a_{c_1,1}+\sum_{k=2}^{n}\left((-1)^{\rho_k}a_{c_k,k}\prod_{j=1}^{k-1}q_{c_j,j}\right)\\ &+\left(\prod_{j=1}^{n}q_{c_j,j}\right)\inf\left\{(-1)^{\rho_{n+1}}a_{i_{n+1},n+1}+\sum_{t=n+2}^{\infty}\left((-1)^{\rho_t}a_{i_t,t}\prod_{r=n+1}^{t-1}q_{i_r,r}\right)\right\},\\ x^{''} =& (-1)^{\rho_1}a_{c_1,1}+\sum_{k=2}^{n}\left((-1)^{\rho_k}a_{c_k,k}\prod_{j=1}^{k-1}q_{c_j,j}\right)\\ &+\left(\prod_{j=1}^{n}q_{c_j,j}\right)\sup\left\{(-1)^{\rho_{n+1}}a_{i_{n+1},n+1}+\sum_{t=n+2}^{\infty}\left((-1)^{\rho_t}a_{i_t,t}\prod_{r=n+1}^{t-1}q_{i_r,r}\right)\right\}, \end{aligned}$$

hold, we have

$$x\in\Lambda^{\tilde{Q}^{'}_{\mathbb{N}_B}}_{c_1c_2\ldots c_n}\quad\text{and}\quad x^{'},x^{''}\in\Lambda^{\tilde{Q}^{'}_{\mathbb{N}_B}}_{c_1c_2\ldots c_n}.$$

It is obvious that the statement is true when $n\in\mathbb{N}_B$. Our lemma is proven. □

From Lemma 0.3, it follows that the following statement is true.

Corollary 0.4. *For any cylinder $\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2\ldots c_n}$, the following equalities hold:*

$$\inf \Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2\ldots c_n} = (-1)^{\rho_1} a_{c_1,1} + \sum_{k=2}^{n} \left((-1)^{\rho_k} a_{c_k,k} \prod_{j=1}^{k-1} q_{c_j,j} \right) - \left(\prod_{j=1}^{n} q_{c_j,j} \right) \left(\tilde{a}_{m_{n+1},n+1} + \sum_{t=n+2}^{\infty} \left(\tilde{a}_{m_t,t} \prod_{r=n+1}^{t-1} \tilde{q}_{m_r,r} \right) \right),$$

$$\sup \Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2\ldots c_n} = (-1)^{\rho_1} a_{c_1,1} + \sum_{k=2}^{n} \left((-1)^{\rho_k} a_{c_k,k} \prod_{j=1}^{k-1} q_{c_j,j} \right) + \left(\prod_{j=1}^{n} q_{c_j,j} \right) \left(\tilde{a}_{0,n+1} + \sum_{t=n+2}^{\infty} \left(\tilde{a}_{0,t} \prod_{r=n+1}^{t-1} \tilde{q}_{0,r} \right) \right);$$

Suppose that $|\cdot|$ denotes the Lebesgue measure of an interval, then the next statement follows from Lemma 0.3 and Corollary 0.4.

Corollary 0.5. *For any cylinder $\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2\ldots c_n}$, the main metric relation is the following*

$$\frac{\left|\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2\ldots c_n c_{n+1}}\right|}{\left|\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2\ldots c_n}\right|} = q_{c_{n+1},n+1} \times$$

$$\frac{\tilde{a}_{m_{n+2},n+2} + \sum_{t=n+3}^{\infty} \left(\tilde{a}_{m_t,t} \prod_{r=n+2}^{t-1} \tilde{q}_{m_r,r} \right) + \tilde{a}_{0,n+2} + \sum_{t=n+3}^{\infty} \left(\tilde{a}_{0,t} \prod_{r=n+2}^{t-1} \tilde{q}_{0,r} \right)}{\tilde{a}_{m_{n+1},n+1} + \sum_{t=n+2}^{\infty} \left(\tilde{a}_{m_t,t} \prod_{r=n+1}^{t-1} \tilde{q}_{m_r,r} \right) + \tilde{a}_{0,n+1} + \sum_{t=n+2}^{\infty} \left(\tilde{a}_{0,t} \prod_{r=n+1}^{t-1} \tilde{q}_{0,r} \right)}.$$

Now let us return to the proof of our theorem.

Let us consider the mutual placement of $\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2\ldots c_{n-1}c}$ and $\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2\ldots c_{n-1}[c+1]}$.

