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STRICTNESS OF $\mathcal{L}_0$–RING COMPLETIONS

LEONARD PAULÍK

Dedicated to Professor J. Jakubík on the occasion of his 70th birthday

ABSTRACT. We investigate strictness of the categorical (Novák) $\mathcal{L}_0$-ring completion and the impact of the Urysohn modification. We show that the rational numbers have $\exp\exp\omega$ nonstrict $\mathcal{L}_0^*$-field completions.

Background information on $\mathcal{L}$–structures and particularly on $\mathcal{L}$–rings and their completions can be found in [1] – [9]. For the reader's convenience we briefly recall some basic notions and notation.

By $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ we denote the set of all natural, rational and real numbers, respectively. The set of all mappings of $\mathbb{N}$ into $\mathbb{N}$ is denoted by NIN and the subset of all strictly monotone mappings is denoted by MON. By an *$\mathcal{L}_0$-space* $\boldsymbol{X} = (X, \mathscr{L})$ we understand a set X equipped with a sequential convergence $\mathscr{L} \subset X^{\mathbb{N}} \times X$ satisfying axioms of convergence concerning constants, subsequences and uniqueness of limits. In the case when the Urysohn axiom is not satisfied, we also assume that if $\langle x_n \rangle$ and $\langle y_n \rangle$ converge to the same point, then the mixed sequence $\langle x_1, y_1, x_2, y_2, \dots \rangle$ converges to the same point, and if we change finitely many points in a convergent sequence, this new sequence converges to the same point as the original one. The fact that a sequence $\langle x_n \rangle$ is $\mathscr{L}$–converging to x is denoted by $\big(\langle x_n \rangle, x\big) \in \mathscr{L}$. If $S = \big\langle S(n) \big\rangle$ is a sequence and $s \in \mathrm{MON}$, then $S \circ s$ denotes the subsequence of S the n-th term of which is $S\big(s(n)\big)$. If X is a commutative ring (with unit) and $\mathscr{L}$ is compatible with the ring structure (if sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ converge to x and y, respectively, then $\langle x_n - y_n \rangle$ converges to $x - y$ and $\langle x_n y_n \rangle$ converges to xy), then $(X, \mathscr{L})$ is called an *$\mathcal{L}_0$-ring.* We say that *S is Cauchy* if for each $s, t \in \mathrm{MON}$ the sequence $S \circ s - S \circ t$ converges to zero and $\boldsymbol{X}$ is said to be complete if for every Cauchy sequence $S \in X^{\mathbb{N}}$ there is $x \in X$ such that $(S, x) \in \mathscr{L}$. By a completion of an $\mathcal{L}_0$-ring we understand a complete $\mathcal{L}_0$-ring in which the original one is embedded as a subalgebra and a topologically dense subspace (iterated sequential closure).

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Let $\mathscr{M}$ be the usual metric convergence on $\mathbb{R}$ and $\mathscr{M} \upharpoonright \mathbb{Q}$ its restriction to $\mathbb{Q}$. Denote $\boldsymbol{Q} = (\mathbb{Q}, \mathscr{M} \upharpoonright \mathbb{Q})$.

Let $\boldsymbol{Y} = (Y, \mathscr{K})$ be an $\mathcal{L}_0$-space and let X be a dense subset (each $y \in Y$ is a limit of a sequence ranging in X). Consider the following two conditions:

(s) Let $(\langle y_n \rangle, y) \in \mathscr{K}$, $y_n \in Y \setminus X$, $n \in \mathbb{N}$. Then there are a subsequence $\langle y'_n \rangle$ of $\langle y_n \rangle$ and sequences S_k in X, $k \in \mathbb{N}$, such that $(\langle S_k(n) \rangle, y'_k) \in \mathscr{K}$, $k \in \mathbb{N}$, and for each $g \in \mathrm{NIN}$ we have $(\langle S_n(g(n)) \rangle, y) \in \mathscr{K}$;

(ss) Let $(\langle y_n \rangle, y) \in \mathscr{K}$, $y_n \in Y \setminus X$, $n \in \mathbb{N}$. Then there are sequences S_k in X, $k \in \mathbb{N}$, such that $(\langle S_k(n) \rangle, y_k) \in \mathscr{K}$, $k \in \mathbb{N}$, and for each $g \in \mathrm{NIN}$ we have $(\langle S_n(g(n)) \rangle, y) \in \mathscr{K}$.

Clearly, (ss) implies (s) and if Y is a first-countable topological space, then (ss) holds. Both conditions guarantee that the convergence of sequences ranging in $Y \backslash X$ is in certain sense controlled from X (cf. [7], [2]). If (s) is satisfied, then $\boldsymbol{X} = (X, \mathscr{K} \upharpoonright X)$ will be called a *strict subspace* of $\boldsymbol{Y}$ and $\boldsymbol{Y}$ will be called a *strict extension* of $\boldsymbol{X}$. If (ss) holds, then we will speak of a *strong strictness*.

Strictness in connection with completions of various classes of $\mathcal{L}_0$-groups (including $\mathcal{L}_0$-rings and $\mathcal{L}_0$-fields) has been investigated, e.g., in [7] and [8]. The present paper is devoted to the strictness of $\mathcal{L}_0$-rings. We start with examining the categorical (Novák) completion of $\mathcal{L}_0$-rings.

Let $\boldsymbol{X} = (X, \mathscr{L})$ be an $\mathcal{L}_0$-ring. Assume that $\boldsymbol{X}$ satisfies the following condition:

(Cr) if $\langle x_n \rangle$ is $\mathscr{L}$–converging to zero and $\langle y_n \rangle$ is a Cauchy sequence, then $\langle x_n y_n \rangle$ is $\mathscr{L}$–converging to zero.

Clearly, (Cr) is a necessary condition for $\boldsymbol{X}$ to have an $\mathcal{L}_0$-ring completion and, as shown in [5] (cf. [4]), if X is a field, then the condition is also sufficient.

THEOREM 1. *Let $\boldsymbol{X} = (X, \mathscr{L})$ be an $\mathcal{L}_0$-ring satisfying (Cr) and let X be a field. Then the categorical (Novák) $\mathcal{L}_0$-ring completion $\rho\boldsymbol{X} = (\rho X, \mathscr{L}_2)$ of $\boldsymbol{X}$ is strongly strict.*

P r o o f. Recall that ρX is the ring of all equivalence classes of Cauchy sequences (two Cauchy sequences are equivalent if their difference $\mathscr{L}$–converges to zero and each point $x \in X$ is identified with the equivalence class $[\langle x \rangle]$ of all sequences $\mathscr{L}$–converging to x). It can be considered as a vector space over the scalar field X; let $\{1\} \cup B$ be a Hamel basis of ρX over X. Further, $\mathscr{L}_2$ is defined as follows: $(\langle y_n \rangle, y) \in \mathscr{L}_2$ iff there are $m \in \mathbb{N}$, $\{b_1, \dots, b_m\} \subset B$ and Cauchy sequences $S_k \in X^{\mathbb{N}}$, $k = 0, 1, \dots, m$, such that

$$y_n = S_0(n) + S_1(n) b_1 + \dots + S_m(n) b_m, \quad n \in \mathbb{N},$$
$$y = [S_0] + [S_1] b_1 + \dots + [S_m] b_m,$$

where $[S_k]$ is the equivalence class of sequences containing S_k, $k = 0, 1, \dots, m$.

We have to verify condition (ss) for $Y = \rho X$ and $\mathscr{K} = \mathscr{L}_2$. Let $(\langle y_n \rangle, y) \in \mathscr{L}_2$ such that $y_n \in \rho X \backslash X$, $n \in \mathbb{N}$. We know that $b_k \in \rho X \backslash X$, $k = 1, \dots, m$. Since X is dense in ρX, there are sequences $T_k \in X^{\mathbb{N}}$ such that $(\langle T_k \rangle, b_k) \in \mathscr{L}_2$, $k = 1, \dots, m$. For each $k \in \mathbb{N}$, define the sequence $\mathbb{R}_k \in X^{\mathbb{N}}$ as follows

$$\mathbb{R}_k(n) = S_0(k) + S_1(k)T_1(n+k) + \cdots + S_m(k)T_m(n+k), \quad n \in \mathbb{N}.$$

Then each sequence $\langle \mathbb{R}_k(n) \rangle$ is $\mathscr{L}_2$-converging to y_k. Let $g \in \mathrm{NIN}$. Then the sequence $\langle \mathbb{R}_n(g(n)) \rangle$ is $\mathscr{L}_2$-converging to y. Hence ρX is a strongly strict completion of $\boldsymbol{X}$. □

Assume, moreover, that $\mathscr{L}$ satisfies the Urysohn axiom and the following condition (see [4]):

(Cq) Let $\langle S(n) \rangle$ be a sequence no subsequence of which is $\mathscr{L}$–Cauchy. Then there exist $s, t \in \mathrm{MON}$ such that no subsequence of the sequence $\langle S(s(n)) - S(t(n)) \rangle$ is $\mathscr{L}$–Cauchy.

