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A NOTE ON EXPANSIONS OF RATIONAL NUMBERS BY CERTAIN SERIES

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ABSTRACT. This paper deals with representations of rational numbers defined in terms of numeral systems that are certain generalizations of the classical q-ary numeral system.

In [1], expansions of the following form of real numbers from the interval $[0,1]$ were considered:

$$\frac{\varepsilon_1}{q_1}+\frac{\varepsilon_2}{q_1q_2}+\cdots+\frac{\varepsilon_k}{q_1q_2\cdots q_k}+\cdots. \tag{1}$$

Here $Q\equiv(q_k)$ is a fixed sequence of positive integers, $q_k>1$, and (Θ_k) is a sequence of the sets $\Theta_k\equiv\{0,1,\dots,q_k-1\}$, as well as $\varepsilon_k\in\Theta_k$.

Note that the last-mentioned expansion under the condition $q_k=const=q$ for all positive integers k, where $1<q\in\mathbb{N}$ ($\mathbb{N}$ is the set of all positive integers), is the q-ary expansion of real numbers from $[0,1]$, i.e.,

$$\frac{\varepsilon_1}{q}+\frac{\varepsilon_2}{q^2}+\cdots+\frac{\varepsilon_k}{q^k}+\cdots,$$

where $\varepsilon_k\in\{0,1,\dots,q-1\}$. In this case, a number is a rational number if and only if a sequence (ε_k) is periodic.

However, the problem of the rationality/irrationality of numbers defined in terms of generalizations of the q-ary numeral system is difficult. A version of this problem for expansions of form (1) was introduced in the paper [1] and has been studied by a number of researchers. For example, G. Cantor, P. A. Diananda, A. Oppenheim, P. Erdős, J. Hančl, E. G. Straus, P. Rucki, P. Kuhapatanakul, V. Laohakosol and other scientists studied this problem (see references). Known results include different conditions satisfied by (ε_k) and/or (q_k).

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In this paper, the main attention is given to calculating values of ε_k for an arbitrary rational number and a fixed sequence (q_k). That is, the problem of the rationality is investigated for the case of a numeral system whose base is an infinite sequence of positive integers.

By $x = \Delta^Q_{\varepsilon_1\varepsilon_2\ldots\varepsilon_k\ldots}$ denote a number $x \in [0,1]$ represented by series (1). Suppose $c_1, c_2, \ldots, c_m$ is an ordered tuple of integers such that $c_i \in \{0, 1, \ldots, q_i - 1\}$ for $i = \overline{1, m}$. *A cylinder* $\Delta^Q_{c_1c_2\ldots c_m}$ *of rank* m *with base* $c_1c_2\ldots c_m$ is a set of the form

$$\Delta^Q_{c_1c_2\ldots c_m} \equiv \{x : x = \Delta^Q_{c_1c_2\ldots c_m\varepsilon_{m+1}\varepsilon_{m+2}\ldots\varepsilon_{m+k}\ldots}\}.$$

That is, any cylinder $\Delta^Q_{c_1c_2\ldots c_m}$ is a closed interval of the form

$$\left[\Delta^Q_{c_1c_2\ldots c_m000\ldots}, \Delta^Q_{c_1c_2\ldots c_m[q_{m+1}-1][q_{m+2}-1][q_{m+3}-1]\ldots}\right].$$

Define *the shift operator* σ *of expansion* (1) as

$$\sigma(x) = \sigma\left(\Delta^Q_{\varepsilon_1\varepsilon_2\ldots\varepsilon_k\ldots}\right) = \sum_{k=2}^{\infty} \frac{\varepsilon_k}{q_2q_3\cdots q_k} = q_1\Delta^Q_{0\varepsilon_2\ldots\varepsilon_k\ldots}.$$

It is easy to see that

$$\begin{aligned}\sigma^n(x) &= \sigma^n\left(\Delta^Q_{\varepsilon_1\varepsilon_2\ldots\varepsilon_k\ldots}\right)\\ &= \sum_{k=n+1}^{\infty} \frac{\varepsilon_k}{q_{n+1}q_{n+2}\cdots q_k} = q_1\cdots q_n\Delta^Q_{\underbrace{0\ldots0}_{n}\varepsilon_{n+1}\varepsilon_{n+2}\ldots}.\end{aligned}$$

