Tatra Mountains Math. Publ. **5** (1995), 151–168

TATRA MOUNTAINS *Mathematical Publications*

UNDECIDABLE ELEMENTARY THEORIES OF CLASSES OF GENERALIZED PASCAL TRIANGLES

IVAN KOREC

Dedicated to Professor J. Jakubík on the occasion of his 70th birthday

ABSTRACT. Generalized Pascal triangles (GPT) are mappings of (some cofinite subsets of) $\mathbb{N}\times\mathbb{N}$ into finite sets associated to some finite algebras analogously as the classical Pascal triangle can be associated to the (infinite) algebra $\langle\mathbb{N};+,0\rangle$. Elementary theories of various classes of structures related to GPT are investigated. It is shown that even for GPT of very simple structure (e.g., nilpotent ones or GPT modulo 2) these theories are undecidable.

1. Introduction

Generalized Pascal triangles (GPT) can be used to describe computations of one-dimensional cellular automata from finite initial configurations. (However, they were originally defined in [K1] studying of the structure of real-time systolic trellis automata, see [CGS].) Every GPT is determined by a finite algebra $\mathcal{A}$ and a word w; they code the local transition function (and the quiescent state) and the initial configuration, respectively. Formal definitions are contained in the next section. We shall assume that elements of considered algebras are nonnegative integers. Then GPT are partial binary operations on $\mathbb{N}$; they are total if $|w|=1$.

We shall consider structures (with base set $\mathbb{N}$) which contain GPT among their basic (partial) operations, and classes of such structures. Besides a GPT, a considered structure may contain some further operations or relations. They always will be explicitly mentioned and will be the same for all structures of any considered class (only GPT will vary). Three very natural possibilities are to join either $+$ or $\leq$ or $\mathbf{s}$ (successor). However, sometimes these symbols are definable (in their usual meaning) from a GPT itself, and it is not important whether they are explicitly joined or not.

AMS Subject Classification (1991): Primary 03B25; Secondary 68Q80.
Key words: elementary theories, decidability, generalized Pascal triangles.

There are rather simple GPT G for which the arithmetic operations $+$, $\times$ are first order definable in $\langle \mathbb{N}\ G\rangle$; such GPT are, e.g., the Pascal's triangle modulo 6 or GPT$(\mathcal{A}, 1)$ for a 3–element algebra (see [K3], [K5]). Of course, the elementary theories of these structures are undecidable. Three simple sufficient conditions for decidability of the elementary theory of (the structure associated to) one GPT are

(i) $\mathrm{card}(\mathcal{A}) \leq 2$;
(ii) $\mathcal{A}$ is simple semilinear (i.e., all its GPT are definable in Presburger arithmetic), particularly, $\mathcal{A}$ is GPT-nilpotent (explained below);
(iii) the binary operation of $\mathcal{A}$ is addition modulo a prime.

It is not substantial whether only total GPT or also partial GPT are considered, and whether $+$ (or $\leq$ or $\mathbf{s}$) occurs in the considered structures (this statement is not obvious).

If a class of (structures associated to) GPT contains an element whose (elementary) theory is undecidable, then the whole class also has undecidable theory. However, below we shall mostly consider classes each element of which has decidable theory. Nevertheless, elementary theories of *classes* of such structures can be undecidable. In the theorems of Section 3 examples of such classes with fixed algebra $\mathcal{A}$ (only the initial word varies) will be given. They will cover all three decidability conditions mentioned above, and their combination with commutativity and other conditions. Various possibilities for additional operations or relations will also be considered.

The above mentioned classes obviously must be infinite, hence the role of partial GPT is substantial in the presented theorems. (There are only finitely many total GPT for a given algebra.) It would not be difficult to give theorems where the algebra is not fixed and only total GPT are considered, but they seem to be much less interesting.

2. Preliminaries

The symbol $\mathbb{N}$ will denote the set of nonnegative integers and $\mathbb{Z}$ the set of all integers. The set $\{0, 1, \ldots, k-1\}$ will be denoted by $\mathbb{N}_k$. If $\mathbf{A}$ is an alphabet (i.e., a finite nonempty set), then $\mathbf{A}^+$ will denote the set of all nonempty words in the alphabet $\mathbf{A}$. The symbol $\mathbf{A}^*$ will denote the set $\mathbf{A}^+ \cup \{\varepsilon\}$, where ε is the empty word. The length of a word w will be denoted $|w|$. The ith symbol of w will be denoted by $w(i)$; the starting symbol is $w(0)$ and hence the last symbol is $w(|w|-1)$. The reflection of a word w will be denoted w^{R}. Also some further usual notations from the theory of formal languages will be used. (Repeated concatenation must be distinguished from the power of an integer by the context.)

By an algebra we shall understand an algebra $\mathcal{A} = \langle \mathbf{A}; *, \mathrm{o}\rangle$ of signature $(2,0)$ which satisfies the identity $\mathrm{o} * \mathrm{o} = \mathrm{o}$. Usually we shall consider only finite algebras. The exceptions will be explicitly mentioned.

DEFINITION 2.1. (i) For every $n \in \mathbb{N} \setminus \{\emptyset\}$ we denote

$$\mathbb{D}_n = \bigl\{(x,y) \in \mathbb{N} \times \mathbb{N} \mid \ x + y \geq n - 1\bigr\}.$$

(ii) To every algebra $\mathcal{A} = \langle \mathbf{A}; *, \mathrm{o}\rangle$ and every word $w \in \mathbf{A}^+$ the function $G = \mathrm{GPT}(\mathbf{A}, w)$ will be the mapping $G : \mathbb{D}_{|w|} \to \mathbf{A}$ defined by

$$G(x,y) = \begin{cases} w(x)\,, & \text{if} \quad x + y = |w| - 1\,, \\ \mathrm{o} * G(0, y-1)\,, & \text{if} \quad x = 0,\ y \geq |w|\,, \\ G(x-1, 0) * \mathrm{o}\,, & \text{if} \quad y = 0,\ x \geq |w|\,, \\ G(x-1, y) * G(x, y-1)\,, & \text{if} \quad x + y \geq |w|\,,\ x > 0\,,\ y > 0\,. \end{cases}$$

(iii) Functions of the form $\mathrm{GPT}(\mathcal{A}, w)$ for a finite algebra $\mathcal{A}$ and a word $w \in \mathbf{A}^+$ will be called *generalized Pascal triangles* (abbreviation: GPT).

(iv) A function $G = \mathrm{GPT}(\mathcal{A}, w)$ will be called *nilpotent* if $G(x,y) = \mathrm{o}$ for all but finitely many pairs $(x,y) \in \mathbb{N} \times \mathbb{N}$. A finite algebra $\mathcal{A} = \langle \mathbf{A}; *, \mathrm{o}\rangle$ will be called GPT-*nilpotent* if all $\mathrm{GPT}(\mathcal{A}, w)$, $w \in \mathbf{A}^+$ are nilpotent.

Notice that a GPT G satisfying $G(x,y) = c$ for a constant c and all but finitely many pairs $(x,y) \in \mathbb{N}\times\mathbb{N}$ need not be nilpotent (for any algebra $\mathcal{A}$). The GPT-nilpotency of finite algebras is undecidable in general. A sufficient condition for GPT-nilpotency of $\mathcal{A} = \langle \mathbf{A}; *, \mathrm{o}\rangle$ is given by the identities $x * \mathrm{o} = \mathrm{o} * x = \mathrm{o}$.