Let $\mathscr{L}_2^*$ be the *Urysohn modification* of $\mathscr{L}_2$ (i.e., $(S, x) \in \mathscr{L}_2^*$ whenever for each $s \in \mathrm{MON}$ there exists $t \in \mathrm{MON}$ such that $(S \circ s \circ t, x) \in \mathscr{L}_2$). Then $(\rho X, \mathscr{L}_2^*)$ is the categorical (Novák) $\mathcal{L}_0^*$-ring completion of $\boldsymbol{X}$ (cf. [4]).

COROLLARY. *$(\rho X, \mathscr{L}_2^*)$ is a strict extension of $(X, \mathscr{L})$.*

R e m a r k . Constructions using Hamel basis are extensively used in the theory of convergence structures to define convergences compatible with the underlying linear space [3], [4], [5], [7] and [8]. The basic ideas of this method were explored in the pioneering work [9] of J. J a k u b í k .

It is known that the Urysohn modification has a non-trivial impact on the construction of a completion (see [4]). Comparing Theorem 1 and Corollary we see that there is a correlation between the Urysohn modification and the relationship between (s) and (ss). The "closer" $\mathscr{L}^*$ is to $\mathscr{L}$, the more likely (ss) follows from (s). Let $\boldsymbol{X} = (X, \mathscr{L})$ be a commutative $\mathcal{L}_0^*$-group and let $\nu\boldsymbol{X} = (\nu X, \mathscr{L}_1^*)$ be its categorical (Novák) $\mathcal{L}_0^*$-group completion. Observe that $(\nu X, \mathscr{L}_1^*)$ is a strongly strict extension of $(X, \mathscr{L})$ and $\mathscr{L}_1^*$ is rather "close" to $\mathscr{L}_1$. Indeed (cf. Lemma 3.3 in [7]), if $(\langle y_n \rangle, y) \in \mathscr{L}_1^*$ and $y_n \in \nu X \setminus X$, $n \in \mathbb{N}$, then there are $k \in \mathbb{N}$ and a finite nonvoid set $\{a_1, \dots, a_n\} \subset \nu X \setminus X$ such that each y_n belongs to some coset $X + a_i$, $i = 1, \dots, k$, and if $\langle y_{s(n)} \rangle$ is the subsequence of $\langle y_n \rangle$ ranging in the same coset $X + a_i$, then there is an $\mathscr{L}$–Cauchy sequence S_i in X such that $y_{s(n)} = y + S_i(n) - [S_i]$, $n \in \mathbb{N}$. Hence $\langle y_n \rangle$ is "mixed" of finitely many $\mathscr{L}_1$-convergent sequences (simple ones).

Our second result shows that in the case of the categorical (Novák) $\mathcal{L}_0^*$-ring completion the relationship between $\mathscr{L}_2$ and $\mathscr{L}_2^*$ is much more complicated (however, we do not know whether (s) does or does not imply (ss)).

Denote $\mathscr{L} = \mathscr{M} \restriction \mathbb{Q}$. Then $\rho\mathbb{Q} = \mathbb{R}$.

THEOREM 2. *Let $\rho Q = (\mathbb{R}, \mathscr{L}_2^*)$ be the categorical (Novák) completion of $Q = (\mathbb{Q}, \mathscr{L})$. Then there is a sequence*

$$S(n) = S_1(n)e + S_2(n)\pi\,, \quad n \in \mathbb{N},$$

such that $(S, 0) \in \mathscr{L}_2^*$ and there are non-degenerated closed intervals $I_i \subset \mathbb{R}$, $i = 1, 2$, such that each point of the interval I_i, $i = 1, 2$ is a limit point of some subsequence of S_i, $i = 1, 2$, respectively.

Proof. Arrange the set $\mathbb{Q} \cap (-1, 1)$ into a one-to-one sequence $\langle q_n \rangle$. Let $\langle d_n \rangle$ be a sequence of positive real numbers converging to zero. For each $n \in \mathbb{N}$, let $S_1(n)$, $S_2(n)$ be rational numbers such that

$$S_1(n) \in (q_n\pi - d_n\,, \quad q_n\pi + d_n)\,, \tag{1}$$

$$S_2(n) \in (-q_n e - d_n\,, -q_n e + d_n)\,. \tag{2}$$

Consider the sequence

$$S(n) = S_1(n)e + S_2(n)\pi\,, \quad n \in \mathbb{N}.$$

Since the numbers e and π are $\mathbb{Q}$-linearly independent, for infinitely many $n \in \mathbb{N}$ we have $S(n) \neq 0$. Let $s \in \mathrm{MON}$. Since the sequence $\langle q_{s(n)} \rangle$ is bounded, there exists $t \in \mathrm{MON}$ such that the sequence $\langle q_{s(t(n))} \rangle$ converges to some real number $r \in [-1, 1]$. From $d_n \to 0$ and (1), (2) we can conclude that the sequence $\langle S_1(s(t(n))) \rangle$ is $\mathscr{L}_2$-converging to $r\pi \in [-\pi, \pi]$ and the sequence $\langle S_2(s(t(n))) \rangle$ converges to $-re \in [-e, e]$. Hence $(S, 0) \in \mathscr{L}_2^*$.
Consider an arbitrary real number $r \in [-\pi, \pi]$. Then there is a subsequence $\langle q'_n \rangle$ of $\langle q_n \rangle$ such that $\langle q'_n \rangle$ is $\mathscr{M}$-converging to $r\pi^{-1}$ and hence $\langle q'_n \pi \rangle$ is converging to r. From $d_n \to 0$ and (1) it follows that there is $s \in \mathrm{MON}$ such that the sequence $S_1 \circ s$ is $\mathscr{L}_2$-converging to r. Hence each point of the interval $I_1 = [-\pi, \pi]$ is the limit point in ρQ of some subsequence of S_1. Analogously we can prove that each point of the interval $I_2 = [-e, e]$ is the limit point of some subsequence of S_2. □

Observe that if S is $\mathscr{L}_2^*$-Cauchy (e.g., $\mathscr{L}_2^*$-convergent), then there are $m \in \mathbb{N}$, a finite set $\{b_1, \dots, b_m\} \subset \rho X$ and sequences $S_i \in X^{\mathbb{N}}$, $i = 0, 1, \dots, m$, such that

$$S(n) = S_0(n) + S_1(n)b_1 + \dots + S_m(n)b_m\,, \quad n \in \mathbb{N},$$

but (as in Theorem 2) the sequences S_i can be very far from being $\mathscr{L}$–Cauchy.

On the other hand, since $(\mathbb{R}, \mathscr{M})$ is a strongly strict extension of $Q = (\mathbb{Q}, \mathscr{L})$, from $\mathscr{L}_2^* \subset \mathscr{M}$ it follows easily that $\rho Q = (\mathbb{R}, \mathscr{L}_2^*)$ is a strongly strict extension of Q, too.

PROBLEM 1. *Does there exist an $\mathcal{L}_0^*$-ring $\boldsymbol{X} = (X, \mathscr{L})$ the categorical (Novák) $\mathcal{L}_0^*$-ring completion $\rho \boldsymbol{X} = (\rho X, \mathscr{L}_2^*)$ of which fails to be strongly strict?*

Our final result concerns nonstrict completions of $\mathbb{Q}$. The notion of strictness in situations where a space is topologically dense (iterated closure) in its extension has been studied in [2]. In Problem 2.4. (ii) the author asked whether $\boldsymbol{Q}$ possesses a nonstrict $\mathcal{L}_0^*$-field completion in which $\mathbb{Q}$ is dense. We show that if $\mathbb{Q}$ has to be only topologically dense, then the answer to this question would be "yes". In fact, we do not know whether $\mathbb{Q}$ is in the completion dense or only topologically dense in our construction. The first sequential closure of $\mathbb{Q}$ in the completion yields an extension of $\boldsymbol{Q}$ which fails to be strict.