Suppose that $\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2\ldots c_{n-1}c} \bigcap \Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2\ldots c_{n-1}[c+1]}$ is not the empty or one-element set; then we have the following:

- if cylinders $\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2\ldots c_{n-1}c}$ and $\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2\ldots c_{n-1}[c+1]}$ are *"left-to-right"* situated, then
$$\kappa_1 = \sup \Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2\ldots c_{n-1}c} - \inf \Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2\ldots c_{n-1}[c+1]} > 0;$$
- if cylinders $\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2\ldots c_{n-1}c}$ and $\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2\ldots c_{n-1}[c+1]}$ are *"right-to-left"* situated, then
$$\kappa_2 = \sup \Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2\ldots c_{n-1}[c+1]} - \inf \Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2\ldots c_{n-1}c} > 0.$$

In addition, in the first case, we get

$$\kappa_1<\kappa_2=|\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2...c_{n-1}c}|+|\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2...c_{n-1}[c+1]}|-|\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2...c_{n-1}c}\cap\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2...c_{n-1}[c+1]}|=W.$$

In the second case, $\kappa_2<\kappa_1=W$.

Suppose that $\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2...c_{n-1}c}\bigcap\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2...c_{n-1}[c+1]}$ is the empty or one-element set; then we have the following:

- if cylinders $\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2...c_{n-1}c}$ and $\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2...c_{n-1}[c+1]}$ are *"left-to-right"* situated, then
$$\nu_1=\inf\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2...c_{n-1}[c+1]}-\sup\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2...c_{n-1}c}=-\kappa_1\ge 0;$$
- if cylinders $\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2...c_{n-1}c}$ and $\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2...c_{n-1}[c+1]}$ are *"right-to-left"* situated, then
$$\nu_2=\inf\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2...c_{n-1}c}-\sup\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2...c_{n-1}[c+1]}=-\kappa_2\ge 0.$$

Also, in the first case, we have

$$\nu_1>\nu_2=V=-|\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2...c_{n-1}c}|-|\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2...c_{n-1}[c+1]}|-\varpi,$$

where ϖ is the Lebesgue measure of the interval situated between

$$\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2...c_{n-1}c}\quad\text{and}\quad\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2...c_{n-1}[c+1]}.$$

In the second case,

$$V=\nu_1<\nu_2.$$

Let us consider the difference

$$\begin{aligned}\kappa_1&\equiv\sup\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2...c_{n-1}c}-\inf\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2...c_{n-1}[c+1]}\\&=a_{c,n}(-1)^{\rho_n}\prod_{j=1}^{n-1}q_{c_j,j}+q_{c,n}\left(\prod_{j=1}^{n-1}q_{c_j,j}\right)\left(\sum_{n<t\notin\mathbb{N}_B}^{\infty}\left(\hat{a}_{i_t,t}\prod_{r=n+1}^{t-1}\hat{q}_{i_r,r}\right)\right)\\&\quad-a_{c+1,n}(-1)^{\rho_n}\prod_{j=1}^{n-1}q_{c_j,j}+q_{c+1,n}\left(\prod_{j=1}^{n-1}q_{c_j,j}\right)\left(\sum_{n<t\in\mathbb{N}_B}\left(\check{a}_{i_t,t}\prod_{r=n+1}^{t-1}\check{q}_{i_r,r}\right)\right)\\&=\left(\prod_{j=1}^{n-1}q_{c_j,j}\right)\left(-q_{c,n}(-1)^{\rho_n}+q_{c,n}\sum_{n<t\notin\mathbb{N}_B}\left(a_{m_t,t}\prod_{r=n+1}^{t-1}\hat{q}_{i_r,r}\right)\right.\\&\quad\left.+q_{c+1,n}\sum_{n<t\in\mathbb{N}_B}\left(a_{m_t,t}\prod_{r=n+1}^{t-1}\check{q}_{i_r,r}\right)\right).\end{aligned}$$

Using

$$\omega_1 = \sum_{n<t\notin\mathbb{N}_B}^{\infty}\left(a_{m_t,t}\prod_{r=n+1}^{t-1}\hat{q}_{i_r,r}\right) \qquad \text{and} \qquad \omega_2 = \sum_{n<t\in\mathbb{N}_B}\left(a_{m_t,t}\prod_{r=n+1}^{t-1}\check{q}_{i_r,r}\right),$$

we get

$$\kappa_1 = (q_{c,n}(-1)^{1+\rho_n} + q_{c,n}\omega_1 + q_{c+1,n}\omega_2)q_{c_1,1}q_{c_2,2}\dots q_{c_{n-1},n-1}.$$

