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METRIC BELL INEQUALITIES, RELATIVE MEASURE OF PROBABILITY AND THE GEOMETRY OF HIDDEN VARIABLE SPACE

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ABSTRACT. Metric inequalities for correlations in Bell's scheme of LHV can be interpreted as a consequence of the isotropy of the hidden variable space in relation to the space elements defined by measuring devices. The destroyed isotropy of such a space leads to the concept of relative measure of probability, which allows us to describe correctly the quantum mechanical results. For retaining the Einstein's locality the necessity of the only relative description of the space-like correlations must be accepted.

1. Introduction

The metric content of Bell inequalities for correlations in singlet systems, which has been recognized during last years [2]–[5], has caused a growing attention [6]–[18]. It has been proved recently that similar metric relations hold for correlations in precessing spin systems or in the case of oscillations of K and B mesons [17] (here instants of time, when the measurements of the pertinent characteristics of both particles are performed, play the role of space elements) and also for correlations of phases of interfering particles [18]. Thus, it appears that the metricity of certain correlation functions induced by Bell's scheme of hidden variables is its very general characteristic.

This fact makes possible to consider the relation between the Bell scheme of local hidden variables (=LHV) and quantum mechanics from a new and more general point of view [16]. Indeed, the key-question of hidden variables can be now formulated in such a way: why correlation functions in the Bell scheme of LHV are limited by the metric of spherical or Riemannian geometry while inequalities for quantum mechanical correlation functions have a nonmetric form? The adequate answer to this question may serve as guidance for the inevitable

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extension of Bell's LHV scheme. Our contribution is devoted to the analysis of the above question.

The plan of our exposition is as follows. The main results about metric properties of the corresponding correlation functions in the LHV scheme are briefly recapitulated in Section 2. The difference between this scheme and quantum mechanics is considered in Section 3. It is indicated that the metricity of certain correlation functions is a quite natural consequence of the invariant description of the LHV picture and expresses the isotropy of the LHV space with respect to "space elements" defined by the measuring devices. At the same time the destroyed isotropy of the LHV space leads to the necessity of introduction of a new concept – relative measure of probability [6] – which is capable of describing the quantum mechanical results correctly.

The properties of relative measure of probability (=RM) are considered in Section 4. It appears that in relation to the experiments performed up to now [19] there exist different possible interpretations of RM [16]. We then discuss one of the alternatives, i.e., RM as necessity of the only relative description of the space-like quantum correlations in more details, emphasizing nonclassical features of such a statistics. In the closing Section the relation of RM to the other "no-go" theorems for LHV is briefly recapitulated and some more general problems are discussed.

2. Bell's LHV scheme and metric inequalities for correlation functions

We shall start our consideration from the usual description of the correlations of two particles in the Bell LHV scheme [20]

$$P(a,b) = \int A(a,\lambda)\, B(b,\lambda)\, \rho(\lambda)\, \mathrm{d}\lambda\,, \tag{1}$$

here a and b represent certain space elements, which characterize measured variables, $A(a,\lambda)$ and $B(b,\lambda)$ denote the results of the measurement of the properties of the first and the second particle, respectively (acquiring values of $+1$ and -1 according to convention), λ's are hidden parameters and $\rho(\lambda)$ is a positively defined, normalized measure of probability.

The space elements a and b can be, e.g. – directions on which the spin projection of particles or linear polarizations of photons are measured; – instants of time [17], when the measurements of spin projections in singlet precessing systems are performed (the same relates, with some caution, to measuring flavour characteristics in systems of K and B mesons); – values of phases [18] of interfering particles, etc.

The correlation function is expressed usually as a relation of the corresponding mean values

$$P(a,b) = \frac{\langle ++\rangle + \langle --\rangle - \langle +-\rangle - \langle -+\rangle}{\langle ++\rangle + \langle --\rangle + \langle +-\rangle + \langle -+\rangle}\,, \tag{2}$$

here $\langle ++\rangle$ denotes the mean value for $A=+1$ and $B=+1$, etc.

It is supposed further that the measurements on both particles are performed at space-like intervals and then the principle of "locality" is formulated (as we shall see later, it is only a necessary condition for the description to be local)

$$A(a,\lambda)\neq \mathcal{F}(b), \quad \text{and} \quad B(b,\lambda)\neq \mathcal{F}(a)\,, \tag{3}$$

and also independence of probability measure of real experimental arrangement

$$\rho \neq \rho\,(a,b)\,, \tag{4}$$

which express isotropy of the hidden variable space in relation to the space elements a and b.

This last relation does not hold, generally, neither in the quantum nor in the classical statistics. It is not fulfilled in the cases when the measurement of variable itself supposes destroying the initial symmetry of the system [16]. In the case of the space-like correlations, however, it looks quite reasonable.

As far as we are trying to construct the LHV scheme as close as possible to that of quantum mechanics we shall specify some properties of the quantum mechanical correlation function which can be realized in the LHV scheme, too.

From the symmetry of the considered systems it follows

$$P^{QM}(a,a)=\pm 1\,, \tag{5}$$

where the upper sign relates, e.g., to the case of the correlations of photon linear polarization in singlet state and/or correlations of phases of interfering bosons, while the lower sign corresponds to the correlations of spin projections in singlet $\left(s=\frac{1}{2}\right)$ stationary states, or precessing in the constant magnetic field, to the correlations of phases of interfering fermions, etc.