Consider $\mathbb{R}$ as a field extension of $\mathbb{Q}$ and the field $\mathbb{C}$ of complex numbers as an algebraically closed field extension of $\mathbb{Q}$. Let B be a transcendence basis of $\mathbb{R}$ over $\mathbb{Q}$ (i.e., a maximal set of algebraically independent elements of $\mathbb{R}$ over $\mathbb{Q}$, the cardinality of which is $\exp \omega$, cf. [10]). Without loss of generality we can assume that $2 < b < 3$ for all $b \in B$. Let $\{S_\alpha ; \alpha \in \exp \omega\}$ be a partition of B into disjoint infinite countable subset S_α of B. Consider each S_α as a one-to-one sequence. Let f be a mapping of $\exp \omega$ into $\{0, 1\}$.

LEMMA. *There is an $\mathcal{L}_0$-ring convergence $\mathscr{L}_f$ on $\mathbb{R}$ such that $\mathscr{L}_2^* \subset \mathscr{L}_f$, $\big(S_\alpha, f(\alpha)\big) \in \mathscr{L}_f$ for all $\alpha \in \exp \omega$ and for $S \in \mathbb{Q}^{\mathbb{N}}$ we have $(S, x) \in \mathscr{L}_f$ iff $(S, x) \in \mathscr{M}$.*

P r o o f . The assertion can be proved analogously as Lemma 3.3.1 in [8]. First, let $\mathscr{L}_f$ be the smallest $\mathcal{L}$–ring convergence on $\mathbb{R}$ such that $\mathscr{L}_2^* \subset \mathscr{L}_f$ and $\big(S_\alpha, f(\alpha)\big) \in \mathscr{L}_f$ for each $\alpha \in \exp \omega$. The existence of $\mathscr{L}_f$ is guaranteed by Lemma 1, Lemma 2 and Remark 4 in [6]. Indeed, $\mathscr{L}_f^{\leftarrow}(0) = \{S \in \mathbb{R}^{\mathbb{N}} ; (S, 0) \in \mathscr{L}_f\}$ consists of all sequences of the form $\sum\limits_{i=1}^{m} T(i,1) \dots T\big(i, k(i)\big)$, where $m, k(i) \in \mathbb{N}$, and $T(i,j)$ either $\mathscr{L}_2^*$-converges to zero, or it is of the form $\langle x \rangle \big(S_\alpha \circ s - f(\alpha)\big)$, $x \in \mathbb{R}$, $\alpha \in \exp \omega$, $s \in \mathrm{MON}$. Further, $\mathscr{L}_f$ has unique limits iff $\mathscr{L}_f^{\leftarrow}(0)$ does not contain any constant sequence except $\langle 0 \rangle$ (recall that $(S, x) \in \mathscr{L}_f$ iff $\big(S - \langle x \rangle, 0\big) \in \mathscr{L}_f$). So, suppose that $\langle a \rangle \in \mathscr{L}_f^{\leftarrow}(0)$, $a \in \mathbb{R}$. An analysis of the sequence of equations $a = \sum\limits_{i=1}^{m} T(i,1)(n) \dots T\big(i, k(i)\big)(n)$, $n \in \mathbb{N}$, reveals that $a = 0$. (Hint: passing to suitable subsequences of $T(i,j)$ we can guarantee that if $\big(T(i,j), 0\big) \in \mathscr{L}_2^*$ then $\big(T(i,j), 0\big) \in \mathscr{L}_2$ and two sequences of the form $S_\alpha \circ s_1$ and $S_\alpha \circ s_2$ are either identical, or the sets $\{S_\alpha\big(s_1(n)\big) ; n \in \mathbb{N}\}$ and $\{S_\alpha\big(s_2(n)\big) ; n \in \mathbb{N}\}$ are disjoint. Now, from the properties of the transcendence basis B (cf. [10]) it follows that $a = 0$.)

Second, since all $T(i,j)$ defining $\mathscr{L}_f^{\leftarrow}(0)$ are bounded sequences of real numbers, all $\mathscr{L}_1$-convergent sequences are bounded. Let $S \in \mathbb{Q}^{\mathbb{N}}$. Then there exists

$s \in \mathrm{MON}$ such that $S \circ s$ is $\mathscr{M}$-converging to some $x \in \mathbb{R}$. Hence if S is $\mathscr{L}_f$-convergent, then $(S, x) \in \mathscr{L}_2^* \subset \mathscr{M}$. □

THEOREM 3. *There are exactly* $\exp\exp\omega$ *nonhomeomorphic nonstrict* $\mathcal{L}_0^*$*-field completions of* $\boldsymbol{Q}$.

Proof. Let f be a mapping of $\exp\omega$ into $\{0,1\}$. Consider the nonstrict extension $(\mathbb{R}, \mathscr{L}_f)$ of $\boldsymbol{Q} = (\mathbb{Q}, \mathscr{L})$ constructed in the proof of Lemma. Let $\mathscr{L}_f'$ be a coarse $\mathcal{L}_0$-ring convergence on $\mathbb{R}$ coarser then $\mathscr{L}_f$. According to Theorem 4 in [6], $\mathscr{L}_f'$ is an $\mathcal{L}_0$-field convergence. Clearly, $\mathscr{L}_f'$ satisfies the Urysohn axiom. Now, the product convergence $\mathscr{L}_f' \times \mathscr{L}_f'$ on C $\big(\big(\langle(x_n, y_n)\rangle, (x,y)\big) \in \mathscr{L}_f' \times \mathscr{L}_f'$ iff $\big(\langle x_n\rangle, x\big), \big(\langle y_n\rangle, y\big) \in \mathscr{L}_f'\big)$ is compatible with the field structure of C. Again, let $\mathscr{L}''$ be a coarse $\mathcal{L}_0$-field convergence on C, coarser than $\mathscr{L}_f' \times \mathscr{L}_f'$. Since C is algebraically closed, it follows from Theorem 1.2.1 in [1] that $(C, \mathscr{L}'')$ is a complete $\mathcal{L}_0^*$-field. It is easy to see that $\mathscr{L}_f' \subset \mathscr{L}'' \restriction \mathbb{R}$ and $\mathscr{L}'' \restriction \mathbb{Q} = \mathscr{L}$ (cf. Lemma 1.4 in [8]). Let X be the smallest sequentially closed subset in $(C, \mathscr{L}'')$ such that $\mathbb{Q} \subset X$ and let $\mathscr{K}_f = \mathscr{L}'' \restriction X$. Then $(X, \mathscr{K}_f)$ is an $\mathcal{L}_0^*$-field completion of $(\mathbb{Q}, \mathscr{L})$ which fails to be strict. If f_1 and f_2 are different mappings of $\exp\omega$ into $\{0,1\}$, then the corresponding completions $(X, \mathscr{K}_{f_1})$ and $(X, \mathscr{K}_{f_2})$ are nonhomeomorphic. Since $\boldsymbol{Q}$ can have at most $\exp\exp\omega$ $\mathcal{L}_0^*$-field completions (cf. Corollary 3.3 in [3]), the assertion of the theorem follows. □

We do not know whether $\mathbb{R}$ is sequentially closed in $(X, \mathscr{K}_f)$.

Let Λ be the class of all $\mathcal{L}_0^*$-ring convergences on C constructed in the same way as $\mathscr{L}''$ in the proof of Theorem 3. We conclude with the following problems:

PROBLEM 2. *Does there exist a convergence in* Λ *which is a product convergence?*

PROBLEM 3. *Does there exist a convergence* $\mathscr{L}_{\mathbb{R}}$ *in* Λ *such that* $\mathbb{R}$ *is sequentially closed in* $(C, \mathscr{L}_{\mathbb{R}})$*?*

Observe that a positive solution to Problem 2 implies a positive solution to Problem 3. If the answer to Problem 2 or to Problem 3 is "yes", then Problem 2.4.(ii) in [2] has a positive solution.

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COMPATIBLE ORDERS OF SEMILATTICES

Chawewan Ratanaprasert

Dedicated to Professor J. Jakubík on the occasion of his 70th birthday

ABSTRACT. Two discrete semimodular semilattices S and S_1 have isomorphic graphs if and only if S is of the form $A \times B$ and S_1 is of the form $A^\partial \times B$ for a lattice A and a semilattice B. We prove that for discrete semilattices S and S_1 this latter condition holds if and only if S and S_1 have isomorphic graphs and the isomorphism preserves the order on some special types of cells and proper cells.