An example of GPT can be found in Figure 1. The corresponding algebra is displayed in the top right hand corner. It is also clear from this figure how the coordinate axes x, y are oriented (x right downwards and y left-downwards). Numbers of rows and columns (hence not x, y) are displayed. Any value $G(x,y)$ is written into the unit square with the top vertex (x,y) (and the bottom vertex $(x+1, y+1)$).

We shall talk about *rows* and *columns* of GPT in the obvious way; rows are finite sequences or words, columns are infinite sequences. They will be enumerated so that (the occurrence) $G(x,y)$ will belong to the $(x+y)$th row and $(x-y)$th column. The 0th column will be called the *axis* of a GPT. The sequences

$$\bigl(G(x,y) \mid y = 0,1,2,\ldots\bigr) \quad \text{and} \quad \bigl(G(x,y) \mid x = 0,1,2,\ldots\bigr)$$

will be called the xth *left diagonal* and the yth *right diagonal*, of GPT G, respectively. All left and right diagonals of any GPT are ultimately periodic. The *left margin* and the *right margin of width* k of a GPT consists of its first k left or right diagonals, respectively (if k is not specified then width 1 is considered).

	o	L	R	0	1	2	3
o	o	o	o	o	o	o	o
L	o	L	o	L	3	L	L
R	o	o	R	o	o	o	o
0	o	o	R	0	3	0	0
1	o	o	2	2	3	3	1
2	o	o	3	3	2	2	2
3	o	o	R	0	1	1	2

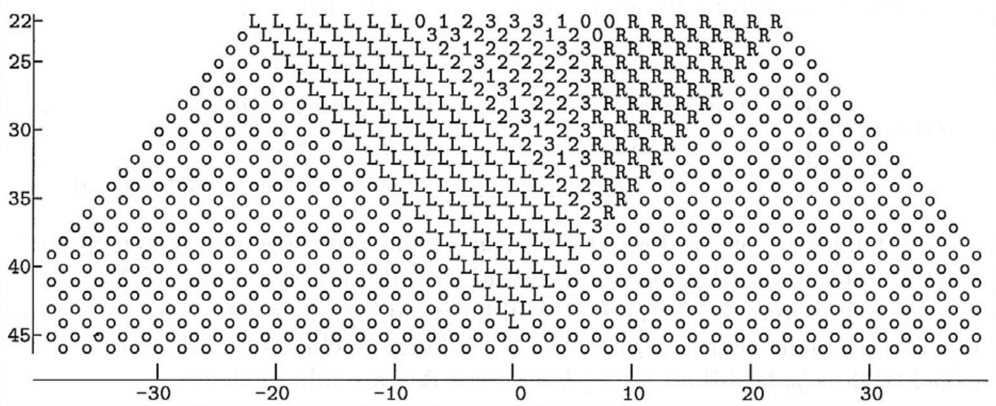

FIGURE 1.

For the notion of (first order) formulae, truth etc. we refer to [Mo] or to some other textbook of mathematical logic. However, sometimes we shall not distinguish between the function or relation and the corresponding symbol for it. The equality sign is considered as a logical symbol. If $\leq$ on $\mathbb{N}$ is defined in a considered structure, we shall also freely use $<$, $>$, $\geq$; moreover, since $0, 1, 2, \ldots$, the successor $\mathbf{s}$, and the predecessor $\mathbf{p}$ are also first order definable, we may use them, too. The letters v, w, x, y, z (possibly with subscripts) will be used as individual variables in the formulae, the symbols $\neg$, $\wedge$, $\vee$, $\Longrightarrow$, $\Longleftrightarrow$ as logical connectives (they are written in the order given by their priorities) and $\exists$, $\forall$ as quantifiers. Relativized quantifiers $(\forall x < y)$, $(\exists x \geq y)$ etc. and other usual abbreviations will also be used. The symbol $\exists!$ will denote "there is exactly one". The symbols $\bigwedge, \bigvee$ will abbreviate conjunction and disjunction of several formulae, respectively; to specify these formulae, metavariables $i, j, k, \ldots$ will also be used.

The symbol $\mathcal{L}(S_1, \ldots, S_r)$ will denote the first order language with nonlogical symbols $S_1, \ldots, S_r$; every S_i is either an operation symbol or a relation symbol, and has a fixed "arity". The kind of a symbol and its "arity" will be usually clear from the context.

By a class of structures we shall always understand a class of $\mathcal{L}$-structures for a first order language $\mathcal{L}$. If M is a $\mathcal{L}$-structure and $\alpha \in \mathcal{L}$, then $M \models \alpha$ denotes that α is true in M. For a class $\mathcal{M}$ of $\mathcal{L}$-structures $\mathcal{M} \models \alpha$ denotes that α is true in every $M \in \mathcal{M}$. The set $\{\alpha \in \mathcal{L} \mid M \models \alpha\}$ will be called the *elementary theory* (or: first order theory) of M; analogously for the class $\mathcal{M}$. *Inessential extensions* of a first order language $\mathcal{L}$ can be obtained by definition of new symbols; new symbols can be eliminated in principle. Definitions of new relation symbols must fulfil only some purely syntactical conditions. Definitions of operation symbols must fulfil also some semantical conditions which can restrict the class of structures where definitions can be applied. If a class of structures is considered, then the denotation of a defined symbol can depend on the considered structure. Sometimes it will depend on the length k of the initial word; to stress that, we shall use (a form of) the letter k in the defined symbol.

To avoid some technical difficulties we shall consider only the finite algebras $\mathcal{A} = \langle \mathbf{A}; *, \circ \rangle$ with the property $\mathbf{A} \subset \mathbb{N}$. (However, we shall not assume that $\mathbf{A}$ is an initial segment of $\mathbb{N}$.) There are several ways how to eliminate this restriction (e.g., to introduce heterogeneous structures with two base sets $\mathbb{N}$, $\mathbf{A}$) but we shall not discuss them in the present paper.

The mathematical structures associated to GPT will have the base set $\mathbb{N}$, and each of them will contain a GPT as a *partial* binary operation. The letter G will be used as binary function symbol for this GPT. Usually partial operations are not considered in first order predicate calculus, and GPT will be the only exceptions in what follows. We could avoid that if we replace a GPT by its graph considered as a ternary relation on $\mathbb{N}$ (this is a standard way how operation symbols can be eliminated). However, we prefer easily readable formulae with the functional symbol G (we shall not use G inside terms, so that no serious difficulties will arise). The formula $G(x,y) = \perp$ denotes that $G(x,y)$ is not defined ($\perp$ is not an element of the considered structure). If we use a ternary predicate instead of G then $G(x,y) = \perp$ can be replaced by an existential formula. Under some special conditions free formula can be used. It suffices when we have constants (given directly or definable by quantifier-free formulae) for all elements of a finite superset of the range of G.

For classes of structures associated to GPT the symbol $\mathcal{GPT}$ will be used with some superscripts and parameters in parentheses. Every special case will be explained when used.

Let us consider the first order language $\mathcal{L}(\oplus, \mathrm{P})$, where $\oplus$ is a binary operation symbol and P is a unary relation symbol. We shall need a result about $\mathcal{L}(\oplus, \mathrm{P})$-structures of the form $\langle \mathbb{N}_m; \oplus_m, P \rangle$ where $P \subseteq \mathbb{N}_m$ and $\oplus_m$ is defined by

$$x \oplus_m y = \begin{cases} x + y, & \text{if } \ x + y < m, \\ m - 1, & \text{otherwise}. \end{cases}$$

LEMMA 2.2. *The elementary theory of the class of structures*

$$\mathcal{P} = \{\langle \mathbb{N}_m; \oplus_m, P\rangle \mid m \in \mathbb{N}\setminus\{0\} \wedge P \subseteq \mathbb{N}_m\}$$

is undecidable.

