

ON $J(r, n)$ -JACOBSTHAL HYBRID NUMBERS

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ABSTRACT. In this paper we introduce a generalization of Jacobsthal hybrid numbers – $J(r, n)$ -Jacobsthal hybrid numbers. We give some of their properties: character, Binet's formula, a summation formula and a generating function.

1. Introduction

The hybrid numbers were introduced by Özdemiş in [6] as a new generalization of complex, hyperbolic and dual numbers.

Let \mathbb{K} be the set of hybrid numbers \mathbf{Z} of the form

$$\mathbf{Z} = a + b\mathbf{i} + c\varepsilon + d\mathbf{h},$$

where $a, b, c, d \in \mathbb{R}$ and $\mathbf{i}, \varepsilon, \mathbf{h}$ are operators such that

$$\mathbf{i}^2 = -1, \quad \varepsilon^2 = 0, \quad \mathbf{h}^2 = 1 \tag{1}$$

and

$$\mathbf{i}\mathbf{h} = -\mathbf{h}\mathbf{i} = \varepsilon + \mathbf{i}. \tag{2}$$

If $\mathbf{Z}_1 = a_1 + b_1\mathbf{i} + c_1\varepsilon + d_1\mathbf{h}$ and $\mathbf{Z}_2 = a_2 + b_2\mathbf{i} + c_2\varepsilon + d_2\mathbf{h}$ are any two hybrid numbers, then equality, addition, subtraction and multiplication by scalar are defined.

Equality: $\mathbf{Z}_1 = \mathbf{Z}_2$ only if $a_1 = a_2, b_1 = b_2, c_1 = c_2, d_1 = d_2$.

Addition: $\mathbf{Z}_1 + \mathbf{Z}_2 = (a_1 + a_2) + (b_1 + b_2)\mathbf{i} + (c_1 + c_2)\varepsilon + (d_1 + d_2)\mathbf{h}$.

Subtraction: $\mathbf{Z}_1 - \mathbf{Z}_2 = (a_1 - a_2) + (b_1 - b_2)\mathbf{i} + (c_1 - c_2)\varepsilon + (d_1 - d_2)\mathbf{h}$.

Multiplication by scalar $s \in \mathbb{R}$: $s\mathbf{Z}_1 = sa_1 + sb_1\mathbf{i} + sc_1\varepsilon + sd_1\mathbf{h}$.

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2010 Mathematics Subject Classification: 11B37, 11B39, 97F50.

Keywords: Jacobsthal numbers, recurrence relations, complex numbers, hyperbolic numbers, dual numbers, hybrid numbers.

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Addition operation in the hybrid numbers is both commutative and associative. Zero is the null element. With respect to the addition operation, the inverse element of \mathbf{Z} is

$$-\mathbf{Z} = -a - b\mathbf{i} - c\varepsilon - d\mathbf{h}.$$

This means that $(\mathbb{K}, +)$ is an Abelian group.

The hybrid numbers multiplication is defined using (1) and (2). Note that using the formulas (1) and (2) we can find the product of any two hybrid units. The following Table 1 presents products of \mathbf{i} , ε , and \mathbf{h} .

TABLE 1.

\cdot	\mathbf{i}	ε	\mathbf{h}
\mathbf{i}	-1	$1 - \mathbf{h}$	$\varepsilon + \mathbf{i}$
ε	$\mathbf{h} + 1$	0	$-\varepsilon$
\mathbf{h}	$-\varepsilon - \mathbf{i}$	ε	1

Using the rules given in Table 1, the multiplication of hybrid numbers can be made analogously as multiplications of algebraic expressions. As you can see, the multiplication operation in the hybrid numbers is not commutative. But it has the property of associativity. Moreover, the set of hybrid numbers is a non-commutative ring with respect to the addition and multiplication operations. The conjugate of a hybrid number \mathbf{Z} is defined by

$$\overline{\mathbf{Z}} = \overline{a + b\mathbf{i} + c\varepsilon + d\mathbf{h}} = a - b\mathbf{i} - c\varepsilon - d\mathbf{h}.$$

The real number

$$\begin{aligned} C(\mathbf{Z}) &= \mathbf{Z}\overline{\mathbf{Z}} = \overline{\mathbf{Z}}\mathbf{Z} = a^2 + (b - c)^2 - c^2 - d^2 \\ &= a^2 + b^2 - 2bc - d^2 \end{aligned} \tag{3}$$

is called the character of the hybrid number \mathbf{Z} .

For the basics on hybrid number theory and algebraic and geometric properties of hybrid numbers, see [6].