The relation (5) can be satisfied in the scheme of LHV if only

$$A(a,\lambda)=B(a,\lambda) \quad \text{for} \quad P(a,a)=1\,, \tag{6a}$$

and

$$A(a,\lambda)=-B(a,\lambda) \quad \text{for} \quad P(a,a)=-1\,. \tag{6b}$$

From (1) and (6a) or (6b) it follows that

$$P(a,b)=P(b,a)\,. \tag{7}$$

Now a function of $d(a,b)$ can be introduced (we shall show latter that it is equal to a certain conditional probability)

$$d(a,b)=\frac{1}{2P(a,a)}\left[P(a,a)-P(a,b)\right], \tag{8}$$

which can be interpreted as a metric distance. It is symmetric due to (7), positively defined, equal zero for identical elements and satisfies the triangular inequality

$$d(a_1,a_2)+d(a_2,a_3)-d(a_1,a_3)\geq 0\,. \tag{9}$$

The metric defined through $d(a,b)$ is not yet fully specified. An important characteristic of the metric is, e.g., its degeneracy. For that we shall investigate the properties of $P(a,b)$ and $d(a,b)$ more thoroughly. As it can be easily realized, there are other properties of $P^{QM}(a,b)$ which must hold in the LHV scheme also.

First it must hold

$$P(a,b) = P(a-b)\,, \tag{10}$$

as in the quantum mechanical case.

Due to the periodicity of $P^{QM}(a-b)$ it is convenient to express the difference $a-b$ through some angle φ_{ab}. It is evident in the case when a and b are vectors, and it can be realized for one-dimensional spaces as time instants and phase values, too. In this case we can represent the space elements as points on a circle and corresponding vectors will join the centre with these points as it is usual in affine geometry.

The equality (10) can be generally interpreted as a consequence of the rotational invariance of the considered quantum systems.

Another important characteristic of $P^{QM}(a-b)$ is its period $\equiv 2\kappa$. It can be derived from the relation

$$P^{QM}(a,b) = -P^{QM}(a,a)\,, \tag{10a}$$

or, equivalently,

$$P^{QM}(\kappa) = -P^{QM}(0)\,, \tag{10b}$$

where we used notation $\kappa = \varphi_{ab}$.

For the considered systems $\kappa = \pi$ for projections of spins, flavour characteristics and phases and $\kappa = \frac{\pi}{2}$ for correlations of photon linear polarizations.

Postulating relation (10a,b) for LHV correlation functions and using inequality (9), it is possible to derive another useful relations for $P(\varphi_{ab})$ [22]

$$P(\varphi) = -P(\kappa-\varphi), \quad P(\kappa-\varphi) = P(\kappa+\varphi), \quad \text{etc.} \quad \varphi \le \kappa\,, \tag{10c}$$

from which an important relation

$$P(2n\kappa) = P(0)\,, \tag{11}$$

$n = 0, 1, 2, \ldots$, can be deduced.

Thus, it appears, that the metric defined with the use of $d(a,b)$ is degenerated, as far as

$$d(a,b) = 0 \quad \text{for} \quad \varphi_{ab} = 2n\kappa, \quad n = 0,1,2,\ldots\,. \tag{12}$$

This degeneracy can be removed by the change of topology of the space, identifying all space elements for which $d(a,b) = 0$. The upper bound of the metric defined with $d(a,b)$ is equal to the metric of ordinary spherical geometry (for $\kappa = \pi$) or of Riemannian geometry on the spherical surfaces (for $\kappa = \frac{\pi}{2}$). It can be realized, when we take space elements $a, b, \ldots$ as unit vectors with beginnings in a common point. Then the distance between their ends, measured on the spherical surface can be expressed as

$$\mathcal{D}(a,b) = \kappa d(a,b)\,. \tag{13}$$

A similar reasoning as above can be done for distance defined in another way. Braunstein and Caves [23] have introduced mean conditional information entropy

$$H(a/b) = -\sum_{\alpha,\beta} p(\alpha,\beta)\ \log p\,(\alpha/\beta)\,, \tag{14}$$

here $p\,(\alpha,\beta)$ is the joint probability and $p\,(\alpha/\beta)$ the conditional probability for $A(a,\lambda)=\alpha$ and $B(b,\lambda)=\beta$ $(\alpha,\beta=\pm 1)$.

The symmetry of the considered quantum systems leads to the relation

$$p^{QM}(\alpha,\beta) = p^{QM}(-\alpha,-\beta)\,, \tag{15}$$

where $\alpha,\beta=\pm 1$. Assuming that the same relation holds in the LHV scheme we can rewrite (14)

$$\begin{aligned} H(a/b) = &-\frac{1}{2}\left[P(a,b)+1\right]\log\Big\{\frac{1}{2}\left[P(a,b)+1\right]\Big\} - \\ &-\frac{1}{2}\left[1-P(a,b)\right]\log\Big\{\frac{1}{2}\left[1-P(a,b)\right]\Big\}. \end{aligned} \tag{16}$$

Using properties of $P(a,b)$ derived above, it can be shown that this function satisfies the first three conditions of metricity and with using Bayes's theorem the triangular inequality can be derived [12, 13, 23]. Using (10c) and (16) it can be also shown that

$$H(a/b) = H(\varphi_{ab}) = 0 \quad \text{for} \quad \varphi_{ab} = n\kappa, \quad n = 0,1,2,\dots\,, \tag{17}$$

i.e., this metric is twice more degenerated than the metric defined by $\mathcal{D}(a,b)$ (Cf.(12)).