The proof will be omitted because the result is known, at least in essential. It can be based on the coding of finite computations of Turing machines by the relations P. A reduction to [McK] can also be used.

3. Undecidability theorems

THEOREM 3.1. *There is a finite* GPT*-nilpotent algebra* $\mathcal{A}$ *such that the elementary theory of the class of structures*

$$\mathcal{GPT}^{\mathrm{s}}(\mathcal{A}) = \{\langle \mathbb{N}; \mathrm{GPT}(\mathcal{A}, w), \mathrm{s}\rangle \mid w \in \mathbf{A}^+\}$$

is undecidable.

Proof. Let $\mathcal{C} = \langle \mathbf{C}; \cdot, c_0\rangle$, $\mathbf{C} = \{c_i \mid 0 \leqq i \leqq r\}$ be an algebra such that the formal language

$$\mathrm{Nilp}(\mathcal{C}) = \{v \in \mathbf{C}^+ \mid \mathrm{GPT}(\mathcal{C}, v) \text{ is nilpotent}\}$$

is not recursive. (In other words, the problem whether $\mathrm{GPT}(\mathcal{C}, v)$ is nilpotent is recursively unsolvable.) An algebra with the desired property exists by Theorem 2.2 of [K-2] (and it can be effectively constructed).

The algebra $\mathcal{A}$ mentioned in the theorem will be constructed in the form $\mathcal{A} = \langle \mathbf{A}; *, \mathrm{o}\rangle$, where $\mathbf{A} = \mathbf{C} \cup \{L, R, \mathrm{o}\}$, the elements L, R, o do not belong to $\mathbf{C}$ and are pairwise distinct and $*$ is defined by the formula

$$x * y = \begin{cases} x \cdot y, & \text{if } x, y \in \mathbf{C}, \\ x, & \text{if } x = y \text{ and } x \notin \mathbf{C}, \\ x \cdot c_0, & \text{if } y = R,\ x \in \mathbf{C} \text{ and } x \cdot c_0 \neq c_0, \\ R, & \text{if } y = R,\ x \in \mathbf{C} \text{ and } x \cdot c_0 = c_0, \\ c_0 \cdot y, & \text{if } x = L,\ y \in \mathbf{C} \text{ and } c_0 \cdot y \neq c_0, \\ L, & \text{if } x = L,\ y \in \mathbf{C} \text{ and } c_0 \cdot y = c_0, \\ \mathrm{o}, & \text{otherwise}. \end{cases}$$

The operation is schematically given also in the following table

$*$	o	L	R	y
o	o	o	o	o
L	o	L	o	$c_0 \cdot y/L$
R	o	o	R	o
x	o	o	$x \cdot c_0/R$	$x * y$

where x, y denote arbitrary elements of $\mathbf{C}$ and z/X denotes z if $z \neq c_0$ and denotes X otherwise. An example (with $\mathbf{C} = \{0,1,2,3\}$) can be found in Figure 1.

If $w \in \mathbf{A}^+$ and $G = \mathrm{GPT}(\mathcal{A}, w)$ then obviously $G(x,y) = \mathrm{o}$ whenever $x \geqq |w|$ or $y \geqq |w|$. Therefore the algebra $\mathcal{A}$ is GPT-nilpotent. We shall show how to every word $v \in \mathbf{C}^+$ a formula φ_v can be constructed such that

$$v \notin \mathrm{Nilp}(\mathcal{C}) \quad \text{if and only if} \quad \mathrm{GPT}^{\mathrm{s}}(\mathcal{A}) \models \varphi_v \,. \tag{3.1.1}$$

Then we shall easily prove that $\mathcal{GPT}^{\mathrm{s}}(\mathcal{A})$ has undecidable elementary theory. To obtain the result in Remark 3.2 (concerning Δ_2 formulae) later, we shall try to construct φ_v as simple as possible (with respect to the quantifier prefix of its prenex normal form). We can define the constant 0, and then we can use terms of the form $\mathrm{s}^n(0)$ for elements of considered algebras; practically we prefer more free notation.

The formula φ_v will be constructed in the form $\alpha \wedge \beta_v \implies \gamma$, where α and γ do not depend on v. For every structure $\langle \mathbb{N}; \mathrm{GPT}(\mathcal{A}, w), \mathrm{s}\rangle$ the formulae α, β_v, γ, will express the following.

α: $w \in L^*\mathbf{C}^*R^*$;

β_v: LvR is a central subword of w;

γ: LR does not occur in (any row of) $\mathrm{GPT}(\mathcal{A}, w)$.

The formula γ is the most simple; it can be

$$\neg\exists x, y\Big(\big(G(x, \mathrm{s}(y)\big) = L \wedge G(\mathrm{s}(x), y) = R\Big)\,.$$

The formula α will express that none of the words o, RL, cL or Rc $(c \in \mathbf{C})$ is a subword of w; it can be

$$\begin{aligned}\neg\exists x, y\Big(G(x,y) = \bot \;\wedge\; \big(G(x, \mathrm{s}(y)\big) &= \mathrm{o} \vee G\big(\mathrm{s}(x), y\big) = \mathrm{o} \\ &\vee \big(G(x, \mathrm{s}(y)) = R \wedge G(\mathrm{s}(x), y) \in \mathbf{C}\big) \\ &\vee \big(G(x, \mathrm{s}(y)) \in \mathbf{C} \cup \{R\} \wedge G(\mathrm{s}(x), y) = L\big)\big)\Big)\,.\end{aligned}$$

Here $G(\mathrm{s}(x),y)\in\mathbf{C}$ is an abbreviation for $\bigvee_{i=0}^{r} G(\mathrm{s}(x),y)=c_i$, and analogously for $\mathbf{C}\cup\{R\}$. To explain the formulae α and γ, notice that $G(x,\mathrm{s}(y))$ and $G(\mathrm{s}(x),y)$ are neighbour elements of a row of G. The formula $G(x,y)=\bot$ expresses that the mentioned row is the initial one (i.e., w in essential).

To write the formula β_v, assume $v=v(0)v(1)\dots v(|v|-1)$, $v(i)\in\mathbf{C}$ for all $0\leqq i<|v|$. Then β_v can be

$$\exists x\Big(G(x,\mathrm{s}^{|v|}(x))=\bot\wedge G(x,\mathrm{s}^{|v|+1}(x))=L\wedge G(\mathrm{s}^{|v|+1}(x),x)=R\wedge \\ \wedge\bigwedge_{i=0}^{|v|-1} G(\mathrm{s}^{i+1}(x),\mathrm{s}^{|v|-i}(x))=v(i)\Big).$$

(Notice that the proof does not require that LvR is a *central* subword of w. However, one more quantifier would be necessary to express that LvR is an *arbitrary* subword of w.)