Therefore,

$$x = \sum_{i=1}^{n} \frac{\varepsilon_i}{q_1q_2\cdots q_i} + \frac{1}{q_1q_2\cdots q_n}\sigma^n(x). \tag{2}$$

Lemma 1. *Let* $x \in (0,1)$ *be a rational number represented by series* (1). *If* $x = \frac{p}{r} = \Delta^Q_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots}$, *then the equality*

$$\varepsilon_n = \left[\frac{\Delta_n}{r}\right] \tag{3}$$

holds for all $n \in \mathbb{N}$, *where*

$$\Delta_n = pq_1q_2\cdots q_n - r(\varepsilon_1q_2q_3\cdots q_n + \varepsilon_2q_3q_4\cdots q_n + \cdots + \varepsilon_{n-1}q_n).$$

Proof. Let $\frac{p}{r}$ be a fixed number, where $(p, r) = 1$, $p < r$, and $p \in \mathbb{N}, r \in \mathbb{N}$. Then

$$\frac{p}{r} = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{q_1q_2\cdots q_n}.$$

Remark 1. Note that, since $x \in \Delta^Q_{c_1c_2\ldots c_m}$ but

$$\Delta^Q_{c_1c_2\ldots c_m[q_{m+1}-1][q_{m+2}-1]\ldots} = \Delta^Q_{c_1c_2\ldots c_{m-1}[c_m+1]000\ldots},$$

we assume that

$$\Delta^Q_{c_1c_2\ldots c_{m-1}[c_m]000\ldots} \le x < \Delta^Q_{c_1c_2\ldots c_{m-1}[c_m+1]000\ldots}.$$

Let us prove this lemma. It is easy to see that

$$\frac{p}{r} \in \Delta^Q_{\varepsilon_1} = \left[\Delta^Q_{\varepsilon_1 000\ldots}, \Delta^Q_{\varepsilon_1 [q_2-1][q_3-1]\ldots}\right] = \left[\frac{\varepsilon_1}{q_1}, \frac{\varepsilon_1+1}{q_1}\right].$$

That is

$$\frac{\varepsilon_1}{q_1} \le \frac{p}{r} < \frac{\varepsilon_1+1}{q_1}, \qquad \varepsilon_1 r \le pq_1 < r(\varepsilon_1+1), \qquad \varepsilon_1 \le \frac{pq_1}{r} < \varepsilon_1 + 1.$$

So,

$$\varepsilon_1 = \left[\frac{p}{r}q_1\right], \qquad \text{where } [x] \text{ is the integer part of } x.$$

Now we get

$$\frac{p}{r} \in \Delta^Q_{\varepsilon_1\varepsilon_2} = \left[\frac{q_2\varepsilon_1+\varepsilon_2}{q_1q_2}, \frac{q_2\varepsilon_1+\varepsilon_2+1}{q_1q_2}\right].$$

Whence,

$$\frac{q_2\varepsilon_1+\varepsilon_2}{q_1q_2} \le \frac{p}{r} < \frac{q_2\varepsilon_1+\varepsilon_2+1}{q_1q_2}, \qquad \varepsilon_2 \le \frac{pq_1q_2 - rq_2\varepsilon_1}{r} < \varepsilon_2 + 1.$$

So,

$$\varepsilon_2 = \left[\frac{pq_1q_2 - rq_2\varepsilon_1}{r}\right].$$

In the third step, we have

$$\frac{p}{r} \in \Delta^Q_{\varepsilon_1\varepsilon_2\varepsilon_3} = \left[\frac{\varepsilon_1q_2q_3+\varepsilon_2q_3+\varepsilon_3}{q_1q_2q_3}, \frac{\varepsilon_1q_2q_3+\varepsilon_2q_3+\varepsilon_3+1}{q_1q_2q_3}\right]$$

and

$$\varepsilon_3 = \left[\frac{pq_1q_2q_3 - r(\varepsilon_1q_2q_3+\varepsilon_2q_3)}{r}\right].$$