Concluding this Section let us make some comments.

C1. Using the relation (15) we can interpret the function of $d(a,b)$ as a conditional probability.
It holds

$$d(a,b) = \begin{cases} p\,(+1,+1)+p\,(-1,-1)\,, & \text{for } P(a,a)=-1, \\ p\,(+1,-1)+p\,(-1,+1)\,, & \text{for } P(a,a)=1, \end{cases} \tag{18}$$

here $p\,(\alpha,\beta)$; $\alpha,\beta=\pm 1$ denotes joint probabilities as above.

C2. In spite of similarity of the functions $\mathcal{D}(a,b)$ and $H(a/b)$ these functions differ in respect to the Bell's LHV scheme substantially. While this scheme allows for $\mathcal{D}(a,b)$ to be an exact spherical or Riemannian metric, it is so for $H(a/b)$. For details see [13].

C3. For specialists in quantum logic it will not be surprising that different space elements which define the measured variables are connected by metric relations in the LHV scheme, as far as all the reasoning is based on an identical probability scheme. Really, the problem of treating such phenomena as precession of spins or flavour variables in the case of oscillations of K and B mesons, correlation of phases of interfering particles has consisted mainly in the choosing of the suitable quantum state and corresponding experimental procedure. But what probably has passed unnoticed is the real content of the discovered metricity which permits us to formulate the question of hidden variables as it is done in the title of the next Section.

3. Why Correlation functions in the Bell's scheme of LHV are limited by metric of spherical or Riemannian geometry while the inequalities for quantum mechanical correlations have a nonmetric form?

Discordance between Bell's LHV scheme and quantum mechanics is usually solved with the use of Bell's Theorem [21], which asserts that quantum mechanics can be completed only by nonlocal hidden variables. In the language of the quantum logic the answer is given in a more abstract and, therefore, more correct manner [24]: the violation of Bell's inequalities implies that a lattice of propositions for a quantum physical system is not distributive [5, 24].

It is true that the broken locality (cf. with (3))

$$A(a,\lambda) = \mathcal{F}(b) \quad \text{and} \quad B(b,\lambda) = \mathcal{F}(a)$$

can cause that the Bell inequalities will not be fulfilled and, consequently, the nondistributivity of the propositions will appear, but it is not true that the concepts of nonlocality and nondistributivity are equivalent, as far as the converse implication has not yet been proven.

Similarly, there exists a close relation between the Bell's scheme of LHV with corresponding metric inequalities and Boolean logic of propositions with its distributivity, but such an abstract relation does not give an adequate answer to the key question above either.

Before proceeding it will be useful to say some words about the used methodology. Up to now we have used pure algebraic methods and the geometrical picture has served only for the better illustration of the results. We shall use this attitude as long as possible because of its generality. But at a certain stage of our investigation we shall be forced to use purely geometrical methods. It is not a surprise, because we are discussing problems which are formulated in geometrical terms and the geometrical methods appear to be more powerful. For explaining some details we shall also use elementary results of theory of information.

Let us begin our consideration from the first part of the posed question - why the correlation functions in the Bell's scheme of LHV are limited by the metric of spherical or Riemannian geometry.

We understand a limiting (the strongest) correlation function of $P(a,b)$ in the usual sense. It is a function which reaches the maximum value of $|P(a,b)|$ for any a and b, or, equivalently, which minimalizes the difference between quantum mechanical and LHV values $|P^{QM}(a,b) - P(a,b)|$. It is not difficult to realize that for such a function $P(a,b)$ the corresponding function of $\mathcal{D}(a,b)$ is an exact metric distance of spherical or Riemannian geometry (see Fig. 1).

It can be proved an useful theorem which characterizes such extremal functions [6].

THEOREM 1. *Let* $\mathcal{D}(0) = 0$, $\mathcal{D}(\kappa) = \kappa$, *and* $A(a,\lambda)$, $B(b,\lambda)$ *and* $\rho(\lambda)$ *guarantee the rotational invariance of* $\mathcal{D}(a,b)$, *i.e.,* $\mathcal{D}(a,b) = \mathcal{D}(\varphi_{ab})$.

Then $\mathcal{D}(a,b)$ *is the exact metric distance of spherical or Riemannian geom-*

etry if and only if the sequence

$$A(a_1,\lambda),\ A(a_2,\lambda)\,,\ \dots\ ,\ A(a_n,\lambda)\,,$$

for an ordered set of space elements

$$a_1, a_2, a_3, \dots, a_n\,; \quad \varphi_{a_1 a_n} \le \kappa\,,$$

changes its sign no more than once for each λ.

For a one-dimensional space of elements the ordering simply means placing of a_i in correspondence with the increasing index, for a spherical surface, e.g., it means that a_i are placed similarly on the main circle.

This theorem can be proven algebraically, exploiting the fact, that the mentioned property of $A(a_i,\lambda)$ is necessary and sufficient condition for the polygon equality

$$\mathcal{D}\,(a_1,a_2) + \mathcal{D}\,(a_2,a_3) + \cdots + \mathcal{D}\,(a_{n-1},a_n) - \mathcal{D}\,(a_1,a_n) = 0 \tag{19}$$

to be held.