2. The $J(r, n)$ -Jacobsthal numbers

In 1965 [4] Horadam introduced a second order linear recurrence sequence $\{W_n\}$ defined by the relation

$$W_0 = a, \quad W_1 = b, \quad W_n = pW_{n-1} - qW_{n-2}, \tag{4}$$

for $n \geq 2$ and arbitrary integers a, b, p, q . This sequence is a certain generalization of famous sequences such as Fibonacci sequence ($a = 0, b = 1, p = 1, q = -1$),

Lucas sequence ($a = 2, b = 1, p = 1, q = -1$) or Pell sequence ($a = 0, b = 1, p = 2, q = -1$). Hence sequences defined by (4) are called sequences of the Fibonacci type. The Jacobsthal sequence $\{J_n\}$ is defined by the second order linear recurrence of the type (4), i.e.,

$$J_n = J_{n-1} + 2J_{n-2} \quad \text{for } n \geq 2 \quad (5)$$

with initial terms

$$J_0 = 0 \quad \text{and} \quad J_1 = 1.$$

The Binet's formula of this sequence has the following form

$$J_n = \frac{1}{3}(2^n - (-1)^n) \quad \text{for } n \geq 0.$$

There are many generalizations of this sequence in the literature. The second order recurrence (5) has been generalized in two ways: the first by preserving the initial conditions and the second by preserving the recurrence relation, see [2], [3], [5], [9]. In [1] a one-parameter generalization of the Jacobsthal numbers was investigated. We recall this generalization.

Let $n \geq 0, r \geq 0$ be integers. The n th $J(r, n)$ -Jacobsthal number $J(r, n)$ is defined as follows

$$J(r, n) = 2^r J(r, n-1) + (2^r + 4^r) J(r, n-2) \quad \text{for } n \geq 2 \quad (6)$$

with initial conditions

$$J(r, 0) = 1, \quad J(r, 1) = 1 + 2^{r+1}.$$

It is easily seen that $J(0, n) = J_{n+2}$. By (6) we obtain

$$\begin{aligned} J(r, 0) &= 1, \\ J(r, 1) &= 2 \cdot 2^r + 1, \\ J(r, 2) &= 3 \cdot 4^r + 2 \cdot 2^r, \\ J(r, 3) &= 5 \cdot 8^r + 5 \cdot 4^r + 2^r, \\ J(r, 4) &= 8 \cdot 16^r + 10 \cdot 8^r + 3 \cdot 4^r, \\ J(r, 5) &= 13 \cdot 32^r + 20 \cdot 16^r + 9 \cdot 8^r + 4^r. \end{aligned}$$

We will now recall some properties of the $J(r, n)$ -Jacobsthal numbers.

THEOREM 2.1 (Binet's formula [1]). *For $n \geq 0$ the n th $J(r, n)$ -Jacobsthal number is given by*

$$J(r, n) = \frac{\sqrt{4 \cdot 2^r + 5 \cdot 4^r} + 3 \cdot 2^r + 2}{2\sqrt{4 \cdot 2^r + 5 \cdot 4^r}} \lambda_1^n + \frac{\sqrt{4 \cdot 2^r + 5 \cdot 4^r} - 3 \cdot 2^r - 2}{2\sqrt{4 \cdot 2^r + 5 \cdot 4^r}} \lambda_2^n,$$

where

$$\lambda_1 = 2^{r-1} + \frac{1}{2}\sqrt{4 \cdot 2^r + 5 \cdot 4^r},$$

$$\lambda_2 = 2^{r-1} - \frac{1}{2}\sqrt{4 \cdot 2^r + 5 \cdot 4^r}.$$

THEOREM 2.2 ([1]). *The generating function of the sequence $\{J(r, n)\}$ has the following form*

$$f(x) = \frac{1 + (1 + 2^r)x}{1 - 2^r x - (2^r + 4^r)x^2}.$$

PROPOSITION 2.3 ([1]). *Let $n \geq 4$, $r \geq 0$ be integers. Then*

$$J(r, n) = (3 \cdot 8^r + 2 \cdot 4^r)J(r, n-3) + (2 \cdot 16^r + 3 \cdot 8^r + 4^r)J(r, n-4).$$

THEOREM 2.4 ([1]). *Let $n \geq 1$, $r \geq 0$ be integers. Then*

$$\sum_{l=0}^{n-1} J(r, l) = \frac{J(r, n) + (2^r + 4^r)J(r, n-1) - 2 - 2^r}{4^r + 2^{r+1} - 1}.$$