From the last expression, we have the following:

- $\kappa_1 > 0$ whenever $\rho_n = 1$, i.e., $n \in \mathbb{N}_B$;
- if $\rho_n = 2$, i.e., $n \notin \mathbb{N}_B$, then:

$$\text{if} \quad (1-\omega_1)q_{c,n} = q_{c+1,n}\omega_2, \quad \text{then} \quad \kappa_1 = 0;$$

$$\text{if} \quad (1-\omega_1)q_{c,n} > q_{c+1,n}\omega_2, \quad \text{then} \quad \kappa_1 < 0;$$

$$\text{if} \quad (1-\omega_1)q_{c,n} < q_{c+1,n}\omega_2, \quad \text{then} \quad \kappa_1 > 0.$$

Really, since

$$0 \le \omega_1 \le 1 \quad \text{and} \quad 0 \le \omega_2 \le 1, \quad \text{for} \quad n \notin \mathbb{N}_B$$

we have

$$-q_{c,n} \le \frac{\kappa_1}{q_{c_1,1}q_{c_2,2}\dots q_{c_{n-1},n-1}} \le q_{c+1,n}.$$

Let us consider the difference

$$\begin{aligned}
\kappa_2 &\equiv \sup\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2\dots c_{n-1}[c+1]} - \inf\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2\dots c_{n-1}c}\\
&= (-1)^{\rho_n}a_{c+1,n}\prod_{j=1}^{n-1}q_{c_j,j} + q_{c+1,n}\left(\prod_{j=1}^{n-1}q_{c_j,j}\right)\left(\sum_{n<t\notin\mathbb{N}_B}\left(a_{m_t,t}\prod_{r=n+1}^{t-1}\hat{q}_{i_r,r}\right)\right)\\
&\quad - a_{c,n}(-1)^{\rho_n}\prod_{j=1}^{n-1}q_{c_j,j} + q_{c,n}\left(\prod_{j=1}^{n-1}q_{c_j,j}\right)\left(\sum_{n<t\in\mathbb{N}_B}\left(a_{m_t,t}\prod_{r=n+1}^{t-1}\check{q}_{i_r,r}\right)\right)\\
&= \big(q_{c,n}(-1)^{\rho_n} + q_{c+1,n}\omega_1 + q_{c,n}\omega_2\big)\left(\prod_{j=1}^{n-1}q_{c_j,j}\right).
\end{aligned}$$

Hence, we obtain the following:

- $\kappa_2 > 0$ whenever $\rho_n = 2$, i.e., $n \notin \mathbb{N}_B$;
- if $\rho_n = 1$, i.e., $n \in \mathbb{N}_B$, then:

$$\begin{aligned}
&\text{if} \quad (1-\omega_2)q_{c,n} = q_{c+1,n}\omega_1, \quad \text{then} \quad \kappa_2 = 0;\\
&\text{if} \quad (1-\omega_2)q_{c,n} > q_{c+1,n}\omega_1, \quad \text{then} \quad \kappa_2 < 0;\\
&\text{if} \quad (1-\omega_2)q_{c,n} < q_{c+1,n}\omega_1, \quad \text{then} \quad \kappa_2 > 0.
\end{aligned}$$

So,

$$-q_{c,n} \le \frac{\kappa_2}{q_{c_1,1}q_{c_2,2}\cdots q_{c_{n-1},n-1}} \le q_{c+1,n}.$$

Finally, we obtain the following results:

- If $\rho_n = 1$, then cylinders $\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2\dots c_{n-1}c}$ and $\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2\dots c_{n-1}[c+1]}$ are right-to-left situated. However, the set $\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2\dots c_{n-1}c} \bigcap \Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2\dots c_{n-1}[c+1]}$ is the empty set or the one-element set or an interval. It depends on the matrix $\tilde{Q}'_{\mathbb{N}_B}$.
- If $\rho_n = 2$, then cylinders $\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2\dots c_{n-1}c}$ and $\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2\dots c_{n-1}[c+1]}$ are left-to-right situated. However, the set $\Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2\dots c_{n-1}c} \bigcap \Lambda^{\tilde{Q}'_{\mathbb{N}_B}}_{c_1c_2\dots c_{n-1}[c+1]}$ is one of the following sets (that depends on $\tilde{Q}'_{\mathbb{N}_B}$): the empty set, the one-element set, an interval.

Since Lemma 0.3, Corollaries 0.4 and 0.5 are true, the theorem is proven. □

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