The considered property of $A(a_i,\lambda)$ has a simple geometrical content. For $\kappa = \pi$ it corresponds to the situation when both a_i and λ are vectors in some n-dimensional space and the procedure of evaluating of $A(a_i,\lambda) \equiv A(\overrightarrow{a}_i, \overrightarrow{\lambda})$ is defined as

$$A(\overrightarrow{a}_i, \overrightarrow{\lambda}) = \mathrm{sign}(\overrightarrow{a}_i \cdot \overrightarrow{\lambda})\,. \tag{20}$$

Using this geometric picture it is not difficult to satisfy relations (5), (6a), (6b) and/or (10a) and (10b) if we describe one particle by $\overrightarrow{\lambda}_1$ and the second one by $\overrightarrow{\lambda}_2$ with condition $\overrightarrow{\lambda}_1 \parallel \overrightarrow{\lambda}_2$ or $\overrightarrow{\lambda}_1 \uparrow\downarrow \overrightarrow{\lambda}_2$ in accordance with the symmetry of the system.

For $\kappa = \frac{\pi}{2}$, what corresponds, e.g., to the measurement of photon linear polarization the corresponding expressions are more complicated, nevertheless they can be also written down as signs of scalar products

$$A(a,\lambda) \equiv A(\overrightarrow{a}, \overrightarrow{\lambda}) = \mathrm{sign}(\overrightarrow{\lambda} \cdot \tilde{\lambda}^a)\,, \tag{21}$$

where $\tilde{\lambda}^a$ is constructed in such a way that $\overrightarrow{a}$ is a bissectrisse of the angle $\overrightarrow{\lambda}\tilde{\lambda}^a$. For fulfilling the relations (5), (6a), (6b) and/or (10a), (10b) the relation $\overrightarrow{\lambda}_1 \| \overrightarrow{\lambda}_2$ or $\overrightarrow{\lambda}_1 \perp \overrightarrow{\lambda}_2$ must be used.

We have introduced here the explicit expressions of (21) because we shall especially use this example further.

It is worth noting that the supposed properties of $\rho\,(\lambda)$ are very general up to now: we demand only $\rho\,(\lambda) > 0$, $\int \rho\,(\lambda)\,\mathrm{d}\lambda = 1$ and preserving the rotational invariance of $P(\varphi_{ab})$.

It appears that the LHV scheme acquires an intelligible content when it is expressed in geometrical terms. Let us proceed further and geometrize the measure of probability $\rho\,(\lambda)$. We shall use affine geometry for this purpose.

For that we suppose generally that the hidden parameters λ, which determine behaviour of each particle, are vectors in a n-dimensional space. For physical examples introduced above it is enough to use $n = 2$ or 3.

We place the beginnings of the vectors $\overrightarrow{\lambda}$ in the origin of the coordinate system. The ends of considered vectors then define some surface $\mathcal{S}$ with dimension $n-1$. As far as there exists one-to-one correspondence between $\overrightarrow{\lambda}_1$ (hidden parameter describing the first particle) and $\overrightarrow{\lambda}_2$ (which relates to the second one), there does not arise a misunderstanding, if we omit the indices "1" or "2" (due to the coupling the subsequent relations hold for both $\overrightarrow{\lambda}_1$ and $\overrightarrow{\lambda}_2$).

We shall describe the distribution of $\rho(\lambda)$ as

$$\rho(\lambda)\,\mathrm{d}\lambda \sim (\overrightarrow{\lambda} \cdot \overrightarrow{n})\,\mathrm{d}S\,, \tag{22}$$

where $\mathrm{d}\,S$ denotes an element of the surface $\mathcal{S}$, and $\overrightarrow{n}$ is a normal to the surface at the "point" $\overrightarrow{\lambda}$. The direction of $\overrightarrow{n}$ must be chosen so that $(\overrightarrow{\lambda} \cdot \overrightarrow{n}) \geq 0$. Then the normalization condition takes a form

$$\oint_{\mathcal{S}} (\overrightarrow{\lambda} \cdot \overrightarrow{n})\,\mathrm{d}S = 1\,. \tag{23}$$

Which $\rho(\lambda)$, $A(a,\lambda)$ and $B(b,\lambda)$ must be chosen for satisfying the conditions of Theorem 1? We know that for reaching the extremal values of $\mathcal{D}(a,b)$ and for satisfying relations $\mathcal{D}(0) = 0$ and $\mathcal{D}(\kappa) = \kappa$, it is enough to use functions of signs of the scalar products (20) or (21). Then for guaranteeing the rotational invariance of the correlation function (10) it is sufficient to express all terms as geometric invariants with respect to rotations. Due to the homogeneity and isotropy of the spaces with constant curvature, we can use any elementary geometry which can be introduced in such spaces: Euclidean, Lobatchevskian or Riemannian.

As far as the signs of the scalar product are invariant with respect to the rotations, what remains is an invariant description of $\rho(\lambda)$. It is clear that this invariance of $\rho(\lambda)$ will be guaranteed only if $\mathcal{S}$ will be identified with the spherical surface S^{n-1}. In this case

$$\overrightarrow{\lambda} \cdot \overrightarrow{n} = \mathrm{const}\,,$$

and hidden vectors are equally distributed in the space. Such a picture corresponds to the classical description of the singlet systems and the condition $\rho \neq \rho(a,b)$ is satisfied here automatically.