THEOREM 2.5 (Convolution identity [1]). *Let n, m, r be integers such that $m \geq 2, n \geq 1, r \geq 0$. Then*

$$J(r, m+n) = 2^r J(r, m-1)J(r, n) + (4^r + 8^r)J(r, m-2)J(r, n-1).$$

3. The $J(r, n)$ -Jacobsthal hybrid numbers

The n th Horadam hybrid number H_n is defined as

$$H_n = W_n + \mathbf{i}W_{n+1} + \varepsilon W_{n+2} + \mathbf{h}W_{n+3}.$$

For special values of W_n we obtain the definitions of

(i) n th Fibonacci hybrid number FH_n

$$FH_n = F_n + \mathbf{i}F_{n+1} + \varepsilon F_{n+2} + \mathbf{h}F_{n+3},$$

(ii) n th Pell hybrid number PH_n

$$PH_n = P_n + \mathbf{i}P_{n+1} + \varepsilon P_{n+2} + \mathbf{h}P_{n+3},$$

(iii) n th Jacobsthal hybrid number JH_n

$$JH_n = J_n + \mathbf{i}J_{n+1} + \varepsilon J_{n+2} + \mathbf{h}J_{n+3}.$$

Interesting results on the Horadam hybrid numbers and the Jacobsthal hybrid numbers obtained recently can be found in [7], [8]. In this paper we introduce a $J(r, n)$ -Jacobsthal hybrid number JH_n^r .

For a non-negative integer n the n th $J(r, n)$ -Jacobsthal number JH_n^r is defined as

$$JH_n^r = J(r, n) + \mathbf{i}J(r, n+1) + \varepsilon J(r, n+2) + \mathbf{h}J(r, n+3), \quad (7)$$

where $J(r, n)$ is given by (6).

Using the above definitions, we can write initial $J(r, n)$ -Jacobsthal hybrid numbers, i.e.,

$$\begin{aligned} JH_0^r &= 1 + \mathbf{i}(2 \cdot 2^r + 1) + \varepsilon(3 \cdot 4^r + 2 \cdot 2^r) + \mathbf{h}(5 \cdot 8^r + 5 \cdot 4^r + 2^r), \\ JH_1^r &= 2 \cdot 2^r + 1 + \mathbf{i}(3 \cdot 4^r + 2 \cdot 2^r) + \varepsilon(5 \cdot 8^r + 5 \cdot 4^r + 2^r) \\ &\quad + \mathbf{h}(8 \cdot 16^r + 10 \cdot 8^r + 3 \cdot 4^r), \\ JH_2^r &= 3 \cdot 4^r + 2 \cdot 2^r + \mathbf{i}(5 \cdot 8^r + 5 \cdot 4^r + 2^r) \\ &\quad + \varepsilon(8 \cdot 16^r + 10 \cdot 8^r + 3 \cdot 4^r) \\ &\quad + \mathbf{h}(13 \cdot 32^r + 20 \cdot 16^r + 9 \cdot 8^r + 4^r). \end{aligned} \quad (8)$$

The next theorems present recurrence relations for the $J(r, n)$ -Jacobsthal hybrid numbers.

THEOREM 3.1. *Let $n \geq 2$, $r \geq 0$ be integers. Then*

$$JH_n^r = 2^r JH_{n-1}^r + (2^r + 4^r) JH_{n-2}^r,$$

where JH_0^r , JH_1^r are given in (8).

Proof. Using (7) and (6), we have

$$\begin{aligned} &2^r JH_{n-1}^r + (2^r + 4^r) JH_{n-2}^r \\ &= 2^r (J(r, n-1) + \mathbf{i}J(r, n) + \varepsilon J(r, n+1) + \mathbf{h}J(r, n+2)) \\ &\quad + (2^r + 4^r) (J(r, n-2) + \mathbf{i}J(r, n-1) + \varepsilon J(r, n) + \mathbf{h}J(r, n+1)) \\ &= J(r, n) + \mathbf{i}J(r, n+1) + \varepsilon J(r, n+2) + \mathbf{h}J(r, n+3) \\ &= JH_n^r. \end{aligned}$$

□

THEOREM 3.2. *Let $n \geq 4$, $r \geq 0$ be integers. Then*

$$JH_n^r = (3 \cdot 8^r + 2 \cdot 4^r)JH_{n-3}^r + (2 \cdot 16^r + 3 \cdot 8^r + 4^r)JH_{n-4}^r.$$

Proof. Let $A = 3 \cdot 8^r + 2 \cdot 4^r$, $B = 2 \cdot 16^r + 3 \cdot 8^r + 4^r$. Using Proposition 2.3, we obtain

$$\begin{aligned} JH_n^r &= J(r, n) + \mathbf{i}J(r, n+1) + \varepsilon J(r, n+2) + \mathbf{h}J(r, n+3) \\ &= A \cdot J(r, n-3) + B \cdot J(r, n-4) \\ &\quad + \mathbf{i}(A \cdot J(r, n-2) + B \cdot J(r, n-3)) \\ &\quad + \varepsilon(A \cdot J(r, n-1) + B \cdot J(r, n-2)) \\ &\quad + \mathbf{h}(A \cdot J(r, n) + B \cdot J(r, n-1)) \\ &= A(J(r, n-3) + \mathbf{i}J(r, n-2) + \varepsilon J(r, n-1) + \mathbf{h}J(r, n)) \\ &\quad + B(J(r, n-4) + \mathbf{i}J(r, n-3) + \varepsilon J(r, n-2) + \mathbf{h}J(r, n-1)). \end{aligned}$$