G. Birkhoff ([1, Problem 8]) asked for necessary and sufficient conditions on a lattice $L = (L; \vee, \wedge)$ in order that every lattice $M = (L; \vee^*, \wedge^*)$ whose (unoriented) graph is isomorphic with the graph of L be lattice-isomorphic to L. For the case when the lattices L and M are supposed to be distributive or modular, the problem was solved by Jakubík and Kolibiar (see [2, 4, 8, 9, 11, 12, 13]). In [8] Jakubík also showed that if one of L or M is modular (distributive), then so is the other. Duffus and Rival [3] solved the problem for those graded lattices which are determined by the ordered subset of their atoms and coatoms.

In [12] Jakubík proved that for discrete modular lattices L and M on the same underlying set L, the graphs $G(L)$ and $G(M)$ are isomorphic if and only if the following condition holds:

(a) there exist lattices $A = (A; \leqslant)$, $B = (B; \leqslant)$ and a direct product representation $\psi \colon L \to A \times B$ via which L is isomorphic with $A \times B$ and M is isomorphic with $A^\partial \times B$ where A^∂ stands for the dual of A.

Note that this yields a solution to Birkhoff's problem within the class of discrete modular lattices, since a modular lattice L will be uniquely determined by its graph if and only if every direct factor of L is self-dual.

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Jakubík proved in [5] that for discrete lattices (with no assumption of modularity) Condition (a) is equivalent to

(b) $\boldsymbol{L}$ and $\boldsymbol{M}$ have isomorphic graphs and all proper cells of $\boldsymbol{L}$ and all proper cells of $\boldsymbol{M}$ are either preserved or reversed (see below for the definitions).

In [15] Kolibiar proved that for discrete semimodular semilattices $\boldsymbol{S}$ and $\boldsymbol{S}_1$ on the same underlying set S, the graphs $G(\boldsymbol{S})$ and $G(\boldsymbol{S}_1)$ are isomorphic if and only if the following condition holds:

(c) there exist a lattice $\boldsymbol{A} = (A; +, \cdot)$, a semilattice $\boldsymbol{B} = (B; \cup)$ and a map $\psi\colon S \to A \times B$ via which ψ is a subdirect embedding of $\boldsymbol{S}$ into $\boldsymbol{A} \times \boldsymbol{B}$ and $\boldsymbol{S}_1$ into $\boldsymbol{A}^{\partial} \times \boldsymbol{B}$.

In this paper we give new characterizations of (c) and derive Kolibiar's result as a corollary.

An order $\leqslant_1$ is said to be a *compatible order* of a semilattice $\boldsymbol{S} = (S; \leqslant)$ if $\leqslant_1$ is a subsemilattice of $\boldsymbol{S}^2$.

In [14], it is proved that if $\leqslant_1$ is a compatible order of a semilattice $(\boldsymbol{S}; \vee, \leqslant)$, then the relations θ_1, θ_2 on $\boldsymbol{S}$ defined by $(*)$ are congruence relations on $(S; \vee)$:

$$\left.\begin{array}{l} a\ \theta_1\ b \text{ if and only if } a \leqslant u \geqslant b \text{ and } a \leqslant_1 u \geqslant_1 b \\ a\ \theta_2\ b \text{ if and only if } a \leqslant v \geqslant b \text{ and } a \geqslant_1 v \leqslant_1 b \end{array}\right\} \text{ for some } u, v \in S. \qquad (*)$$

LEMMA 1 [14]. *Let $\psi\colon \boldsymbol{S} \to \boldsymbol{S}' \times \boldsymbol{S}''$ be a subdirect representation of a semilattice $\boldsymbol{S}$. Denote $\psi(x)$ by (x_1, x_2). Given $a, b \in S$, set $a \leqslant_1 b$ if $a_1 \geqslant b_1$ and $a_2 \leqslant b_2$. Then $\leqslant_1$ is a compatible order of $\boldsymbol{S}$.*

The order $\leqslant_1$ of the lemma above is said to have *stemmed from a subdirect representation* of $\boldsymbol{S}$. If $\boldsymbol{S} = (S; \vee, \leqslant)$ and $\boldsymbol{S}_1 = (S; \vee_1, \leqslant_1)$ are semilattices and $\leqslant_1$ stems from a subdirect representation of $\boldsymbol{S}$, we write $\boldsymbol{S}_1 \# \boldsymbol{S}$.

THEOREM 1 [14]. *Let $\leqslant_1$ be a compatible order of a semilattice $\boldsymbol{S} = (S; \vee, \leqslant)$. The following conditions are equivalent:*

(i) *$\leqslant_1$ stems from a subdirect representation of $\boldsymbol{S}$;*

(ii) *each interval $\{x \in S \mid a \leqslant_1 x \leqslant_1 b\}$ is a convex subset of $\boldsymbol{S}$;*

(iii) *if $a \leqslant b \leqslant c$, then $a \leqslant_1 c$ implies $a \leqslant_1 b \leqslant_1 c$, and $c \leqslant_1 a$ implies $c \leqslant_1 b \leqslant_1 a$.*

Note that condition (i) can be reformulated in the following way (as follows from the proof of Theorem 1): for the congruence relations θ_1, θ_2 corresponding to $\leqslant_1$, see $(*)$, we have $\theta_1 \cap \theta_2 = \omega$, where ω is the least congruence relation, and $\leqslant_1$ stems from the subdirect representation of $\boldsymbol{S}$ given by θ_1 and θ_2.

LEMMA 2. *If $\leqslant_1$ is a compatible order of a semilattice $\boldsymbol{S} = (S; \vee \leqslant)$, and θ_1, θ_2 are the corresponding congruence relations, then $\leqslant_1 \subseteq (\theta_1 \cap \leqslant) \circ (\theta_2 \cap \geqslant)$.*

Moreover, if $\leqslant_1$ *fulfils the conditions of Theorem 1, then* $\leqslant_1 = (\theta_1 \cap \leqslant) \circ (\theta_2 \cap \geqslant)$.

COROLLARY 1. *Let* $\leqslant_1$ *be a compatible order of a semilattice* $\boldsymbol{S} = (S; \vee, \leqslant)$*, let* θ_1, θ_2 *be the corresponding congruence relations and let* $\leqslant_1$ *fulfil the conditions of Theorem 1. For* $a, b \in S$,

(i) *if* $a \leqslant b$ *and* $a\ \theta_1\ b$*, then* $a \leqslant_1 b$*, and*

(ii) *if* $a \leqslant b$ *and* $a\ \theta_2\ b$*, then* $b \leqslant_1 a$.

THEOREM 2. *Let* $\boldsymbol{S} = (S; \vee, \leqslant)$ *and* $\boldsymbol{S}_1 = (S; \vee_1, \leqslant_1)$ *be semilattices. Then the following are equivalent:*

(i) $\boldsymbol{S}\#\boldsymbol{S}_1$ *and* $\boldsymbol{S}_1\#\boldsymbol{S}$;

(ii) *there are* $\theta_1, \theta_2 \in \operatorname{Con} \boldsymbol{S} \cap \operatorname{Con} \boldsymbol{S}_1$ *such that* $\leq\ = (\theta_1 \cap \leqslant_1) \circ (\theta_2 \cap \geqslant_1)$ *and* $\leqslant_1 = (\theta_1 \cap \leqslant) \circ (\theta_2 \cap \geqslant)$;

(iii) *there is a lattice* $(X; +, \cdot)$*, a semilattice* $(Y; \cup)$ *and a map* $\psi : S \to X \times Y$ *such that* ψ *is a semilattice embedding of* $\boldsymbol{S}$ *into* $(X; +) \times (Y; \cup)$ *and* ψ *is a semilattice embedding of* $\boldsymbol{S}_1$ *into* $(X; \cdot) \times (Y; \cup)$.