We can now formulate the answer to the first part of the posed question in the following way. For obtaining the limiting correlation function $P(\overrightarrow{a}, \overrightarrow{b})$ or $\mathcal{D}(\overrightarrow{a}, \overrightarrow{b})$ in the Bell scheme of LHV the conditions of Theorem 1 must be satisfied. It can be easily done if the invariant description in space with constant curvatures is used as far as here the condition $\rho \neq \rho(a,b)$ is guaranteed by natural isotropy of these spaces. In such a case the function $\mathcal{D}(\overrightarrow{a}, \overrightarrow{b})$ realizes the isomorphic mapping of the spherical surface defined by vectors $\overrightarrow{a}_1, \overrightarrow{a}_2, \dots,$

onto itself and the appearance of the metric of spherical or Riemannian geometries is a logical consequence of it (there are not other geometries on spherical surfaces in such spaces [25]).

Let us proceed to the second part of the problem – why the quantum correlations have a nonmetric form.

From the preceding consideration it follows that for the correct description of quantum phenomena the Bell's scheme of LHV must be extended. We propose preserving a general statistical scheme in such an extension as it is formulated in (1) together with the condition of "locality" expressed by (3).

For justification of the accepted postulates of (1) and of (3) we shall start our consideration from the case of correlations of photon linear polarizations, which can serve as an instructive concrete example. It will also demonstrate the usefulness of Minkowskian geometry in physics (cf. with the use of Minkowskian geometry 3 + 1 in special theory of relativity).

As far as the linear polarizations of photons can be classically described by vectors lying in the plane perpendicular to the photon momentum, we shall express the hidden variables λ as vectors with common beginnings lying in the plane. The ends of such vectors then will define $\mathcal{S}$ as a plane curve.

In accordance with (21), we define a projection of linear polarization of the first photon on $\overrightarrow{a}$ as

$$A(\overrightarrow{a}, \overrightarrow{\lambda_1}) = \operatorname{sign}(\overrightarrow{\lambda_1} \cdot \tilde{\lambda}_1^a)\,, \tag{24a}$$

and, similarly, of the second photon on $\overrightarrow{b}$ as

$$B(\overrightarrow{b}, \overrightarrow{\lambda_2}) = \operatorname{sign}(\overrightarrow{\lambda_2} \cdot \tilde{\lambda}_2^b)\,. \tag{24b}$$

The relation between $\overrightarrow{\lambda_1}$ and $\overrightarrow{\lambda_2}$ is postulated as $\overrightarrow{\lambda}_1 \| \overrightarrow{\lambda}_2$ for singlet systems with even parity and as $\overrightarrow{\lambda_1} \perp \overrightarrow{\lambda_2}$ for singlet systems with odd parity. When only one variable is used, then the relations (24a) and (24b) can be rewritten as follows

$$A(\overrightarrow{a}, \lambda) = \operatorname{sign}(\overrightarrow{\lambda} \cdot \tilde{\lambda}^a)\,, \tag{25a}$$

$$B(\overrightarrow{b}, \lambda) = \pm \operatorname{sign}(\overrightarrow{\lambda} \cdot \tilde{\lambda}^b)\,, \tag{25b}$$

where the upper sign relates to the even parity states and the lower to the odd ones.

Procedures (24a), (24b) or (25a) and (25b) guarantee fulfilling the following relations for $P(\overrightarrow{a}, \overrightarrow{b})$ and $\mathcal{D}(\overrightarrow{a}, \overrightarrow{b})$ in accordance with quantum mechanics for any positive and normalized $\rho(\lambda)$

$$P(\overrightarrow{a}, \overrightarrow{b}) = \begin{cases} \pm 1\,, & \text{if } \varphi_{ab} = n\pi; \quad n = 0, 1, 2, \ldots \\ 0\,, & \text{if } \varphi_{ab} = \frac{2n+1}{4}\pi; \quad n = 0, 1, \ldots \\ \mp 1\,, & \text{if } \varphi_{ab} = \frac{2n+1}{2}\pi; \quad n = 0, 1, 2, \ldots \end{cases} \tag{26a}$$

and, consequently,

$$\mathcal{D}(\overrightarrow{a}, \overrightarrow{b}) = \begin{cases} 0\,, & \text{if } \varphi_{ab} = n\pi; \quad n = 0, 1, 2, \ldots \\ \frac{\pi}{4}\,, & \text{if } \varphi_{ab} = \frac{2n+1}{4}\pi; \quad n = 0, 1, 2, \ldots \\ \frac{\pi}{2}\,, & \text{if } \varphi_{ab} = \frac{2n+1}{2}\pi; \quad n = 0, 1, 2, \ldots \end{cases} \tag{26b}$$

The relations (26b) for $\mathcal{D}(\overrightarrow{a}, \overrightarrow{b})$ correspond to the distance between given elements $\overrightarrow{a}$ and $\overrightarrow{b}$ as it is defined in Riemannian geometry on the circle.