Hence we have

$$JH_n^r = A \cdot JH_{n-3}^r + B \cdot JH_{n-4}^r. \quad \square$$

THEOREM 3.3. *Let $n \geq 0$, $r \geq 0$ be integers. Then*

$$\begin{aligned} JH_n^r - \mathbf{i}JH_{n+1}^r - \varepsilon JH_{n+2}^r - \mathbf{h}JH_{n+3}^r \\ = J(r, n) + J(r, n+2) - 2J(r, n+3) - J(r, n+6). \end{aligned}$$

Proof.

$$\begin{aligned} JH_n^r - \mathbf{i}JH_{n+1}^r - \varepsilon JH_{n+2}^r - \mathbf{h}JH_{n+3}^r \\ = J(r, n) + \mathbf{i}J(r, n+1) + \varepsilon J(r, n+2) + \mathbf{h}J(r, n+3) \\ - \mathbf{i}(J(r, n+1) + \mathbf{i}J(r, n+2) + \varepsilon J(r, n+3) + \mathbf{h}J(r, n+4)) \\ - \varepsilon(J(r, n+2) + \mathbf{i}J(r, n+3) + \varepsilon J(r, n+4) + \mathbf{h}J(r, n+5)) \\ - \mathbf{h}(J(r, n+3) + \mathbf{i}J(r, n+4) + \varepsilon J(r, n+5) + \mathbf{h}J(r, n+6)) \\ = J(r, n) + J(r, n+2) - (1 - \mathbf{h})J(r, n+3) \\ + (\varepsilon + \mathbf{i})J(r, n+4) - (\mathbf{h} + 1)J(r, n+3) \\ - (\varepsilon + \mathbf{i})J(r, n+4) - J(r, n+6) \\ = J(r, n) + J(r, n+2) - 2J(r, n+3) - J(r, n+6). \end{aligned}$$

\square

The next theorem presents the character of the $J(r, n)$ -Jacobsthal hybrid numbers.

THEOREM 3.4. *Let $n \geq 0$ be integer. Then*

$$\begin{aligned} C(JH_n^r) &= [1 - 4^r(2^r + 4^r)^2]J^2(r, n) \\ &\quad + [1 - 2^{r+1} - (2^r + 2 \cdot 4^r)^2]J^2(r, n + 1) \\ &\quad - [2(2^r + 4^r)(1 + 4^r + 2 \cdot 8^r)]J(r, n)J(r, n + 1). \end{aligned}$$

Proof. Using (3), we have

$$\begin{aligned} C(JH_n^r) &= J^2(r, n) + J^2(r, n + 1) - 2J(r, n + 1)J(r, n + 2) - J^2(r, n + 3) \\ &= J^2(r, n) + J(r, n + 1)(J(r, n + 1) - 2J(r, n + 2)) - J^2(r, n + 3) \\ &= J(r, n + 1)\left(J(r, n + 1) - 2(2^r J(r, n + 1) + (2^r + 4^r)J(r, n))\right) \\ &\quad - (2^r J(r, n + 2) + (2^r + 4^r)J(r, n + 1))^2 + J^2(r, n). \end{aligned}$$

After simple calculations we get

$$\begin{aligned} C(JH_n^r) &= (1 - 2^{r+1})J^2(r, n + 1) - 2(2^r + 4^r)J(r, n)J(r, n + 1) \\ &\quad - ((2^r + 2 \cdot 4^r)J(r, n + 1) + 2^r(2^r + 4^r)J(r, n))^2 + J^2(r, n) \\ &= [1 - 4^r(2^r + 4^r)^2]J^2(r, n) \\ &\quad + [1 - 2^{r+1} - (2^r + 2 \cdot 4^r)^2]J^2(r, n + 1) \\ &\quad - [2(2^r + 4^r)(1 + 4^r + 2 \cdot 8^r)]J(r, n)J(r, n + 1). \end{aligned}$$

□

THEOREM 3.5. *Let $n \geq 0, r \geq 0$ be integers. Then*

- (i) $JH_n^r + \overline{JH_n^r} = 2J(r, n),$
- (ii) $(JH_n^r)^2 = 2J(r, n)JH_n^r - C(JH_n^r).$

Proof.