It remains to prove (3.1.1). For every $v\in\mathbf{C}^+$ we have

$$\langle\mathbb{N};\mathrm{GPT}(\mathcal{A},w),\mathrm{s}\rangle\models(\alpha\wedge\beta_v)\quad\text{if and only if}\quad\big(\exists k\in\mathbb{N}\setminus\{0\}\big)\,(w=L^kvR^k).$$

So it suffices to consider the words w of this form in what follows. Let us consider the relationship between the rows of $H=\mathrm{GPT}(\mathcal{C},v)$ and $G=\mathrm{GPT}(\mathcal{A},L^kvR^k)$ for a word $v\in C^+$ and integer $k\in\mathbb{N}$. Let us associate the initial row of H to the initial row of G (i.e., v to $w=L^kvR^k$ inessential). Further, let us associate the next row of H to the next row of G, etc. More formally (for every integer $n\geqq 1$) the $(|v|+n)$th row of H will be associated to the $(|w|+n)$th row of G. If a row of H belongs to $c_0^*uc_0^*$, where u is its substantial part (i.e., u is empty or $u(0)\neq c_0$ and $u(|u|-1)\neq c_0$), then the associated row of G belongs to $\mathrm{o}^*L^*c_0^*uc_0^*R^*\mathrm{o}^*$. Notice that LR can occur in a row of G only if u is empty. On the other hand, if u is empty and the groups of L and R are nonempty, then LR occurs in one of the consecutive rows of G.

If $\mathrm{GPT}(\mathcal{C},v)$ is not nilpotent then the word LR cannot occur in any row of any $\mathrm{GPT}(\mathcal{A},L^kvR^k)$ for any $k\in\mathbb{N}$. Therefore the formula γ is true in every structure $\langle\mathbb{N};\mathrm{GPT}(\mathcal{A},L^kvR^k),\mathrm{s}\rangle$, $k\in\mathbb{N}\setminus\{0\}$, and hence φ_v is true in $\mathcal{GPT}^{\mathrm{s}}(\mathcal{A})$.

If $\mathrm{GPT}(\mathcal{C},v)$ is nilpotent then for sufficiently large k the word LR is a subword of a row of $\mathrm{GPT}(\mathcal{A},L^kvR^k)$. Hence γ is false in the structure $\langle\mathbb{N};\mathrm{GPT}(\mathcal{A},L^kvT^k),\mathrm{s}\rangle$. Since $\alpha\wedge\beta_v$ is true in this structure the formula φ_v is false in it. Therefore φ_v is false in $\mathcal{GPT}^{\mathrm{s}}(\mathcal{A})$.

So (3.1.1) is proved. By the assumption the set $\mathrm{Nilp}(\mathcal{C})$ is not recursive. If the elementary theory of $\mathcal{GPT}^{\mathrm{s}}(\mathcal{A})$ is decidable then we would decide $v\in\mathrm{Nilp}$ so that we construct φ_v, and find out whether it belongs to the mentioned theory. This is a contradiction. $\square$

Remark 3.2. Let us consider the algebra $\mathcal{A}$ from Theorem 3.1, the language $\mathcal{L}(G, \mathrm{s}, 0)$ and the class

$$\mathcal{GPT}^{\mathrm{s},0}(\mathcal{A}) = \{\langle \mathbb{N}; \mathrm{GPT}(\mathcal{A}, w), \mathrm{s}, 0\rangle \mid w \in A^+\}$$

Then already the set of Δ_2-formulae of the elementary theory of $\mathcal{GPT}^{\mathrm{s},0}(\mathcal{A})$ is undecidable. To prove that, let us transform φ_v into prenex normal form. If we rename the bounded variables we shall transform φ_v into the formula

$$\neg\exists x_1 \exists y_1 \overline{\alpha} \wedge \exists x_2 \overline{\beta} \implies \neg\exists x_3 \exists y_3 \overline{\gamma},$$

where $\overline{\alpha}$, $\overline{\beta}$, $\overline{\gamma}$ are quantifier-free. By further transformation we can obtain

$$\forall x_3 \forall y_3 \forall x_2 \exists x_1 \exists y_1 (\neg\overline{\alpha} \wedge \overline{\beta} \implies \neg\overline{\gamma}) \quad \text{or} \quad \exists x_1 \exists y_1 \forall x_2 \forall x_3 \forall y_3 (\neg\overline{\alpha} \wedge \overline{\beta} \implies \neg\overline{\gamma});$$

there are also some other possibilities. Therefore φ_v is (equivalent with) a Π_2-formula and a Σ_2-formula, i.e., it is a Δ_2-formula of the language $\mathcal{L}(G, \mathrm{s}, 0)$. □

The successor operation s in Theorem 3.1 can be eliminated if we use the properties of domains of GPT in a suitable way. Later we shall also need similar results. Therefore now we shall investigate definability in the class

$$\mathcal{D} = \{\langle \mathbb{N}; \mathbb{D}_k\rangle \mid k \geqq 3\};$$

notice that $\mathbb{D}_k$ are domains of GPT; the condition $k \geqq 3$ is a technical one; it makes some constant (particularly, 0, 1) definable. We shall use the symbol Dk for (any of the binary relations) $\mathbb{D}_k$. In other words, $\mathcal{D}$ will be considered as a class of $\mathcal{L}(\mathrm{Dk})$-structures.

We shall define restricted versions of the usual relations and operations, like $\leqq$, s etc. The restrictions will depend on k, and therefore we shall add the letter k (sometimes as a subscript) to the commonly used symbols. If an expression in floors $\lfloor\ \rfloor$ occurs in a formula it ought to be considered as a unique symbol (the floors including). However, its usual meaning can be used as a mnemonics; $\lfloor\ \rfloor$ also denote "integer part" if necessary.

$$x \leqq_k z \iff \forall y(\mathrm{Dk}(x, y) \implies \mathrm{Dk}(z, y));$$
$$x <_k z \iff x \leqq_k z \wedge \neg z \leqq_k x.$$

Notice that $\leqq_k$ is a quasiordering; then classes of the corresponding equivalence relation are $\{0\}, \{1\}, \ldots, \{k-2\}$, $\{k-1, k, k+1, \ldots\}$. The relation $<_k$ is a partial ordering in which $0 <_k 1 <_k \cdots <_k k-2$ and $k-2 <_k x$ for every $x > k-2$; the distinct elements $x, y > k-2$ are incomparable. As usual, we shall also use $>_k$ and $\geqq_k$; notice that $x \leqq_k y$ does not mean $x <_k y \vee x = y$.

Further we can define:

$$\begin{aligned} x = 0 &\iff \forall y\,(x = y \vee x <_k y)\,; \\ x \prec_k y &\iff x <_k y \wedge \neg\exists z\,(x <_k z \wedge z <_k y)\,; \\ y = \mathbf{pk}(x) &\iff x = 0 \wedge y = 0 \vee x \prec_k y\,; \\ x = \lfloor k-2 \rfloor &\iff \exists y, z\,(y \neq z \wedge x \prec_k y \wedge x \prec_k z)\,; \\ y = \mathbf{sk}(x) &\iff x <_k \lfloor k-2 \rfloor \wedge x \prec_k y \vee \lfloor k-2 \rfloor \leqq_k x \wedge y = x\,; \end{aligned}$$

The functions $\mathbf{pk}$, $\mathbf{sk}$ are "k-restricted" predecessor and successor, respectively. Their values coincide with those of $\mathbf{p}$ and $\mathbf{s}$ for all $x < k-2$. Details for greater x must be read from the defining formulae.

$$1 = \mathbf{sk}(0)\,, \qquad \lfloor k-3 \rfloor = \mathbf{pk}\big(\lfloor k-2 \rfloor\big)\,.$$

These constants have their usual meaning provided the condition $k \geqq 3$ is fulfilled. (However, if we define $2 = \mathbf{sk}(1)$ or $\lfloor k-4 \rfloor = \mathbf{pk}\big(\lfloor k-3 \rfloor\big)$ then for $k = 3$ we would obtain $2 = 1$, $\lfloor k-4 \rfloor = \lfloor k-3 \rfloor$. For $k = 1$ even the above definition of 0 loses sense!)

$$\begin{aligned} y = \mathbf{mk}(x) \iff{}& x = 0 \wedge y = 0 \vee \\ &\vee x \neq 0 \wedge \mathrm{Dk}(x,y) \wedge \forall z\big(z <_k y \implies \neg\,\mathrm{Dk}(x,z)\big)\,. \end{aligned}$$

For $x > 0$ the expression $\mathbf{mk}(x)$ denotes the minimal y for which $(x,y) \in \mathbb{D}_k$. This value for $x = 0$ would be $k-1$. However, $k-1$ is not definable and therefore we choose $\mathbf{mk}(0) = 0$.