Now we have met a problem of invariance, which is closely related to the properties of the geometry used. In a plane there are two nonequivalent elementary geometries preserving a quadratic form under rotation: Euclidean geometry with invariant of $x^2 + y^2 = \text{const}$, and Minkowskian geometry with invariants $x^2 - y^2 = \pm\,\text{const}$ [26]. As far as the former one is a metric space, the latter is not, it will be interesting to compare results which follow from the use of both of them [11]. The invariant description in terms of Euclidean geometry guarantees fulfilling the conditions of Theorem 1 and we thus obtain

$$\mathcal{D}(\varphi_{ab}) = \varphi_{ab}, \quad 0 \leq \varphi_{ab} \leq \frac{\pi}{2}, \tag{27a}$$

or

$$P(\varphi_{ab}) = \pm 1 \mp \frac{4\varphi_{ab}}{\pi}, \tag{27b}$$

on the same interval. Here φ_{ab} denotes the angle between vectors $\overrightarrow{a}$ and $\overrightarrow{b}$; the upper sign relates to the even states and the lower to the odd states, respectively.

The full functional dependence of $\mathcal{D}(\varphi_{ab})$ and $P(\varphi_{ab})$ on φ_{ab} can be obtained by direct computations or by the use of relation (10c). We remind that in the case of linear photon polarizations $\kappa = \frac{\pi}{2}$.

Let us postulate now the invariance of the LHV space with respect to the hyperbolic rotations, preserving forms $x^2 - y^2 = \pm\,\text{const}$:

$$\begin{aligned} x' &= x \cosh\theta - y \sinh\theta, \\ y' &= -x \sinh\theta + y \cosh\theta. \end{aligned} \tag{28}$$

It is not difficult to rewrite all expressions needed as invariants with respect to (28). In such a case we have

$$A(\overrightarrow{a}, \lambda) = \operatorname{sign}\big(\|\lambda^a\|_h^2\big), \tag{29a}$$

and

$$B(\overrightarrow{b}, \lambda) = \pm \operatorname{sign}\big(\|\lambda^b\|_h^2\big), \tag{29b}$$

where $\|\lambda^r\|_h^2$ denotes the square of the hyperbolic norm of

$$\|\lambda^r\|_h^2 = \lambda_x^2 - \lambda_y^2,$$

when axis x is identified with $\overrightarrow{r}$ ($\overrightarrow{r} = \overrightarrow{a}, \overrightarrow{b}$).

A curve which defines distribution of $\rho(\lambda)$ must be evidently composed of four branches of hyperbolas

$$x^2 - y^2 = \pm c_1,$$

here c_1 is a constant.

It appears that in the considered case a question arises of orientation of the Minkowski coordinate system in relation to $\overrightarrow{a}$ and $\overrightarrow{b}$. Such a problem appears always when we use Minkowskian geometry in the concrete physical situation. In special theory of relativity, for example, the used coordinate system must be also oriented in a definite way, but here it looks not so striking because the opposite sign in the invariant $x^2 + y^2 + z^2 - c^2t^2$ has a time variable.

In our case the correct quantum mechanical result for the correlations of the projection of the photon linear polarizations in singlet systems can be obtained if we put the coordinate axis x or y parallel to $\overrightarrow{a}$ or $\overrightarrow{b}$. Let us suppose that $\overrightarrow{x} \| \overrightarrow{a}$. Then the normalized probability density takes a form

$$\rho(\lambda)\,\mathrm{d}\lambda \to \rho_{\overrightarrow{a}}(\lambda)\,\mathrm{d}\lambda = c_2(\overrightarrow{n} \cdot \overrightarrow{\lambda})_e = \frac{1}{4}\frac{\|\lambda^a\|_h^2}{\|\lambda\|_e^2}\,, \tag{30}$$

here c_2 is constant. The subscript "e" in the scalar product stresses the using of scalar product as it is defined in Euclidean geometry, $\|\lambda\|_e^2$ is the square of the ordinary Euclidean norm, and $\|\lambda^a\|_h^2$ is the square of hyperbolic norm for $\overrightarrow{a} \| \overrightarrow{x}$.

Using the basic relation (1) and (29a), (29b) with (30) we obtain

$$P(\overrightarrow{a}, \overrightarrow{b}) = \pm\|b^a\|_h^2\,, \tag{31a}$$

or, after interchanging $\overrightarrow{a} \leftrightarrow \overrightarrow{b}$

$$P(\overrightarrow{a}, \overrightarrow{b}) = \pm\|b^a\|_h^2 = \pm\|a^b\|_h^2 = \pm\cos 2\varphi_{ab}\,, \tag{31b}$$

in accordance with the quantum mechanical results. In the expressions (31a) and (31b) the upper sign relates to the even states and the lower to the odd states, respectively, φ_{ab} is an angle between vectors $\overrightarrow{a}$ and $\overrightarrow{b}$.

Without going into details let us notice that the other quantum correlations as projections of spins of stationary singlet states ($s = 1/2$) and of precessing ones in the constant magnetic field, projections of the flavour characteristic in the case of oscillating K and B mesons, the correlations of interfering bosons or fermions can be correctly described with the use of the relative measure of probability $\rho_{\overrightarrow{r}}(\lambda)$ in the space of the LHV with destroyed isotropy, too [6, 16, 17].

From the macroscopic point of view our space is isotropic, however. It is reflected by the fact that $P^{QM}(a,b)$ or $\mathcal{D}^{QM}(a,b)$ are functions of the difference of $a - b$ only. How to reconcile this symmetry with the symmetry of the LHV space, which appears to be different (cf., e.g., the transformations of (28))?