P r o o f. (i)$\Rightarrow$(ii) Assume that $\boldsymbol{S}\#\boldsymbol{S}_1$ and $\boldsymbol{S}_1\#\boldsymbol{S}$. Then, by Lemma 1 and Lemma 2, the congruence relations θ_1, θ_2 of $\boldsymbol{S}$ and $\overline{\theta}_1$, $\overline{\theta}_2$ of $\boldsymbol{S}_1$ defined as in $(*)$ fulfil

$$\leq\ = (\overline{\theta}_1 \cap \leqslant_1) \circ (\overline{\theta}_2 \cap \geqslant_1) \quad \text{and} \quad \leq_1\ = (\theta_1 \cap \leqslant) \circ (\theta_2 \cap \geqslant).$$

By the definitions of θ_1 and $\overline{\theta}_1$, we have $\theta_1 = \overline{\theta}_1$. It remains to show that $\theta_2 = \overline{\theta}_2$.

Let $a\ \theta_2\ b$. Then $a \leqslant u \geqslant b$ and $a \geqslant_1 u \leqslant_1 b$ for some $u \in S$. It follows that $a \vee_1 b \leqslant u \vee_1 b = b$ since $\leqslant$ is compatible with $\vee_1$. Similarly we have $a \vee_1 b \leqslant a$. Hence $a \leqslant_1 a \vee_1 b \geqslant_1 b$ and $a \geqslant a \vee_1 b \leqslant b$ imply $a\ \overline{\theta}_2\ b$. Thus $\theta_2 \subseteq \overline{\theta}_2$. Analogously, $\overline{\theta}_2 \subseteq \theta_2$.

(ii) $\Rightarrow$ (iii) Assume that (ii) holds. Then $\leqslant$ is compatible with $\vee_1$ and $\leqslant_1$ is compatible with $\vee$. Let $(a, b) \in \theta_1 \cap \theta_2$. Then $a \leqslant a \vee b \geqslant b$ and $\leqslant_1 = (\theta_1 \cap \leqslant) \circ (\theta_2 \cap \geqslant)$ imply that $a \leqslant_1 a \vee b \leqslant_1 b$ and $b \leqslant_1 a \vee b \leqslant_1 a$ which yields $a = b$; i.e., $\theta_1 \cap \theta_2 = \omega$.

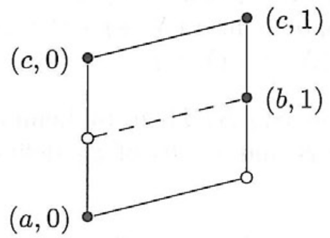
Now we will show that the operation join of $(S/\theta_1; \leqslant_1)$ is the meet operation of $(S/\theta_1; \leqslant)$; or equivalently, it is enough to show that for any $a, b \in S$, $[a \vee b]\,\theta_1 = [a]\,\theta_1$ if and only if $[a \vee_1 b]\,\theta_1 = [b]\,\theta_1$. Let $[a \vee b]\,\theta_1 = [a]\,\theta_1$. Then $[b]\,\theta_1 \leqslant [a \vee_1 b]\,\theta_1$ since $[b]\,\theta_1 \leqslant [a]\,\theta_1$ and $\leqslant$ is compatible with $\vee_1$. Since $b \leqslant_1 a \vee_1 b$ and $\leqslant_1 = (\theta_1 \cap \leqslant) \circ (\theta_2 \cap \geqslant)$, we have $u \in S$ such that $b\ \theta_1\ u\ \theta_2\ a \vee_1 b$ and $b \leqslant u \geqslant a \vee_1 b$; so $[b]\,\theta_1 = [u]\,\theta_1 \geqslant [a \vee_1 b]\,\theta_1$. Hence $[b]\,\theta_1 = [a \vee_1 b]\,\theta_1$. We can prove the converse analogously. Therefore $(S/\theta_1; \vee, \vee_1)$ is a lattice.

Analogously, we can prove that $(S/\theta_2; \leqslant)$ is isomorphic to $(S/\theta_2; \leqslant_1)$; or equivalently, the join operation of $(S/\theta_2; \leqslant)$ is the join operation of $(S/\theta_2; \leqslant_1)$.

Since the natural map is an embedding of $\boldsymbol{S}$ into $(S/\,\theta_1;\vee)\times(S/\,\theta_2;\vee)$ and $\boldsymbol{S}_1$ into $(S/\,\theta_1;\vee_1)\times(S/\,\theta_2;\vee_1)$, we have $(S/\,\theta_1;\vee,\vee_1)$ and $(S/\,\theta_2;\vee)$ are the required lattice and semilattice, respectively.

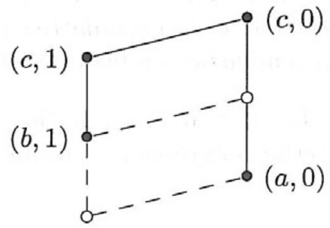
(iii) $\Rightarrow$ (i) Let $\leqslant$ and $\leqslant_1$ denote the order relations on $(X;+)\times(Y;\cup)$ and $(X;\cdot)\times(Y;\cup)$ respectively, and let $\leqslant_2$ denote the order relation on $(X;+,\cdot)$ and $\leqslant_3$ denote the order relation on $(Y;\cup)$.

Let $\boldsymbol{T}=(T;\leqslant)$ and $\boldsymbol{T}_1=(T;\leqslant_1)$ be images of the subdirect representation ψ and let (a_1,a_2), (b_1,b_2), (c_1,c_2) be elements in T with $(a_1,a_2)\leqslant(b_1,b_2)$. Then $a_1\leqslant_2 b_1$ and $a_2\leqslant_3 b_2$. So $a_1\cdot c_1\leqslant_2 b_1\cdot c_1$ and $a_2\cup c_2\leqslant_3 b_2\cup c_2$; i.e., $(a_1\cdot c_1,a_2\cup c_2)\leqslant(b_1\cdot c_1,b_2\cup c_2)$, which shows that $\leqslant$ is compatible with the operation of $(X;\cdot)\times(Y;\cup)$. By analogy, $\leqslant_1$ is compatible with the operation of $(X;+)\times(Y;\cup)$. Hence $\boldsymbol{S}\#\boldsymbol{S}_1$ and $\boldsymbol{S}_1\#\boldsymbol{S}$ follow from the subdirect representation. □



$(S;\vee,\leq)$

Figure 1.



$(S;\vee_1,\leq_1)$

Figure 2

Figure 1 and Figure 2 show that if $\boldsymbol{S}_1$ above is a compatible ordered set of $\boldsymbol{S}$ which stems from a 2-factor subdirect representation of $\boldsymbol{S}$, then it does not necessarily follow that the graph $G(\boldsymbol{S})$ and $G(\boldsymbol{S}_1)$ are isomorphic. In this paper, we shall prove the following theorem for a pair of semilattices.

THEOREM 3. *Let $\boldsymbol{S}=(S;\vee,\leqslant)$ and $\boldsymbol{S}_1=(S;\vee_1,\leqslant_1)$ be discrete semilattices. Then $\boldsymbol{S}\#\boldsymbol{S}_1$ and $\boldsymbol{S}_1\#\boldsymbol{S}$ if and only if the following conditions hold:*

(A) $G(\boldsymbol{S})=G(\boldsymbol{S}_1)$,

(B) *if either $\boldsymbol{S}$ or $\boldsymbol{S}_1$ contains a cell of type $\Diamond(1,n)$, say $C=\{u\prec x\prec v\succ y_n\cdots\succ y_1\succ u\}$, then the other contains one of the following four cells of type $\Diamond(1,n)$: $C, C^{\partial}, D=\{y_1\prec_1 y_2\prec_1\cdots\prec_1 y_n\prec_1 v\prec_1 x\succ_1 u\succ_1 y_1\}$ or D^{∂}, and*

(C) *in both $\boldsymbol{S}$ and $\boldsymbol{S}_1$, all proper cells of type $\bigvee(m,n)$ with $m>1$ and $n>1$ are preserved or reversed.*

A semilattice $\boldsymbol{S} = (S; \leqslant)$ is called *discrete* if each bounded chain in $\boldsymbol{S}$ is finite.

Let $\boldsymbol{S} = (S; \leqslant)$ be a semilattice. For $a, b \in S$ with $a \leqslant b$ an interval $[a, b]$ is the set of all elements $x \in S$ satisfying $a \leqslant x \leqslant b$. We call $[a, b]$ a *prime interval* (or equivalently, b *covers* a, in symbols $a \prec b$) if $\bigl|[a, b]\bigr| = 2$.

By the graph $G(\boldsymbol{S})$ we mean the (undirected) graph whose vertex set is S and whose edges are those pairs $\{a, b\}$, which satisfy either $a \prec b$ or $b \prec a$.