If $k \geqq 3$ is odd then the equation $\mathbf{mk}(x) = x$ has the unique positive solution $\frac{k-1}{2}$, and the equation $\mathbf{mk}(x) = \mathbf{sk}(x)$ has no solution. If k is even then the equation $\mathbf{mk}(x) = x$ has no positive solution, and the equation $\mathbf{mk}(x) = \mathbf{sk}(x)$ has the unique solution $\frac{k}{2} - 1 = \lfloor \frac{k-1}{2} \rfloor$. Therefore we can define:

$$x = \lfloor \tfrac{k-1}{2} \rfloor \iff x > 0 \wedge \mathbf{mk}(x) = x \vee \mathbf{mk}(x) = \mathbf{sk}(x)\,.$$

(The above definition can be modified for $k = 2$ but not for $k = 1$. Notice that for example the constant $\lfloor \frac{k}{3} \rfloor$ is not definable in a similar way.)

The above considerations can be summarized as follows; to be strict, the listed symbols ought to be defined more formally, but we shall not do that.

LEMMA 3.3. *The following predicates and functions are definable in the class* $\mathcal{D} = \big\{\langle \mathbb{N}; \mathbb{D}_k \rangle \mid k \geqq 3\big\}$:

$$<_k,\ \leqq_k,\ \prec_k,\ \mathbf{sk},\ \mathbf{pk},\ \mathbf{mk},\ 0,\ 1,\ \lfloor k-3 \rfloor,\ \lfloor k-2 \rfloor,\ \lfloor \tfrac{k-1}{2} \rfloor\,.$$

Since we can define $\mathrm{Dk}(x,y) \iff \exists z\big(G(x,y) = z\big)$ all mentioned predicates are also definable in the class $\big\{\langle \mathbb{N}; \mathrm{GPT}(\mathcal{A}, w)\rangle \mid w \in \mathbf{A}^+,\ |\,w\,| \geqq 3\big\}$.

Let us now try to eliminate $\mathbf{s}$ from Theorem 3.1, i.e., to prove that the elementary theory of

$$\mathcal{GPT}(\mathcal{A}) = \{\langle \mathbb{N}; \mathrm{GPT}(\mathcal{A}, w)\rangle \mid w \in A^+\}$$

is undecidable. The first idea is simply to replace $\mathbf{s}$ by $\mathbf{sk}$ in the formulae α, β_v, γ. If $\mathbf{s}(x) = \mathbf{sk}(x)$ for all $x \leqq \lfloor k-2 \rfloor$ it would be possible because the values of $\mathbf{s}$ for $x > \lfloor k-2 \rfloor$ were not used in essential. However, $\mathbf{s}(x) = \mathbf{sk}(x)$ only for $x < \lfloor k-2 \rfloor$, and the value $\mathbf{s}(k-2)$ was used. This difficulty can be avoided if we replace the condition $w \in L^*\mathbf{C}^*R^*$ by the condition $w \in \mathbf{A}L^*\mathbf{C}^*R^*\mathbf{A}$, and we ignore the margins of $\mathrm{GPT}(\mathcal{A}, w)$. The formula γ will be replaced by

$$\neg\exists x, y\big(G(\mathbf{sk}(x), \mathbf{sk}^2(y)) = L \wedge G(\mathbf{sk}^2(x), \mathbf{sk}(y)) = R\big)\,.$$

Analogously in α the terms $G(x, y)$, $G(x, \mathbf{s}(y))$ ought to be replaced by the terms $G(\mathbf{sk}(x), \mathbf{sk}(y))$ and $G(\mathbf{sk}(x), \mathbf{sk}^2(x))$, respectively, and similar changes must be made also in β_v. We shall not formulate this statement as a theorem because we shall provide a stronger result. Besides elimination of $\mathbf{s}$, we shall also arrange commutativity and idempotency of the algebra $\mathcal{A}$, and (the axial) symmetry of considered GPT.

THEOREM 3.4. *There is a finite algebra $\mathcal{B} = \langle \mathbf{B}; \star, 0\rangle$ which satisfies the identities*

$$x \star y = y \star x\,, \qquad 0 \star x = 0\,, \qquad x \star x = x\,, \tag{3.4.1}$$

and such that the elementary theory of the class

$$\mathcal{GPT}(\mathcal{B}) = \{\langle \mathbb{N}; \mathrm{GPT}(\mathcal{B}, w)\rangle \mid w \in \mathbf{B}^+ \quad \text{and} \quad w^{\mathrm{R}} = w\}$$

is undecidable.

Proof. Let $\mathcal{C}$ and $\mathcal{A}$ be like in Theorem 3.1 and its proof. Let us define

$$\mathbf{B} = \{0\} \cup \{1, 2, 3\} \times \mathbf{A}\,,$$

and let $\mathcal{B} = \langle \mathbf{B}; \star, 0\rangle$ where the operation $\star$ is defined as follows:

$$0 \star z = z \star 0 = 0 \quad \text{for every} \quad z \in \mathbf{B} \quad \text{and}$$

$$(i, x) \star (j, y) = \begin{cases} (i, x)\,, & \text{if} \quad i = j,\ x = y\,,\\ 0\,, & \text{if} \quad i = j,\ x \neq y\,,\\ (i, x * y)\,, & \text{if} \quad j \equiv i + 1 \pmod 3\,,\\ (j, y * x)\,, & \text{if} \quad i \equiv j + 1 \pmod 3 \end{cases}$$

for every $i, j \in \{1, 2, 3\}$ and $x, y \in \mathbf{A}$. (Provided $\mathbf{A} \subseteq \mathbb{N}$, the property $\mathbf{B} \subseteq \mathbb{N}$ can be arranged if (i, x) is replaced by $3x + i$.)

An example of $\mathcal{B}$ can be found in Figure 2. The algebra and GPT correspond to those given in Figure 1. Since it was necessary to use only one letter to denote an element, the pairs (i, o), $i = 1,2,3$ were replaced by o, p, q. The pairs belonging to L and R were replaced similarly by consecutive capitals. The other pairs (i, x) are replaced by the above convection, with hexadecimal digits A, B, C where (see also the column of pairs written by the right-hand side of the table of $\mathcal{B}$). Moreover, dots are used instead of 0 in the displayed GPT.

The constructed algebra satisfies all three identities from the theorem. We shall show only a less trivial case of commutativity. If $u = (i, x)$, $v = (j, y)$ and $i = j$ then $u \star v = 0 = v \star u$. Otherwise $j \equiv i + 1 \pmod 3$ or $i \equiv j+1 \pmod 3$; let us take the first case for example. Then $u \star v = (i, x * y) = v \star u$ by the fourth and the fifth line of the definition of $\star$, respectively.