It turns out that we must restrict the definition given by (1) by the following condition

$$P^{QM}(\overrightarrow{a}, \overrightarrow{b}) \equiv P_{\overrightarrow{r}}(\overrightarrow{a}, \overrightarrow{b}) = \int A(\overrightarrow{a}, \lambda)\, B(\overrightarrow{b}, \lambda)\, \rho_{\overrightarrow{r}}(\lambda)\, \mathrm{d}\lambda\,, \tag{32}$$

where $\overrightarrow{r} = \overrightarrow{a}$ or $\overrightarrow{b}$.

In the next Section we shall see that (32) can be generalized in a way: as $\overrightarrow{r}$ all elements, for which transformation of $\rho_{\overrightarrow{a}} \to \rho_{\overrightarrow{r}}$ or $\rho_{\overrightarrow{b}} \to \rho_{\overrightarrow{r}}$ can be interpreted as ordinary rotations in Euclidean space, can be taken.

We have introduced this new concept, $\rho_{\vec{r}}(\lambda)$, which we already called a relative measure of probability in our first paper on the subject [6]. Such a concept reminds in some sense of N. Bohr's approach to the problem of quantum mechanics [27], when he stressed the necessity of relating the theoretical description of the quantum phenomena to the concrete physical situation defined by measuring devices.

The relative measure of probability permits us to overcome the limits of Bell's inequality and that of Braunstein and Caves due to the destroyed isotropy of the LHV space with respect to the space elements a and b [6, 10, 13]. It practically means that we cannot describe the correlation on three vectors in general positions with the use of only one $\rho_{\vec{r}}$. This fact is also responsible for the nondistributivity of the quantum logic propositions in this model.

It is evident that the possibility of the description of a set of correlation measurements with one $\rho_{\vec{r}}$ does not depend on the fact if these correlations can be measured in one run or not. In practice any function of $P(\vec{a}, \vec{b})$ must be measured in a separate experimental run. Due to the definition (32) we can describe with one common probability measure $\rho_{\vec{r}}$ (and, therefore, to calculate the joint probabilities) all correlations with one common vector $\vec{a}$ or $\vec{b}$. It is in accord with the result of analysis of A. Fine, which has proved this property of LHV by the algebraic method without any reference to the concrete model [28].

As we have already shown [13] and have demonstrated practically in considering the Greenberger–Horne–Zeilinger correlations [16], the possibility of using one probability measure for the considered correlated system does not depend on the commutativity of the measured operators either. We shall return to this question in the last Section in connection with the others "no-go" theorems for hidden variables.

4. Relative measure of probability and its physical implications

It has appeared that for obtaining correct values for the quantum correlations we have been forced to destroy the isotropy of LHV space with respect to the space elements defined by measuring devices and to abandon the original Bell's scheme of LHV. In which way can the introduced RM be interpreted?

The most direct possibility is an interpretation as a consequence of so-called nonlocal influences [29, 30]. Really, we can imagine that before the measurement the system is described by the absolute $\rho(\lambda)$ which is independent of a and b. Then the origin of $\rho_a(\lambda)$ or $\rho_b(\lambda)$ can be explained as a result of the interaction of the measuring device with the system as it is demonstrated in Fig. 2. Such an explanation, however, is explicitly nonlocal in the case of space-like correlations, because the measurement of the projection of the first particle changes the distribution of λ_2 of the second remote one. As a consequence the unavoidable inconsistencies as in the orthodox interpretation of quantum mechanics will appear. Because of the nonsymmetric actions of both apparatuses (really, the first device changes the state of both particles and the second one only registers this change) there arise problems when the order of the events is reversed. Space-like events do not obey absolute time ordering and in the case of time dependent phenomena [17] undesired retroaction appears; it is not clear how to explain the phenomenon when both measurements are done in some reference frame at the same moment, etc.

Careful analysis of the experiments performed with photons [19] permits another logically possible explanation of the quantum correlations in this special case. The low efficiency of the photon detectors used allows one to explain the mentioned correlations as a result of the subtle interplay between registered and unregistered pairs [31, 32].

Actually, we can start our consideration again with $\rho(\lambda) = \text{const}$ as in the preceding case and explain the distorted distribution with respect to the orientations of devices as a result of special choice during the measurement (See Fig. 3). Nevertheless, in this model both detectors cannot act independently, because in such a case the resulting probability density is a product of $\rho_a(\lambda_1)$ and $\rho_b(\lambda_2)$ and it is impossible to define similar (or identical) functions of ρ_a and ρ_b and at the same time to obtain constant counting rate for registered pairs which is independent of φ_{ab}. But it is possible, with some caution, to formulate such a hidden variable theory which can explain the experiments performed with these low efficiency detectors. The deficiency of this interpretation is its limited universality [1].

In our first work [6] we have proposed another interpretation of the relative measure, we have taken it as a necessity of the only relative description of the space-like correlations. In such an interpretation the space elements defined by measuring devices play a role of reference frame to which the description of the physical system can be related. We have used certain features of the methodology of the special theory of relativity in this task. It seems reasonable to use the following assumptions in our case.

A1. Principle of covariant description in any reference frame used. In practice it means that all procedures of $A(a, \lambda^i)$, $B(b, \lambda^i)$ and also $\rho(\lambda^i)$ must have the same functional dependence on the corresponding variables λ^i. This will guarantee the equivalence of the reference frames and actions of both apparatuses can be described symmetrically.