(i) By the definition of the conjugate of a hybrid number we get

$$\begin{aligned} JH_n^r + \overline{JH_n^r} &= J(r, n) + \mathbf{i}J(r, n + 1) + \varepsilon J(r, n + 2) + \mathbf{h}J(r, n + 3) \\ &\quad + J(r, n) - \mathbf{i}J(r, n + 1) - \varepsilon J(r, n + 2) - \mathbf{h}J(r, n + 3) \\ &= 2J(r, n). \end{aligned}$$

(ii) Using formula (7) and Table 1, we have

$$\begin{aligned}
 (JH_n^r)^2 &= J^2(r, n) - J^2(r, n+1) + J^2(r, n+3) \\
 &\quad + 2\mathbf{i}J(r, n)J(r, n+1) + 2\varepsilon J(r, n)J(r, n+2) \\
 &\quad + 2\mathbf{h}J(r, n)J(r, n+3) + (\varepsilon\mathbf{i} + \mathbf{i}\varepsilon)J(r, n+1)J(r, n+2) \\
 &\quad + (\mathbf{i}\mathbf{h} + \mathbf{h}\mathbf{i})J(r, n+1)J(r, n+3) \\
 &\quad + (\varepsilon\mathbf{h} + \mathbf{h}\varepsilon)J(r, n+2)J(r, n+3) \\
 &= J^2(r, n) - J^2(r, n+1) + J^2(r, n+3) + 2J(r, n+1)J(r, n+2) \\
 &\quad + 2(\mathbf{i}J(r, n)J(r, n+1) + \varepsilon J(r, n)J(r, n+2) \\
 &\quad + \mathbf{h}J(r, n)J(r, n+3)) \\
 &= -J^2(r, n) - J^2(r, n+1) + J^2(r, n+3) \\
 &\quad + 2J(r, n+1)J(r, n+2) + 2J(r, n)JH_n^r \\
 &= 2J(r, n)JH_n^r - C(JH_n^r).
 \end{aligned}$$

□

We will present the Binet's formula for the $J(r, n)$ -Jacobsthal hybrid numbers.

THEOREM 3.6. *Let $n \geq 0$, $r \geq 0$ be integers. Then*

$$\begin{aligned}
 JH_n^r &= C_1\lambda_1^n (1 + \mathbf{i}\lambda_1 + \varepsilon\lambda_1^2 + \mathbf{h}\lambda_1^3) \\
 &\quad + C_2\lambda_2^n (1 + \mathbf{i}\lambda_2 + \varepsilon\lambda_2^2 + \mathbf{h}\lambda_2^3),
 \end{aligned}$$

where

$$C_1 = \frac{\sqrt{4 \cdot 2^r + 5 \cdot 4^r} + 3 \cdot 2^r + 2}{2\sqrt{4 \cdot 2^r + 5 \cdot 4^r}},$$

$$C_2 = \frac{\sqrt{4 \cdot 2^r + 5 \cdot 4^r} - 3 \cdot 2^r - 2}{2\sqrt{4 \cdot 2^r + 5 \cdot 4^r}}$$

and

$$\lambda_1 = 2^{r-1} + \frac{1}{2}\sqrt{4 \cdot 2^r + 5 \cdot 4^r},$$

$$\lambda_2 = 2^{r-1} - \frac{1}{2}\sqrt{4 \cdot 2^r + 5 \cdot 4^r}.$$

Proof. By Theorem 2.1 we get

$$J(r, n) = C_1\lambda_1^n + C_2\lambda_2^n.$$

Hence

$$\begin{aligned}
 JH_n^r &= J(r, n) + \mathbf{i}J(r, n+1) + \varepsilon J(r, n+2) + \mathbf{h}J(r, n+3) \\
 &= C_1\lambda_1^n + C_2\lambda_2^n + \mathbf{i}(C_1\lambda_1^{n+1} + C_2\lambda_2^{n+1}) \\
 &\quad + \varepsilon(C_1\lambda_1^{n+2} + C_2\lambda_2^{n+2}) \\
 &\quad + \mathbf{h}(C_1\lambda_1^{n+3} + C_2\lambda_2^{n+3}) \\
 &= C_1\lambda_1^n(1 + \mathbf{i}\lambda_1 + \varepsilon\lambda_1^2 + \mathbf{h}\lambda_1^3) \\
 &\quad + C_2\lambda_2^n(1 + \mathbf{i}\lambda_2 + \varepsilon\lambda_2^2 + \mathbf{h}\lambda_2^3),
 \end{aligned}$$

which ends the proof. \square

THEOREM 3.7. *Let $n \geq 1$, $r \geq 0$ be integers. Then*

$$\begin{aligned}
 \sum_{l=0}^{n-1} JH_l^r &= \frac{JH_n^r + (2^r + 4^r)JH_{n-1}^r - (2 + 2^r)(1 + \mathbf{i} + \varepsilon + \mathbf{h})}{4^r + 2^{r+1} - 1} \\
 &\quad - (\mathbf{i} + \varepsilon(2 + 2^{r+1}) + \mathbf{h}(2^{r+2} + 3 \cdot 4^r + 2)).
 \end{aligned} \tag{9}$$