Let $\boldsymbol{S} = (S; \leqslant)$ and $\boldsymbol{S}_1 = (S_1; \leqslant_1)$ be semilattices. It is said that $G(\boldsymbol{S})$ is isomorphic with $G(\boldsymbol{S}_1)$ if there is a bijection $f\colon S \to S_1$ such that for all $a, b \in S$, $\{a, b\}$ is an edge of $G(\boldsymbol{S})$ if and only if $\bigl\{f(a), f(b)\bigr\}$ is an edge of $G(\boldsymbol{S}_1)$. Throughout this paper we assume, without loss of generality, that $S = S_1$ and f is the identity map whenever $G(\boldsymbol{S})$ is isomorphic to $G(\boldsymbol{S}_1)$, whence $G(\boldsymbol{S}) = G(\boldsymbol{S}_1)$.

If $G(\boldsymbol{S}) = G(\boldsymbol{S}_1)$, then a set $C \subseteq S$ is said to be *preserved* if, whenever $a, b \in C$ and $a \prec b$, then $a \prec_1 b$.

Let $u, v, x_1, \dots, x_m, y_1, \dots, y_n$ be distinct elements in S such that

(i) $u \prec x_1 \prec \dots \prec x_m \prec v$, $u \prec y_1 \prec \dots \prec y_n \prec v$, and

(ii) either v is the least upper bound of x_1 and y_1 (denoted by $v = x_1 \vee y_1$) or u is the greatest lower bound of x_m and y_n (denoted by $u = x_m \wedge y_n$).

Then the set $C = \{u, v, x_1, \dots, x_m, y_1, \dots y_n\}$ is said to be a *cell* of $\boldsymbol{S}$. If $x_1 \vee y_1 = v$, we call C a *cell of type* $\bigvee(m, n)$. Dually, if $x_m \wedge y_n = u$, we call C a *cell of type* $\bigwedge(m, n)$. If $x_1 \vee y_1 = v$ and $x_m \wedge y_n = u$, we call C a *cell of type* $\Diamond(m, n)$. If $m = n$, then C is a *cell of length* $n + 1$. A cell C is called *proper* if either $m > 1$ or $n > 1$.

A semilattice $\boldsymbol{S}$ is said to be *upper semimodular* if $\boldsymbol{S}$ satisfies the following *Upper Covering Condition* (UCC):

(UCC): *if a and b cover c with $a \neq b$, then both a and b are covered by $a \vee b$.*

Let $\boldsymbol{S}$ and $\boldsymbol{S}_1$ be discrete semimodular semilattices. Then $\boldsymbol{S}$ and $\boldsymbol{S}_1$ contain no cells of type $\bigvee(m, n)$ with $m, n \geqslant 1$ & $m + n > 2$; i.e., Conditions (B) and (C) of Theorem 3 always hold. Therefore we obtain one of Kolibiar's results [15] as a corollary.

COROLLARY 2. [15] *Let $\boldsymbol{S}$ and $\boldsymbol{S}_1$ be semimodular semilattices. Then $\boldsymbol{S}$ and $\boldsymbol{S}_1$ satisfy Condition* (c) *if and only if $G(\boldsymbol{S}) = G(\boldsymbol{S}_1)$.*

We now prove Theorem 3 via the following lemmata.

LEMMA 3. *Let $\boldsymbol{S} = (S; \vee, \leqslant)$ and $\boldsymbol{S}_1 = (S; \vee_1, \leqslant_1)$ be discrete semilattices satisfying $\boldsymbol{S} \# \boldsymbol{S}_1$ and $\boldsymbol{S}_1 \# \boldsymbol{S}$. Then Conditions* (A), (B) *and* (C) *hold.*

Proof. Assume that $a \prec b$. Then $a \leqslant b$ implies $a \leqslant a \vee_1 b \leqslant b$ which yields $a = a \vee_1 b$ or $b = a \vee_1 b$; i.e., $b \leqslant_1 a$ or $a \leqslant_1 b$. If $a \leqslant_1 c \leqslant_1 b$ for some

$c \in S$, then it follows by Theorem 1 with $a \leqslant b$ that $a \leqslant c \leqslant b$; hence $a = c$ or $b = c$, which shows that $a \prec_1 b$. Similarly if $b \leqslant_1 c \leqslant_1 a$ for some $c \in S$, then $b \prec_1 a$.

Analogously $a \prec_1 b$ implies $a \prec b$ or $b \prec a$. Hence $G(S) = G(S_1)$.

To show that $\boldsymbol{S}$ and $\boldsymbol{S}_1$ satisfy Condition (B), let $C = \{u \prec x \prec v \succ y_n \succ \cdots \succ y_1 \succ u\}$ be a cell of type $\Diamond(1,n)$ in $\boldsymbol{S}$ (see Figure 3(1)). By the assumption, Condition (A) and the definitions of θ_1 and θ_2 (defined as in $(*)$), we have either $x\ \theta_1\ u\ \theta_1\ y_1$, $x\ \theta_2\ u\ \theta_2\ y_1$, $x\ \theta_1\ u\ \theta_2\ y_1$ or $x\ \theta_2\ u\ \theta_1\ y_1$.

Case 1: $x\ \theta_1\ u\ \theta_1\ y_1$. Then $x \geqslant_1 u \leqslant_1 y_1$. It follows by Corollary 1 that $x\ \theta_1\ v$ and $x \leqslant v$ imply $x \leqslant_1 v$. Since $u\ \theta_1\ x$ implies $y_i\ \theta_1\ v$ for all $i = 1, 2, \ldots, n$, the transitivity of θ_1 yields $y_i\ \theta_1\ y_j$ for all i, j. Thus $y_i\ \theta_1\ y_{i+1}$ and $y_i \prec y_{i+1}$ imply $y_i \prec_1 y_{i+1}$ for all $i = 1, 2, \ldots, n$; i.e., $y_1 \prec_1 y_2 \prec_1 \cdots \prec_1 y_n \prec_1 v$. Therefore C is a cell of $\boldsymbol{S}_1$ of type $\Diamond(1,n)$.

Case 2: $x\ \theta_2\ u\ \theta_2\ y_1$. We can prove analogously to Case 1 that C^{∂} is a cell of $\boldsymbol{S}_1$ of type $\Diamond(1,n)$.

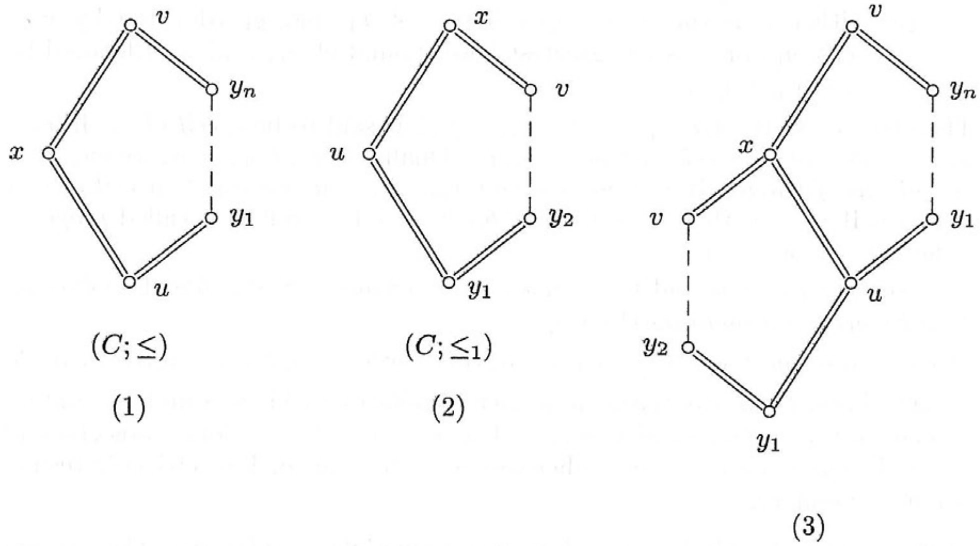


FIGURE 3.

Case 3: $x\ \theta_1\ u\ \theta_2\ y_1$. Then $x\ \theta_2\ v$ and $v\ \theta_1\ y_i$ for all $i = 1, 2, \ldots, n$. It follows from Corollary 1 and the transitivity of θ_1 that $y_1 \prec_1 y_2 \prec_1 \cdots \prec_1 y_n \prec_1 v \prec_1 x \succ_1 u \succ_1 y_1$; i.e., D is a cell of $\boldsymbol{S}_1$ of type $\Diamond(1,n)$ (see Figure 3).