Further proof uses the same idea as the proof of Theorem 3.1, although it is technically more complicated. We shall prove the exact analogy of (3.1.1) (with $\mathcal{B}$ instead $\mathcal{A}$). The formula φ_v also will be built similarly but its component α, β_v, γ and the form of w will be modified. To construct α, let us assume that the length $|v|$ of a word $v \in C^+$ is a multiple of 3 and that it is greater than the maximal element of $\mathbf{A}$. (This will enable us to define constants for all elements of $\mathbf{A}$. In fact, approximately one half of this value would suffice.) To arrange that, we can add several c_0 at the end of v if necessary. Let us define $\overline{v} \in \mathbf{B}^{|v|}$ by the formula

$$\overline{v}(i) = \big(i \,\mathrm{MOD}\, 3 + 1,\ v(i)\big) \quad \text{for all} \quad i < |v|\,,$$

and let us denote

$$L_i = (i, L)\,, \qquad R_i = (i, R)\,, \qquad \mathbf{C}_i = \{i\} \times \mathbf{C} \qquad \text{for} \quad i \in \{1,2,3\}\,.$$

A word w associated to v will be

$$0(L_1L_2L_3)^k\, \overline{v}(R_1R_2R_3)^m\, 0(R_3R_2R_1)^m\, \overline{v}^{\mathrm{R}}(L_3L_2L_1)^k\, 0\,, \quad k, m > 0\,.$$

(If we delete 0's here and forget the first components $i \in \{1,2,3\}$ in every element then the left half of w will be transformed into $L^k v R^m$. This word is very similar to $L^k v R^k$ used in the proof of Theorem 3.1.) Notice that $w^{\mathrm{R}} = w$ and

$$w \in 0(L_1L_2L_3)^+(\mathbf{C}_1\mathbf{C}_2\mathbf{C}_3)^+(R_1R_2R_3)^+0(R_3R_2R_1)^+(\mathbf{C}_3\mathbf{C}_2\mathbf{C}_1)^+(L_3L_2L_1)^+0\,;$$

conversely, if a word $w \in \mathbf{B}^+$ satisfies these conditions then it is associated to some $v \in \mathbf{C}^+$ (where $|v|$ is a multiple of 3). The formula α which expresses them is rather long. To make it shorter, let us define

$$\begin{aligned} y = W(x) \iff\ & 0 <_k x \wedge x <_k \lfloor k-2 \rfloor \wedge y = \\ & = G\big(x, \mathbf{mk}(x)\big) \vee (x = 0 \vee x \geqq_k \lfloor k-2 \rfloor) \wedge y = 0\,. \end{aligned}$$

	0	o	p	q	L	M	N	R	S	T	1	2	3	4	5	6	7	8	9	A	B	C	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
o	0	o	o	q	0	o	q	0	o	q	0	o	q	0	o	q	0	o	q	0	o	q	(1,o)
p	0	o	p	p	o	0	p	o	0	p	o	0	p	o	0	p	o	0	p	o	0	p	(2,o)
q	0	q	p	q	q	p	0	q	p	0	q	p	0	q	p	0	q	p	0	q	p	0	(3,o)
L	0	0	o	q	L	L	N	0	o	q	0	L	q	0	A	q	0	L	q	0	L	q	(1,L)
M	0	o	0	p	L	M	M	o	0	p	o	0	M	o	0	B	o	0	M	o	0	M	(2,L)
N	0	q	p	0	N	M	N	q	p	0	N	p	0	C	p	0	N	p	0	N	p	0	(3,L)
R	0	0	o	q	0	o	q	R	R	T	0	o	T	0	o	9	0	o	C	0	o	T	(1,R)
S	0	o	0	p	o	0	p	R	S	S	R	Q	p	7	0	p	A	0	p	R	0	p	(2,R)
T	0	q	p	0	q	p	0	T	S	T	q	S	0	q	8	0	q	B	0	q	S	0	(3,R)
1	0	0	o	q	0	o	N	0	R	q	1	1	3	0	A	9	0	1	C	0	1	3	(1,0)
2	0	o	0	p	L	0	p	o	0	S	1	2	2	7	0	B	A	0	2	1	0	2	(2,0)
3	0	q	p	0	q	M	0	T	p	0	3	2	3	C	8	0	3	B	0	3	2	0	(3,0)
4	0	0	o	q	0	o	C	0	7	q	0	7	C	4	A	C	0	A	9	0	4	6	(1,1)
5	0	o	0	p	A	0	p	o	0	8	A	0	8	A	5	B	7	0	B	4	0	5	(2,1)
6	0	q	p	0	q	B	0	9	p	0	9	B	0	C	B	6	C	8	0	6	5	0	(3,1)
7	0	0	o	q	0	o	N	0	A	q	0	A	3	0	7	C	7	7	9	0	7	6	(1,2)
8	0	o	0	p	L	0	p	o	0	B	1	0	B	A	0	8	7	8	8	4	0	8	(2,2)
9	0	q	p	0	q	M	0	C	p	0	C	2	0	9	B	0	9	8	9	9	5	0	(3,2)
A	0	0	o	q	0	o	N	0	R	q	0	1	3	0	4	6	0	4	9	A	7	9	(1,3)
B	0	o	0	p	L	0	p	o	0	S	1	0	2	4	0	5	7	0	5	7	B	8	(2,3)
C	0	q	p	0	q	M	0	T	p	0	3	2	0	6	5	0	6	8	0	9	8	C	(3,3)

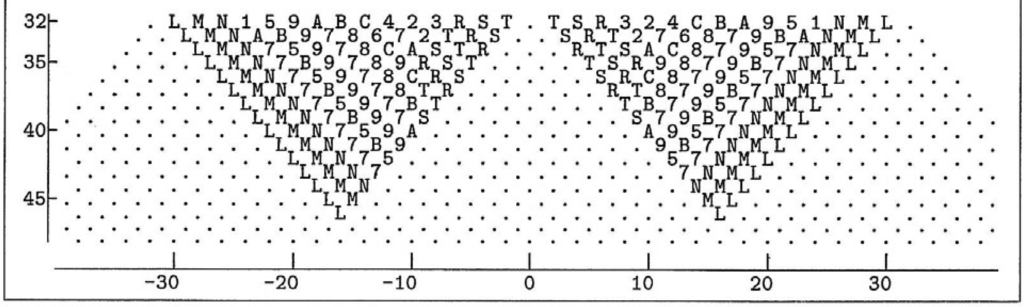

FIGURE 2.