Proof. By the definition of the $J(r, n)$ -Jacobsthal hybrid numbers we have

$$\begin{aligned}
 \sum_{l=0}^{n-1} JH_l^r &= JH_0^r + JH_1^r + \cdots + JH_{n-1}^r \\
 &= J(r, 0) + \mathbf{i}J(r, 1) + \varepsilon J(r, 2) + \mathbf{h}J(r, 3) \\
 &\quad + J(r, 1) + \mathbf{i}J(r, 2) + \varepsilon J(r, 3) + \mathbf{h}J(r, 4) + \cdots \\
 &\quad + J(r, n-1) + \mathbf{i}J(r, n) + \varepsilon J(r, n+1) + \mathbf{h}J(r, n+2) \\
 &= J(r, 0) + J(r, 1) + \cdots + J(r, n-1) \\
 &\quad + \mathbf{i}[J(r, 1) + J(r, 2) + \cdots + J(r, n) + J(r, 0) - J(r, 0)] \\
 &\quad + \varepsilon[J(r, 2) + J(r, 3) + \cdots + J(r, n+1) + J(r, 0) \\
 &\quad + J(r, 1) - J(r, 0) - J(r, 1)] \\
 &\quad + \mathbf{h}[J(r, 3) + J(r, 4) + \cdots + J(r, n+2) \\
 &\quad + J(r, 0) + J(r, 1) + J(r, 2) \\
 &\quad - J(r, 0) - J(r, 1) - J(r, 2)].
 \end{aligned}$$

Using Theorem 2.4, we obtain

$$\begin{aligned}
 \sum_{l=0}^{n-1} JH_l^r &= \frac{1}{4^r + 2^{r+1} - 1} [J(r, n) + (2^r + 4^r)J(r, n-1) - 2 - 2^r \\
 &\quad + \mathbf{i}[J(r, n+1) + (2^r + 4^r)J(r, n) - 2 - 2^r] \\
 &\quad + \varepsilon[J(r, n+2) + (2^r + 4^r)J(r, n+1) - 2 - 2^r] \\
 &\quad + \mathbf{h}[J(r, n+3) + (2^r + 4^r)J(r, n+2) - 2 - 2^r]] \\
 &\quad - \mathbf{i} - \varepsilon(2 + 2^{r+1}) - \mathbf{h}(2^{r+2} + 3 \cdot 4^r + 2) \\
 &= \frac{1}{4^r + 2^{r+1} - 1} [J(r, n) + \mathbf{i}J(r, n+1) + \varepsilon J(r, n+2) \\
 &\quad + \mathbf{h}J(r, n+3) + (2^r + 4^r) \\
 &\quad (J(r, n-1) + \mathbf{i}J(r, n) + \varepsilon J(r, n+1) + \mathbf{h}J(r, n+2)) \\
 &\quad - (2 + 2^r)(1 + \mathbf{i} + \varepsilon + \mathbf{h})] \\
 &\quad - \mathbf{i} - \varepsilon(2^{r+1} + 2) - \mathbf{h}(2^{r+2} + 3 \cdot 4^r + 2) \\
 &= \frac{JH_n^r + (2^r + 4^r)JH_{n-1}^r - (2 + 2^r)(1 + \mathbf{i} + \varepsilon + \mathbf{h})}{4^r + 2^{r+1} - 1} \\
 &\quad - (\mathbf{i} + \varepsilon(2 + 2^{r+1}) + \mathbf{h}(2^{r+2} + 3 \cdot 4^r + 2)).
 \end{aligned}$$

□

In particular, we obtain the following formula for the Jacobsthal hybrid numbers.

COROLLARY 3.8. *Let $n \geq 1$ be an integer. Then*

$$\sum_{l=0}^{n-1} JH_l = \frac{JH_{n+1} - JH_1}{2}.$$

Proof. By formula (9) for $r = 0$ we have

$$\begin{aligned}
 \sum_{l=0}^{n-1} JH_l^0 &= \frac{JH_n^0 + 2JH_{n-1}^0 - 3(1 + \mathbf{i} + \varepsilon + \mathbf{h})}{2} \\
 &\quad - (\mathbf{i} + 4\varepsilon + 9\mathbf{h}) \\
 &= \frac{JH_{n+1}^0 - (3 + 5\mathbf{i} + 11\varepsilon + 21\mathbf{h})}{2}.
 \end{aligned}$$