In the n-th step, we obtain

$$\begin{aligned}\frac{p}{r} \in \Delta^Q_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{n-1}\varepsilon_n} &= \left[\sum_{i=1}^{n-1}\frac{\varepsilon_i}{q_1q_2\cdots q_i} + \frac{\varepsilon_n}{q_1q_2\cdots q_n}, \sum_{i=1}^{n-1}\frac{\varepsilon_i}{q_1q_2\cdots q_i} + \frac{\varepsilon_n+1}{q_1q_2\cdots q_n}\right] \\ &= \left[\frac{\varepsilon_1q_2q_3\cdots q_n + \varepsilon_2q_3q_4\cdots q_n + \cdots + \varepsilon_{n-1}q_n + \varepsilon_n}{q_1q_2\cdots q_n},\right. \\ &\qquad \left.\frac{\varepsilon_1q_2q_3\cdots q_n + \cdots + \varepsilon_{n-1}q_n + \varepsilon_n + 1}{q_1q_2\cdots q_n}\right].\end{aligned}$$

Let ς_n denote the sum

$$\varepsilon_1q_2q_3\cdots q_n + \varepsilon_2q_3q_4\cdots q_n + \cdots + \varepsilon_{n-1}q_n.$$

Then

$$\frac{\varsigma_n+\varepsilon_n}{q_1q_2\cdots q_n} \le \frac{p}{r} < \frac{\varsigma_n+\varepsilon_n+1}{q_1q_2\cdots q_n}, \qquad \varepsilon_n \le \frac{pq_1q_2\cdots q_n - r\varsigma_n}{r} < \varepsilon_n + 1.$$

This completes the proof. $\square$

Denoting by $\Delta_n = pq_1q_2\cdots q_n - r\varsigma_n$, we get

$$\varepsilon_n = \left[\frac{\Delta_n}{r}\right].$$

Also, for $n \geq 2$ the condition $\varsigma_n = \varsigma_{n-1}q_n + \varepsilon_{n-1}q_n$ holds and

$$\Delta_n = q_n(\Delta_{n-1} - r\varepsilon_{n-1}).$$

So, the following statement is true.

Lemma 2. *Let $x \in (0,1)$ be a rational number represented by series* (1). *If $x = \frac{p}{r} = \Delta^Q_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots}$, then the equality*

$$\varepsilon_n = \left[\frac{q_n(\Delta_{n-1} - r\varepsilon_{n-1})}{r}\right]$$

holds for all $1 < n \in \mathbb{N}$, where $\Delta_1 = pq_1$ and $\varepsilon_1 = \left[\frac{\Delta_1}{r}\right]$.

Theorem. *A number $x = \Delta^Q_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots} \in (0,1)$ is a rational number $\frac{p}{r}$, where $p, r \in \mathbb{N}, (p,r) = 1$, and $p < r$, if and only if the condition*

$$\varepsilon_n = \left[\frac{q_n(\Delta_{n-1} - r\varepsilon_{n-1})}{r}\right]$$

holds for all $1 < n \in \mathbb{N}$, where $\Delta_1 = pq_1$, $\varepsilon_1 = \left[\frac{\Delta_1}{r}\right]$, and $[a]$ is the integer part of a.

Proof. *The necessity* follows from previous two lemmas.

Let us prove that *the sufficiency* is true. Suppose that the following sequence of conditions is true:

$$\varepsilon_1 = \left[\frac{p}{r}q_1\right], \quad \varepsilon_2 = \left[\frac{pq_1q_2 - rq_2\varepsilon_1}{r}\right], \quad \varepsilon_3 = \left[\frac{pq_1q_2q_3 - r(\varepsilon_1q_2q_3 + \varepsilon_2q_3)}{r}\right], \ldots$$

$$\varepsilon_n = \left[\frac{pq_1q_2\cdots q_n}{r} - (\varepsilon_1q_2q_3\cdots q_n + \varepsilon_2q_3q_4\cdots q_n + \cdots + \varepsilon_{n-1}q_n)\right] = \left[\frac{\Delta_n}{r}\right], \ldots$$