Hence $W(x)$ is the xth symbol of w provided $0 < x < |w| - 1$; the other values of W are not substantial. Then α can be:

$$\begin{aligned}
&\forall x,y\big(G(x,y)=G(y,x)\big)\wedge\forall x\big(G(x,0)=\bot\vee G(x,0)=0\big)\wedge W(1)=L_1\wedge\\
&\quad\exists x\big(x\neq 0\wedge x=\mathbf{mk}(x)\big)\wedge W\big(\lfloor\tfrac{k-1}{2}\rfloor\big)=0\wedge W\big(\mathbf{pk}(\lfloor\tfrac{k-1}{2}\rfloor)\big)=R_3\wedge\\
&\quad\forall x\Big(0<_k x\wedge x<_k\lfloor\tfrac{k-1}{2}\rfloor\implies\\
&\qquad\implies\Big(W(x)=L_1\implies W\big(\mathbf{sk}(x)\big)=L_2\wedge W\big(\mathbf{sk}^2(x)\big)=L_3\wedge\\
&\qquad\qquad\wedge W\big(\mathbf{sk}^3(x)\big)\in\{L_1\}\cup\mathbf{C}_1\big)\\
&\qquad\wedge\big(W(x)\in\mathbf{C}_1\implies W\big(\mathbf{sk}(x)\big)\in\mathbf{C}_2\wedge W\big(\mathbf{sk}^2(x)\big)\in\mathbf{C}_3\wedge\\
&\qquad\qquad\wedge W\big(\mathbf{sk}^3(x)\big)\in\mathbf{C}_1\cup\{R_1\}\big)\\
&\qquad\wedge\big(W(x)=R_1\implies W\big(\mathbf{sk}(x)\big)=R_2\wedge W\big(\mathbf{sk}^2(x)\big)=R_3\wedge\\
&\qquad\qquad\wedge W\big(\mathbf{sk}^3(x)\big)\in\{R_1,0\}\Big)\Big).
\end{aligned}$$

The first line expresses that G is symmetric (hence $w^{\mathrm{R}}=w$) and that $0L_1$ is an initial segment of w. The second line expresses that $|w|$ is odd and that $R_3 0 R_3$ is a central subword of w. The rest expresses the structure of the left half of w.

The formula γ can be

$$\neg\exists x,y\Big(x\neq 0\wedge\mathbf{sk}(x)<_k y\wedge\bigvee_{i=1}^{3}G\big(x,\mathbf{sk}(y)\big)=L_i\wedge\bigvee_{j=1}^{3}G\big(\mathbf{sk}(x),y\big)=R_j\Big).$$

It expresses that the left half of $\mathrm{GPT}(\mathcal{B},w)$ does not contain any word L_iR_j, $i,j\in\{1,2,3\}$. Analogously the formula β_v can be

$$\begin{aligned}
&\exists x\Big(x\neq 0\wedge\mathbf{sk}^{|v|}(x)<_k\lfloor\tfrac{k-1}{2}\rfloor\wedge W(x)=L_3\wedge W\big(\mathbf{sk}^{|v|+1}(x)\big)=R_1\wedge\\
&\qquad\wedge\bigwedge_{i=0}^{|v|-1}W\big(\mathbf{sk}^{i+1}(x)\big)=\overline{v}(i)\Big).
\end{aligned}$$

It expresses that the left half of w contains $L_3\overline{v}R_1$ (or LvR if we delete the first components).

So the construction of $\varphi_v=(\alpha\wedge\beta_v)\implies\gamma$ is finished. The proof can continue similarly as the proof of Theorem 3.1. □

If we have the addition in the considered structures, and (at least) two elements in the considered algebra, then we can obtain an undecidability result without any further conditions on the algebra:

THEOREM 3.5. *For every algebra $\mathcal{A}=\langle\mathbf{A};*,\circ\rangle$ of cardinality ≥ 2 the elementary theory of the class of structures*

$$\mathcal{GPT}^{+}(\mathcal{A})=\big\{\langle\mathbb{N};\mathrm{GPT}(\mathcal{A},w),+\rangle\mid w\in A^{+}\big\}$$

is undecidable.

P r o o f . We shall interpret (the elementary theory of) the class $\mathcal{P}$ from Lemma 2.2 into (the elementary theory of) the class $\mathcal{GPT}^+(\mathcal{A})$. Let the element o in the initial row w of $\mathrm{GPT}(\mathcal{A}, w)$ be informally understood as "false" and let all other elements be understood as "true". To every structure $\langle \mathbb{N}_m; \oplus_m, P\rangle \in \mathcal{P}$ the structures $\langle \mathbb{N}; \mathrm{GPT}(\mathcal{A}, w), +\rangle$ will be associated, where

$$w \in \mathbf{A}^m \quad \text{and for all } i < m \text{ it holds } w(i) = \mathrm{o} \text{ if and only if } i \notin P .$$

(For $\mathrm{card}(\mathbf{A}) > 2$ many structures are associated to one; it is not substantial.) Let us extend the language $\mathcal{L}(G, +)$ by the definitions

$$x = \lfloor m-1 \rfloor \iff G(x,0) \neq \bot \land (\forall y < x)\big(G(y,0) = \bot\big) ,$$
$$\mathrm{P}^*(x) \iff \exists y\big(x + y = \lfloor m-1 \rfloor \land G(x,y) \neq \mathrm{o}\big) .$$

Every formula $\alpha \in \mathcal{L}(\oplus, \mathrm{P})$ can be transformed into a formula α^* of (an inessential extension of) $\mathcal{L}(G, +)$ so that (every occurrence of) $\oplus$ will be replaced by $+$, the symbol P will be replaced by P^* and the quantifiers will be relativized to "$\leq \lfloor m-1 \rfloor$", i.e.,

$$[\forall x\alpha]^* = \big(\forall x \leq \lfloor m-1 \rfloor\big)\alpha^* \quad \text{and} \quad [\exists x\alpha]^* = \big(\exists x \leq \lfloor m-1 \rfloor\big)\alpha^* .$$

Then for every closed formula $\alpha \in \mathcal{L}(\oplus, \mathrm{P})$ we have

$$\mathcal{P} \models \alpha \quad \text{if and only if} \quad \mathcal{GPT}^+(\mathcal{A}) \models \alpha^* .$$

Since the elementary theory of $\mathcal{P}$ is undecidable, the elementary theory of the class $\mathcal{GPT}^+(\mathcal{A})$ is undecidable, too. □

There is also a two element algebra for which $+$ in Theorem 3.5 can be omitted. It is the algebra $\mathcal{N}_2 = \langle \mathbb{N}_2; \oplus, 0\rangle$ where $\oplus$ is the addition modulo 2. (Notice that $\mathcal{N}_2$ is not GPT-nilpotent.)

THEOREM 3.6. *The elementary theory of the class of structures*

$$\mathcal{GPT}(\mathcal{N}_2) = \big\{\langle \mathbb{N}; \mathrm{GPT}(\mathcal{N}_2, w)\rangle \mid w \in \{0,1\}^+\big\}$$

is undecidable.

P r o o f . We shall interpret the class $\mathcal{P}$ from Lemma 2.2 into the class $\mathcal{GPT}(\mathcal{N}_2)$. The left halves of the initial words will code unary relations P and their right halves (and corresponding margins of GPT) will enable us to define addition of small nonnegative integers. To every structure $\langle \mathbb{N}_m; \oplus_m, P\rangle \in \mathcal{P}$ the structure $\langle \mathbb{N}; \mathrm{GPT}(\mathcal{N}_2, w)\rangle$ will be associated, where $w \in \{0,1\}^{2m+1}$ is defined by

$$w(i) = \begin{cases} 0, & \text{if } i < m \text{ and } i \notin P, \\ 1, & \text{if } i < m \text{ and } i \in P, \\ 1, & \text{if } i = m \text{ or } i = 2m, \\ 0, & \text{if } m < i < 2m. \end{cases}$$

We can also write $w = v10^{|v|-1}1$ where v is a code of P: let $v(i) = 1$ if and only if $i \in P$ is true. More technically, we shall construct a closed formula $\beta \in \mathcal{L}(G)$ and to every closed formula $\alpha \in \mathcal{L}(\oplus, P)$ a closed formula $\alpha^* \in \mathcal{L}(G)$ such that (for every α)

$$\mathcal{P} \models \alpha \quad \text{if and only if} \quad \mathcal{GPT}(\mathcal{N}_2) \models (\beta \implies \alpha^*)\,. \tag{3.6.1}$$

We shall use the symbols from Lemma 3.3 in the construction; by this we understand $k = |w|$, as above.