Using the fact that $J_n(0) = J_{n+2}$ and $JH_0 = \mathbf{i} + \varepsilon + 3\mathbf{h}$, $JH_1 = 1 + \mathbf{i} + 3\varepsilon + 5\mathbf{h}$,

we get

$$\begin{aligned}
 \sum_{l=0}^{n-1} JH_l &= \frac{JH_{n+1} - (3 + 5\mathbf{i} + 11\varepsilon + 21\mathbf{h})}{2} + JH_0 + JH_1 \\
 &= \frac{JH_{n+1} - (3 + 5\mathbf{i} + 11\varepsilon + 21\mathbf{h}) + 2(1 + 2\mathbf{i} + 4\varepsilon + 8\mathbf{h})}{2} \\
 &= \frac{JH_{n+1} - (1 + \mathbf{i} + 3\varepsilon + 5\mathbf{h})}{2} = \frac{JH_{n+1} - JH_1}{2},
 \end{aligned}$$

which ends the proof. \square

THEOREM 3.9. *Let $m \geq 2, n \geq 1, r \geq 0$ be integers. Then*

$$\begin{aligned}
 2JH_{m+n}^r &= 2^r JH_{m-1}^r JH_n^r + (4^r + 8^r) JH_{m-2}^r JH_{n-1}^r \\
 &\quad + J(r, m+n) + J(r, m+n+2) - 2J(r, m+n+3) - J(r, m+n+6).
 \end{aligned}$$

Proof.

$$\begin{aligned}
 &2^r JH_{m-1}^r JH_n^r + (4^r + 8^r) JH_{m-2}^r JH_{n-1}^r \\
 &= 2^r (J(r, m-1) + \mathbf{i}J(r, m) + \varepsilon J(r, m+1) + \mathbf{h}J(r, m+2)) \\
 &\quad \cdot (J(r, n) + \mathbf{i}J(r, n+1) + \varepsilon J(r, n+2) + \mathbf{h}J(r, n+3)) \\
 &\quad + (4^r + 8^r) (J(r, m-2) + \mathbf{i}J(r, m-1) + \varepsilon J(r, m) + \mathbf{h}J(r, m+1)) \\
 &\quad \cdot (J(r, n-1) + \mathbf{i}J(r, n) + \varepsilon J(r, n+1) + \mathbf{h}J(r, n+2)) \\
 &= 2^r (J(r, m-1)J(r, n) + \mathbf{i}J(r, m-1)J(r, n+1) \\
 &\quad + \varepsilon J(r, m-1)J(r, n+2) + \mathbf{h}J(r, m-1)J(r, n+3) \\
 &\quad + \mathbf{i}J(r, m)J(r, n) - J(r, m)J(r, n+1) + (1 - \mathbf{h})J(r, m)J(r, n+2) \\
 &\quad + (\varepsilon + \mathbf{i})J(r, m)J(r, n+3) + \varepsilon J(r, m+1)J(r, n) \\
 &\quad + (\mathbf{h} + 1)J(r, m+1)J(r, n+1) - \varepsilon J(r, m+1)J(r, n+3) \\
 &\quad + \mathbf{h}J(r, m+2)J(r, n) - (\varepsilon + \mathbf{i})J(r, m+2)J(r, n+1) \\
 &\quad + \varepsilon J(r, m+2)J(r, n+2) + J(r, m+2)J(r, n+3)) \\
 &\quad + (4^r + 8^r) (J(r, m-2)J(r, n-1) + \mathbf{i}J(r, m-2)J(r, n) \\
 &\quad + \varepsilon J(r, m-2)J(r, n+1) + \mathbf{h}J(r, m-2)J(r, n+2) \\
 &\quad + \mathbf{i}J(r, m-1)J(r, n-1) - J(r, m-1)J(r, n) \\
 &\quad + (1 - \mathbf{h})J(r, m-1)J(r, n+1) + (\varepsilon + \mathbf{i})J(r, m-1)J(r, n+2) \\
 &\quad + \varepsilon J(r, m)J(r, n-1) + (\mathbf{h} + 1)J(r, m)J(r, n) \\
 &\quad - \varepsilon J(r, m)J(r, n+2) + \mathbf{h}J(r, m+1)J(r, n-1) - (\varepsilon + \mathbf{i})J(r, m+1)J(r, n) \\
 &\quad + \varepsilon J(r, m+1)J(r, n+1) + J(r, m+1)J(r, n+2)).
 \end{aligned}$$