Case 4: $x\ \theta_2\ u\ \theta_1\ y_1$. We can prove analogously to Case 3 that D^{∂} is a cell of $\boldsymbol{S}_1$ of type $\Diamond(1,n)$.

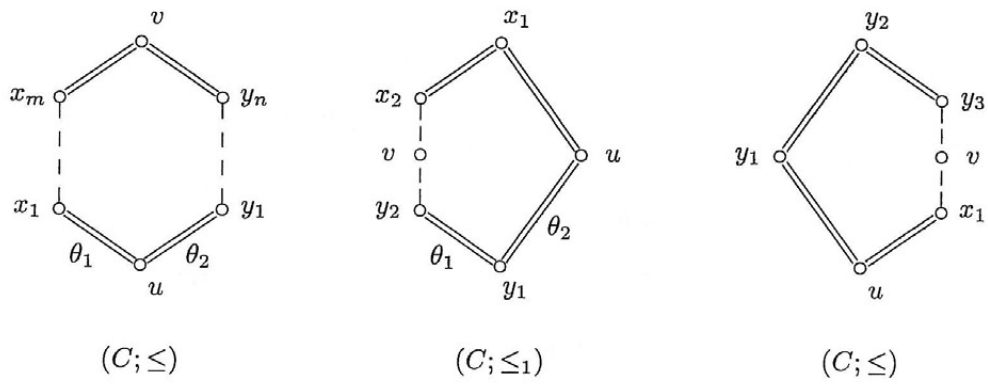


FIGURE 4.

To prove Condition (C), let $A = \{u \prec x_1 \prec \cdots \prec x_m \prec v \succ y_n \succ \cdots \succ y_1 \succ u\}$ be a proper cell of $\boldsymbol{S}$ of type $\bigvee(m,n)$ $(m > 1$ and $n > 1)$. Then $x_1 \vee y_1 = v$. Suppose that $x_1\ \theta_1\ u\ \theta_2\ y_1$. Then $x_i\ \theta_2\ v\ \theta_1\ y_j$ for all $1 \leqslant i < m$ and $1 \leqslant j \leqslant n$, which together with the transitivity of θ_1 and θ_2 implies that $x_i\ \theta_2\ x_k$ and $y_j\ \theta_1\ y_\ell$ for all $1 \leqslant i,\ k \leqslant m$ and $1 \leqslant j,\ \ell \leqslant n$. So $C = \{y_1 \prec_1 y_2 \prec_1 \cdots \prec_1 y_n \prec_1 v \prec_1 x_m \prec_1 \cdots \prec_1 x_1 \succ_1 u \succ_1 y_1\}$ is a cell of $\boldsymbol{S}_1$ of type $\Diamond(1, n+m-1)$ (see Figure 4). By Condition (B), since $y_2\ \theta_1\ y_1\ \theta_2\ u$, we have that D^∂ is a cell of $\boldsymbol{S}$ of type $\Diamond(1, n+m-1)$ which yields $y_2 = y_3 = \cdots = y_n = v$; i.e., $n = 1$, a contradiction.

We will get a similar contradiction if $x_1\ \theta_2\ u\ \theta_1\ y_1$. Therefore, either $x_1\ \theta_1\ u\ \theta_1\ y_1$ or $x_1\ \theta_2\ u\ \theta_2\ y_1$. Hence A is preserved or reversed. □

In the following lemmata, we shall assume that $\boldsymbol{S}$ and $\boldsymbol{S}_1$ are semilattices satisfying Conditions (A), (B) and (C).

LEMMA 4. *Let $a, b, c \in S$ with $a \succ c \prec b$. Then*

(i) *$c \prec_1 a$ implies $b \leqslant_1 a \vee b$, and*
(ii) *$a \prec_1 c$ implies $a \vee b \leqslant_1 b$.*

Proof. We only prove (i) as (ii) follows by duality. Assume $c \prec_1 a$. Since the case $a = b$ is trivial, we assume $a \neq b$.

If $c \prec a \prec a \vee b \succ b \succ c$, then using (B) we obtain immediately $b \prec_1 a \vee b$. If $c \prec a \not\prec a \vee b \not\succ b \succ c$, then condition (C) applies. We may assume that $c \prec a \prec a \vee b \geqslant b \succ c$ (we can prove analogously if $c \prec b \prec a \vee b \geqslant a \succ c$). Then $C = \{c \prec a \prec a \vee b \succ y_n \succ \cdots \succ y_1 \succ b \succ c\}$ for some $y_1, \ldots, y_n \in S$ is a cell of $\boldsymbol{S}$ of type $\Diamond(1,n)$ with $[c,a]$ preserved (reversed). Hence, by Condition (B), either C or $D = \{b \prec_1 y_1 \prec_1 \cdots \prec_1 y_n \prec_1 a \vee b \prec_1 a \succ_1 c \succ_1 b\}$ (resp. C^∂ or D^∂) is a cell of $\boldsymbol{S}_1$ of type $\Diamond(1,n)$ (see Figure 5). In either case we have $b \leqslant_1 a \vee b$ (resp. $a \vee b \geqslant_1 b$). □

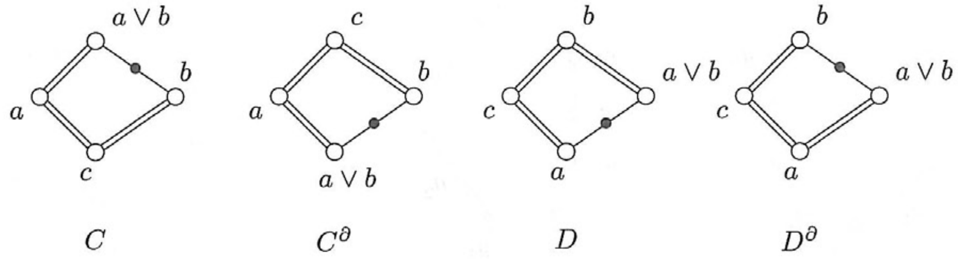


FIGURE 5.

LEMMA 5. *Let $a, b, c \in S$ with $a \prec b$. Then*

(i) *$a \prec_1 b$ implies $a \vee c \leqslant_1 b \vee c$, and*

(ii) *$b \prec_1 a$ implies $b \vee c \leqslant_1 a \vee c$.*

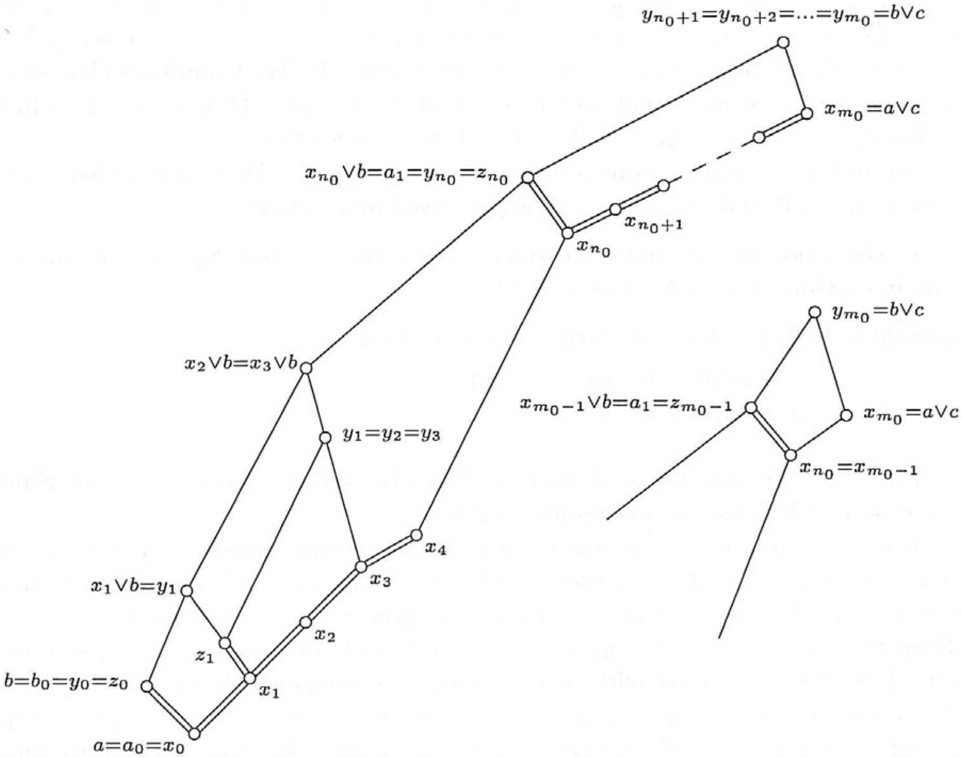


FIGURE 6.