The formula β will express that the form of w is appropriate; it can be

$$\exists x\big(x \neq 0 \wedge x = \mathbf{mk}(x) \wedge G(x,x) = 1\big) \wedge$$
$$\big(\forall x >_k \lfloor \tfrac{k-1}{2} \rfloor\big)\big(x \leq_k \lfloor k{-}2 \rfloor \wedge G\big(x, \mathbf{mk}(x)\big) = 0 \vee x >_k \lfloor k{-}2 \rfloor \wedge G(x,0) = 1\big)\,.$$

The first member expresses that $k = |w|$ is odd and greater than 1, and that the middle symbol of w is equal to 1. The second member expresses the form of the right-hand part of w. To express the property $w(k-1) = 1$, the whole right margin of GPT $(\mathcal{A}, w)$ is mentioned.

To construct the formulae α^* we shall define unary relation symbol Nk^*, Pk^* and a binary operation symbol $\oplus_k^*$ which will correspond to the set $\mathbb{N}_m$ (no relation symbol corresponds to it in $\mathcal{L}(\oplus, \mathrm{P})$) and the symbols P, $\oplus$, respectively.

$$\mathrm{Nk}^*(x) \iff x < \lfloor \tfrac{k-1}{2} \rfloor\,;$$
$$\mathrm{Pk}^*(x) \iff \mathrm{Nk}^* \wedge \big(x = 0 \wedge \big(\forall y >_k \lfloor k-2 \rfloor\big)\big(G(0,y) = 1\big) \vee$$
$$x >_k 0 \wedge G\big(x, \mathbf{mk}(x)\big) = 1\big)\,.$$

The definition of $\oplus_k^*$ will be more complicated. We shall define the operation

$$x \oplus_k^* y = \begin{cases} x + y\,, & \text{if} \quad x + y < \lfloor \frac{k-1}{2} \rfloor\,, \\ \lfloor \frac{k-3}{2} \rfloor\,, & \text{otherwise}\,. \end{cases}$$

Since $\lfloor \frac{k-1}{2} \rfloor = m$ we shall have $x \oplus_m y = x \oplus_k^* y$ for all $x, y \in \mathbb{N}_m$. However, the above definition cannot be immediately rewritten into a defining formula because it contains $+$.

To write the suitable definition we shall use the method which was used in [K4] to define $+$ in the structure $\langle \mathbb{N}; \mathrm{GPT}(\mathcal{N}_2, 1), \mathbf{s}\rangle$. There the symbols $\sqsubseteq_2$ and Pow_2 were defined at first, where $x \sqsubseteq_2 y$ means that the binary digits of x are less than or equal to the corresponding binary digits of y, and $\mathrm{Pow}_2(x)$ means that x is a power of two. Now we shall define the relativizations of the above symbols to the relation symbol Nk^* (which corresponds to the set $\mathbb{N}_m$):

$$y_1 \sqsubseteq_2^k y_2 \iff \wedge\, \mathrm{Nk}^*(y_1) \wedge \mathrm{Nk}^*(y_2) \wedge$$
$$\wedge \big(\forall x >_k \lfloor k-2 \rfloor\big) \big(G(x, y_1) = 0 \implies G(x, y_2) = 0\big);$$
$$\mathrm{Pow}_2^k(y) \iff \mathrm{Nk}^*(y) \wedge y \neq 0 \wedge \forall z\big(z \sqsubseteq_2^k y \implies z = 0 \vee z = y\big)\,.$$

It is important for these definitions that the right margin of width $\lfloor \frac{k-1}{2} \rfloor$ of $B_2 = \mathrm{GPT}(\mathcal{N}_2, 1)$ (i.e., of the Pascal triangle modulo 2) is contained in the (shifted) right margin of $G = \mathrm{GPT}(\mathcal{A}, w)$ of the same width. (Formally, $B_2(x,y) = G(x+k, y)$ for all $x \geq 0$, $y < \lfloor \frac{k-1}{2} \rfloor$.)

Since the usual order of Pow_2^k is given by $<_k$ we can define the relation

$$X_k = \big\{(y_1, y_2, y_3, z) \in \mathrm{Nk}^* \times \mathrm{Nk}^* \times \mathrm{Nk}^* \times \mathrm{Nk}^* \mid y_3 = y_1 + y_2 \quad \text{and} \quad z \text{ is the corresponding vector of carries}\big\}.$$

(Notice that the vector of carries is never bigger than the sum.) The formula which defines X_k can be written by [K4]. We shall not give it here, we only remember that it describes the usual algorithm for addition in binary number system. Then we can define

$$y_3 = y_1 \oplus_k^* y_2 \iff \exists z\, X_k(y_1, y_2, y_3, z) \vee \vee \neg\exists x, z\, X_2(y_1, y_2, x, z) \wedge y_3 = \mathbf{pk}\Big(\lfloor \tfrac{k-1}{2} \rfloor\Big)\,.$$

Now we are prepared to define α^* to every formula $\alpha \in \mathcal{L}(P, \oplus)$. In atomic formulae we simply replace P and $\oplus$ by Pk^* and $\oplus_k^*$, respectively. For logical connectives we define

$$[\neg\alpha]^* = \neg\alpha^*, \ [\alpha \wedge \beta]^* = \alpha^* \wedge \beta^*, \quad \text{and similarly for} \quad \vee, \implies, \iff .$$

The quantifiers will be replaced by quantifiers relativized to Nk^*, i.e., (for arbitrary variable instead of x):

$$[\exists x\alpha]^* = \exists x\big(\mathrm{Nk}^*(x) \wedge \alpha^*\big), \quad [\forall x\alpha]^* = \forall x\big(\mathrm{Nk}^*(x) \implies \alpha^*\big)\,.$$

There is a one-to-one correspondence (described above) between the class $\mathcal{P}$ and the class $\{M \in \mathrm{GPT}(\mathcal{N}_2) \mid M \models \beta\}$; roughly speaking, elements of $\mathcal{P}$ are definable "substructures" of elements of the later class. If $M_2 \in \mathcal{GPT}(\mathcal{N}_2)$ corresponds to $M_1 \in \mathcal{P}$ then ($M_2 \models \beta$ and) for every closed formula $\alpha \in \mathcal{L}(\oplus, \mathrm{P})$ we have

$$M_1 \models \alpha \text{ if and only if } M_2 \models (\beta \implies \alpha^*)\,.$$

Then also $\mathcal{P} \models \alpha$ if and only if $\mathcal{GPT}(\mathcal{N}_2) \models (\beta \implies \alpha^*)$. The elementary theory of $\mathcal{P}$ is undecidable, and so is the elementary theory of $\mathcal{GPT}(\mathcal{N}_2)$. □

The author conjectures that the class $\big\{\mathrm{GPT}(\mathcal{N}_2, w) \mid w \in \mathbb{N}_2^+ \wedge w^{\mathrm{R}} = w\big\}$ can replace the class $\mathcal{GPT}(\mathcal{N}_2)$ in Theorem 3.6.

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Received February 28, 1994

Mathematical Institute
Slovak Academy of Sciences
Štefánikova 49
SK–814 73 Bratislava
SLOVAKIA

E-mail: korec@mau.savba.sk