By simple calculations and using Theorem 2.5 we get

$$\begin{aligned}
& 2^r JH_{m-1}^r JH_n^r + (4^r + 8^r) JH_{m-2}^r JH_{n-1}^r \\
= & 2^r J(r, m-1)J(r, n) + (4^r + 8^r)(J(r, m-2)J(r, n-1) \\
& + \mathbf{i}(2^r J(r, m-1)J(r, n+1) + (4^r + 8^r)J(r, m-2)J(r, n)) \\
& + \varepsilon(2^r J(r, m-1)J(r, n+2) + (4^r + 8^r)J(r, m-2)J(r, n+1)) \\
& + \mathbf{h}(2^r J(r, m-1)J(r, n+3) + (4^r + 8^r)J(r, m-2)J(r, n+2)) \\
& + \mathbf{i}(2^r J(r, m)J(r, n) + (4^r + 8^r)J(r, m-1)J(r, n-1)) \\
& + \varepsilon(2^r J(r, m+1)J(r, n) + (4^r + 8^r)J(r, m)J(r, n-1)) \\
& - \mathbf{h}(2^r J(r, m)J(r, n+2) + (4^r + 8^r)J(r, m-1)J(r, n+1)) \\
& - 2^r J(r, m)J(r, n+1) - (4^r + 8^r)J(r, m-1)J(r, n) \\
& + 2^r J(r, m+1)J(r, n+1) + (4^r + 8^r)J(r, m)J(r, n) \\
& + 2^r J(r, m)J(r, n+2) + (4^r + 8^r)J(r, m-1)J(r, n+1) \\
& + 2^r J(r, m+2)J(r, n+3) + (4^r + 8^r)J(r, m+1)J(r, n+2) \\
& + \mathbf{i}[2^r J(r, m)J(r, n+3) + (4^r + 8^r)J(r, m-1)J(r, n+2) \\
& - (2^r J(r, m+2)J(r, n+1) + (4^r + 8^r)J(r, m+1)J(r, n))] \\
& + \varepsilon[2^r J(r, m)J(r, n+3) + (4^r + 8^r)J(r, m-1)J(r, n+2) \\
& - (2^r J(r, m+2)J(r, n+1) + (4^r + 8^r)J(r, m+1)J(r, n)) \\
& + 2^r J(r, m+2)J(r, n+2) + (4^r + 8^r)J(r, m+1)J(r, n+1) \\
& - (2^r J(r, m+1)J(r, n+3) + (4^r + 8^r)J(r, m)J(r, n+2))] \\
& + \mathbf{h}[2^r J(r, m+1)J(r, n+1) + (4^r + 8^r)J(r, m)J(r, n) \\
& + (2^r J(r, m+2)J(r, n) + (4^r + 8^r)J(r, m+1)J(r, n-1))].
\end{aligned}$$

Using Theorem 2.5 again, we obtain

$$\begin{aligned}
 & 2^r JH_{m-1}^r JH_n^r + (4^r + 8^r) JH_{m-2}^r JH_{n-1}^r \\
 &= 2(J(r, m+n) + \mathbf{i}J(r, m+n+1) + \varepsilon J(r, m+n+2) + \mathbf{h}J(r, m+n+3)) \\
 &\quad - (J(r, m+n) + J(r, m+n+2) - 2J(r, m+n+3) - J(r, m+n+6)) \\
 &= 2JH_{m+n}^r \\
 &\quad - (J(r, m+n) + J(r, m+n+2) - 2J(r, m+n+3) - J(r, m+n+6)).
 \end{aligned}$$

Hence we get the result. \square

Next, we shall give the ordinary generating function of the $J(r, n)$ -Jacobsthal hybrid numbers.

THEOREM 3.10. *The generating function of the $J(r, n)$ -Jacobsthal hybrid number sequence $\{JH_n^r\}$ is*

$$G(t) = \frac{JH_0^r + (JH_1^r - 2^r JH_0^r)t}{1 - 2^r t - (2^r + 4^r)t^2}.$$

Proof. Assume that the generating function of the $J(r, n)$ -Jacobsthal hybrid number sequence $\{JH_n^r\}$ has the form $G(t) = \sum_{n=0}^{\infty} JH_n^r t^n$. Then

$$\begin{aligned}
 & (1 - 2^r t - (2^r + 4^r)t^2)G(t) \\
 &= (1 - 2^r t - (2^r + 4^r)t^2) \cdot (JH_0^r + JH_1^r t + JH_2^r t^2 + \dots) \\
 &= JH_0^r + JH_1^r t + JH_2^r t^2 + \dots \\
 &\quad - 2^r JH_0^r t - 2^r JH_1^r t^2 - 2^r JH_2^r t^3 - \dots \\
 &\quad - (2^r + 4^r)JH_0^r t^2 - (2^r + 4^r)JH_1^r t^3 - (2^r + 4^r)JH_2^r t^4 - \dots \\
 &= JH_0^r + (JH_1^r - 2^r JH_0^r)t,
 \end{aligned} \tag{10}$$

since $JH_n^r = 2^r JH_{n-1}^r + (2^r + 4^r)JH_{n-2}^r$ and the coefficients of t^n for $n \geq 2$ are equal to zero.

Moreover,

$$\begin{aligned}
 JH_0^r &= 1 + \mathbf{i}(2 \cdot 2^r + 1) + \varepsilon(3 \cdot 4^r + 2 \cdot 2^r) \\
 &\quad + \mathbf{h}(5 \cdot 8^r + 5 \cdot 4^r + 2^r), \\
 JH_1^r - 2^r JH_0^r &= 2^r + 1 + \mathbf{i}(4^r + 2^r) + \varepsilon(2 \cdot 8^r + 3 \cdot 4^r + 2^r) \\
 &\quad + \mathbf{h}(3 \cdot 16^r + 5 \cdot 8^r + 2 \cdot 4^r).
 \end{aligned}$$

\square

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Received February 23, 2020

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