It follows from equality (2) that

$$x = \frac{p}{r} = \frac{\varepsilon_1q_2q_3\cdots q_n + \varepsilon_2q_3q_4\cdots q_n + \cdots + \varepsilon_{n-1}q_n + \varepsilon_n}{q_1q_2\cdots q_n} + \frac{\sigma^n\left(\frac{p}{r}\right)}{q_1q_2\cdots q_n},$$

$$\sigma^n\left(\frac{p}{r}\right) = \frac{\Delta_n - r\varepsilon_n}{r} = \frac{\Delta_n}{r} - \varepsilon_n,$$

and

$$\varepsilon_n = \frac{\Delta_n}{r} - \sigma^n\left(\frac{p}{r}\right).$$

It follows from the last-mentioned relationship and relationship (3) that

$$0 \leq \varepsilon_n = \left[\frac{\Delta_n}{r}\right] = \frac{\Delta_n}{r} - \sigma^n\left(\frac{p}{r}\right) = \left[\frac{\Delta_n}{r} - \sigma^n\left(\frac{p}{r}\right)\right].$$

That is,
$$\sigma^n\left(\frac{p}{r}\right) = \left\{\frac{\Delta_n}{r}\right\},$$
where $\{a\}$ is the fractional part of a.

Remark 2. Clearly,
$$0 \le \sigma^n(x) \le 1$$
for an arbitrary $x \in [0,1]$. However, for any number of the form
$$\begin{aligned} x = x_1 &= \Delta^Q_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{n-1}\varepsilon_n 000\ldots} \\ &= \Delta^Q_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{n-1}[\varepsilon_n-1][q_{n+1}-1][q_{n+2}-1][q_{n+3}-1]\ldots} = x_2 \end{aligned}$$
the following conditions hold:
$$\begin{aligned} \sigma^n(x) &= \sigma^n\left(\Delta^Q_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{n-1}\varepsilon_n 000\ldots}\right) \\ &= \sum_{k=n+1}^{\infty} \frac{0}{q_{n+1}q_{n+2}\cdots q_k} = 0, \end{aligned}$$
$$\begin{aligned} \sigma^n(x) &= \sigma^n\left(\Delta^Q_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{n-1}[\varepsilon_n-1][q_{n+1}-1][q_{n+2}-1][q_{n+3}-1]\ldots}\right) \\ &= \sum_{k=n+1}^{\infty} \frac{q_k-1}{q_{n+1}q_{n+2}\cdots q_k} = 1. \end{aligned}$$
Since the condition $\sigma^n(x_2) = 1$ holds, we can use only the first representation of number x, i.e., the form x_1, and for these numbers the condition
$$\sigma^n(x) = \sigma^n(x_1) = 0 \quad \text{holds.}$$

In addition, note that
$$\varsigma_n = \frac{pq_1q_2\cdots q_n - \Delta_n}{r}.$$
Whence for an arbitrary $n \in \mathbb{N}$
$$\begin{aligned} x &= \sum_{k=1}^{n} \frac{\varepsilon_k}{q_1q_2\cdots q_k} + \frac{\sigma^n(x)}{q_1q_2\cdots q_n} \\ &= \frac{\varepsilon_1q_2q_3\cdots q_n + \varepsilon_2q_3q_4\cdots q_n + \cdots + \varepsilon_{n-1}q_n + \varepsilon_n}{q_1q_2\ldots q_n} + \frac{\sigma^n(x)}{q_1q_2\ldots q_n} \\ &= \frac{\varsigma_n + \varepsilon_n}{q_1q_2\cdots q_n} + \frac{\left\{\frac{\Delta_n}{r}\right\}}{q_1q_2\cdots q_n} = \frac{\frac{pq_1q_2\cdots q_n - \Delta_n}{r} + \varepsilon_n}{q_1q_2\cdots q_n} + \frac{\frac{\Delta_n}{r} - \left[\frac{\Delta_n}{r}\right]}{q_1q_2\cdots q_n} \\ &= \frac{pq_1q_2\cdots q_n - \Delta_n + r\varepsilon_n}{rq_1q_2\cdots q_n} + \frac{\frac{\Delta_n}{r} - \varepsilon_n}{q_1q_2\cdots q_n} = \frac{p}{r}. \end{aligned}$$
This completes the proof. □

Let us consider certain examples. Suppose

$$x = \Delta^{(2n+1)}_{\varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots} = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{3\cdot 5\cdot 7\cdots(2n+1)}.$$

Then

$$\frac{1}{4} = \Delta^{(2n+1)}_{035229[11]4\ldots}, \qquad \frac{3}{8} = \Delta^{(2n+1)}_{104341967\ldots}.$$

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