P r o o f . Let $a = x_0 \prec x_1 \prec \cdots \prec x_{m_0} = a \vee c$, and define $y_i = x_i \vee z_{i-1}$ where $y_0 = b = z_0$ and in general z_{i-1} is chosen so that $x_{i-1} \prec z_{i-1} \leqslant y_{i-1}$ (if there is an i $(0 < i < m_0)$ such that $z_i = x_{i+1}$, we will continue the process by considering $z_i = z_{i-1}$ and $y_{i+1} = y_i$ (see Figure 6)). Then, by Lemma 4, the interval $[x_{i+1}, y_{i+1}]$ is preserved (reversed) if the interval $[x_i, y_i]$ is preserved (reversed) since $x_i \prec z_i \leqslant y_i$. Hence by induction $[x_{m_0}, y_{m_0}]$ is preserved (reversed) since $[a, b]$ is preserved (reversed). Since $x_i \leqslant y_i \leqslant x_i \vee b$ for all $0 \leqslant i \leqslant m_0$, we have $a \vee c = x_{m_0} \leqslant y_{m_0} \leqslant x_{m_o} \vee b = a \vee c \vee b = b \vee c$.

Let $a_0 = a$, $b_0 = b$, $x_i^{(\circ)} = x_i$, $y_i^{(\circ)} = y_i$, $z_i^{(\circ)} = z_i$ for all $i = 0, 1, \ldots, m_0$. Note that $z_0^{(\circ)} = b_0$. Let $a_1 = z_{n_0}^{(\circ)}$ where n_0 is the least number such that $0 \leqslant n_0 < m_0$ and $x_{n_0+1} \vee z_{n_0} = y_{n_0+1} = y_{n_0+2} = \cdots = y_{m_0}$ (see Figure 6 and 7).

Case 1: If $a_1 = z_{n_0}^{(\circ)} = x_{n_0} \vee b$ $(= y_{n_0})$, then $y_{m_0} = x_{m_0} \vee z_{n_0} = x_{m_0} \vee (x_{n_0} \vee b) = x_{m_0} \vee b = (a \vee c) \vee b = b \vee c$; hence, by using (B) or (C), the preservation of $[x_{n_0}, y_{n_0}]$ implies the preservation of $[x_{m_0}, y_{m_0}] = [a \vee c, b \vee c]$ (see Figure 6).

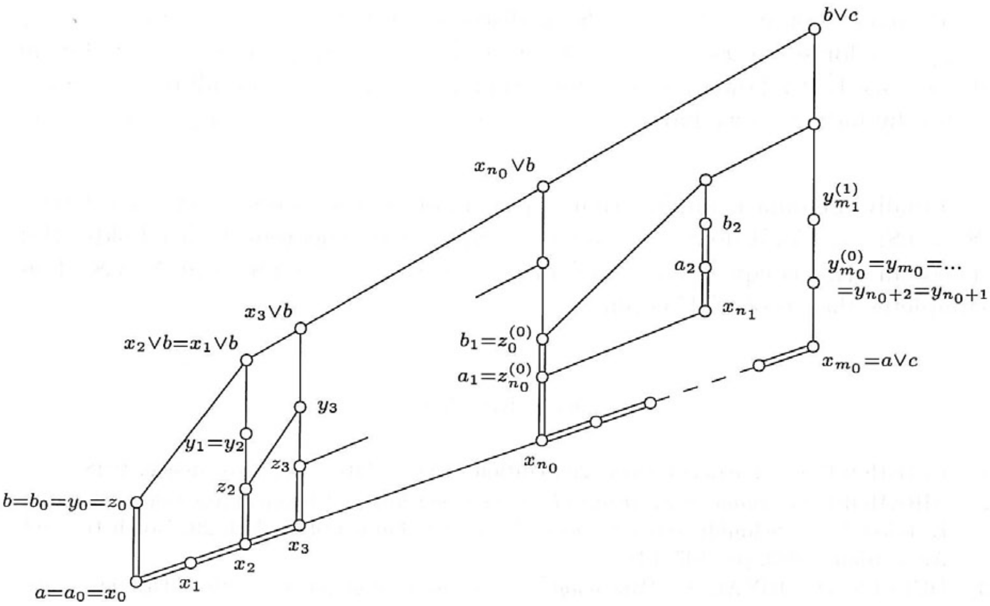


FIGURE 7.

Case 2: If $a_1 = x_{n_0}^{(\circ)} < x_{n_0} \vee b$ (note that $x_{n_0}^{(\circ)} \leqslant y_{n_0} \leqslant x_{n_0} \vee b$), we choose $b_1 (= z_0^{(1)})$ with $a_1 \prec b_1 \leqslant x_{n_0}^{(\circ)} \vee b$ (see Figure 7). Note that since $[a_0, b_0]$ is preserved (reversed) so is $[a_1, b_1]$. Now repeat the construction with a_0 replaced by a_1 and b_0 replaced by b_1. This then produces the elements $z^{(1)}_{n_1}$ and $y^{(1)}_{m_1}$ which are needed to begin the next step of the induction.

(In general, we define $a_{i+1} = z^{(i)}_{n_i-1}$ where n_i is the least number such that $0 \leqslant n_i < m_i$ and $x_{n_i+1} \vee z_{n_i} = y_{n_i+1} = y_{n_i+2} = \cdots = y_{m_i}$ and choose $b_{i+1} = z^{(i+1)}_0$ with $a_{i+1} = z^{(i)}_{n_i} \prec z^{(i+1)}_0 = b_{i+1} \leqslant x^{(i)}_{n_i} \vee b_i$ (for the case $a_{i+1} = z^{(i)}_{n_i} = x^{(i+1)}_{n_i} \vee b_i = y^{(i)}_{n_i}$, (as Case 1) the preservation of $[a_i, b_i]$ implies the preservation of $[y^{(i)}_{m_i}, y^{(i+1)}_{m_{i+1}}]$) and, finally, repeat the process with the covering chain:

$$a_{i+1} = z^{(i)}_{n_i-1} = x^{(i+1)}_0 \prec x^{(i+1)}_1 \prec \cdots \prec x^{(i+1)}_{m_{i+1}} = y^{(i)}_{m_i}\,).$$

Since S is discrete, there exists N such that $y^{(N)}_{m_N} = b \vee c$ and hence we have a chain:

$$a \vee c \leqslant y^{(\circ)}_{m_0} \leqslant y^{(1)}_{m_1} \leqslant \ldots y^{(N)}_{m_N} = b \vee c\,,$$

and each step in this chain is preserved (reversed) since each interval $[a_i, b_i]$ is preserved (reversed). □

LEMMA 6. *Let $a, b, c \in S$ with $a \leqslant_1 b$. Then $a \vee c \leqslant_1 b \vee c$.*

Proof. Let $a \leqslant_1 b$. Since S_1 is discrete, we have $a = x_0 \prec_1 x_1 \prec_1 \cdots \prec_1 x_{n+1} = b$ for some $x_1, \ldots, x_n \in S$. So either $x_i \prec x_{i+1}$ or $x_{i+1} \prec x_i$ for all $0 \leqslant i < n+1$. It follows from Lemma 5 that $x_i \vee c \leqslant_1 x_{i+1} \vee c$ for all $0 \leqslant i < n+1$. Hence by induction we have $a \vee c = x_0 \vee c \leqslant_1 x_1 \vee c \leqslant_1 \cdots \leqslant_1 x_{n+1} \vee c = b \vee c$. □

Finally, Lemma 6 implies that $\leqslant_1$ is a compatible order of the semilattice $S = (S; \leqslant)$. Conditions (B) and (C) imply that Theorem 1 (iii) holds. But Theorem 1(iii) is equivalent to Theorem 2 (i) which is $S \# S_1$ and $S_1 \# S$. This completes the proof of Theorem 3.

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