A2. Principles of relativistic causality hold in quantum phenomena, too. As far as each concrete measurement takes place in finite space-time region, the space-like measurements cannot influence themselves. Hence, all that concerns the second particle when the first one is measured can be interpreted only as a gain of information about it, related to the frame of reference defined by the first apparatus.

The use of both principles is schematically demonstrated in Figs. 4 and 5, where the measurements of the correlations of the photon linear polarizations in states with even parity are presented (for definiteness the angle $\varphi_{ab} = \frac{\pi}{4}$ is chosen).

There arises a question of the relation between different reference frames and, consequently, between λ_1^a and λ_2^b. This relation can be found by specifying the character of the transformations between $\lambda_1^a \leftrightarrow \lambda_2^b$ and of the invariants which such a transformation preserves. For that we propose to use two additional assumptions.

A3. There exists one-to-one correspondence between $\lambda_1^a \leftrightarrow \lambda_2^b$.

A4. The transformation $\lambda_1^a \leftrightarrow \lambda_2^b$ must preserve each concrete event as an invariant, i.e. the result of each concrete measurement must not depend on the frame reference used for its description.

Hence, it must hold

$$A(\overrightarrow{a}, \lambda^a) = A(\overrightarrow{a}, \lambda^b) \quad \text{and} \quad B(\overrightarrow{b}, \lambda^b) = B(\overrightarrow{b}, \lambda^a). \tag{33}$$

These principles were applied in the Figs. 4 and 5, too. We have used here for the better understanding discrete values of λ with $\Delta\lambda = \frac{\pi}{8}$. The continuous distribution of λ can be obtained in the limit $\Delta\lambda \to 0$.

We can imagine the measuring of the correlations as a computer game. Let us have a set of numbers which describe the frequencies of λ_1^a according to $\rho_a(\lambda_1^a)$ depicted in Fig. 4. The measurement of polarization of any particle on $\overrightarrow{a}$ can be described as a tossing by chance one of these numbers, which determines λ_1^a and, consequently, the corresponding projections on $\overrightarrow{a}$: $A(\overrightarrow{a}, \lambda_1^a)$. Due to the connection $\lambda_1^a \leftrightarrow \lambda_2^b$ the first experimenter can predict the result of the projection of the second paired particle in any direction $b_1, b_2, \ldots$, with certainty. The appearance λ_2^b here is not interpreted as a change of the state of the second particle, but only as a gain of information about it, expressed in a reference frame connected with ρ_a. The first experimenter can also exploit (33) to reconstruct the whole picture as it is perceived by the second experimenter by $\rho_b(\lambda_2^b)$ (See Figs. 4 and 5).

It is evident that all these procedures can be reversed. We can start our consideration from the reference frame defined by the second device. As far as our attitude is strictly local, there do not arise additional difficulties when the real order of the events is reversed and/or when the devices change their orientation during the passage of the particles from the source to the detectors [43].

A peculiar property of the transformation $\rho_a(\lambda_1^a) \Leftrightarrow \rho_b(\lambda_2^b)$ is worth noting here. Using assumptions A1 – A4, the unambiguous correspondence between λ_1^a and λ_2^b can be established only for

$$\varphi_{ab} = \frac{2n+1}{4}\pi; \quad n = 0, 1, 2, \ldots,$$

while for

$$\varphi_{ab} = n\,\frac{\pi}{2}; \quad n = 0, 1, 2, \ldots$$

there remains an unambiguity in placement of λ_1^a or λ_2^b for removing of which another bit of information is needed. This situation can be understood on the base of the information theory. We determine, say, the position of λ_2^b in corresponding reference frame exploiting the knowledge of λ_1^a and of the value of the projection $B(b, \lambda_2^a)$ for what, generally, one bit of information is needed. This bit is in disposal for

$$\varphi_{ab} = \frac{2n+1}{4}\pi; \quad n = 0, 1, 2, \dots,$$

because for that value of φ_{ab} we have $H(\varphi_{ab}) = 1$ (the mean conditional entropy is expressed in bits if 2 is used as a base of logarithm in (14) and has a meaning of the information which is stored in the pair and can be revealed by measurement), but not for

$$\varphi_{ab} = n\,\frac{\pi}{2}; \quad n = 0, 1, 2, \dots,$$

where $H(\varphi_{ab}) = 0$.

For other values of φ_{ab} we get $0 < H(\varphi_{ab}) < 1$, which is an intermediate situation between both considered limits. In order to remove the mentioned unambiguity we have used the additional rule for establishing the relation between λ_1^a and λ_2^b. It was the following

- a1) the small changes of φ_{ab} lead to the small changes of the ordering of λ, or, equivalently,
- a2) if there are two possibilities for placing of certain λ, then there is preferred the natural order of indices.

It can be shown, that neither the preceding nor the following investigations are influenced by this special choice.

As far as we interpret the distributions of $\rho_a(\lambda_1^a)$ and $\rho_b(\lambda_2^b)$ as description of the photon singlet state in a different reference frame, we can define an operator which transforms one distribution into another

$$\rho_b(\lambda_2^b) = \hat{R}(\varphi_{ab})\ \rho_a(\lambda_1^a)\,. \tag{34}$$

On the basis of A1 - A4 it can be shown, that there always exists $\left[\hat{R}(\varphi_{ab})\right]^{-1}$ such that

$$\left[\hat{R}(\varphi_{ab})\right]^{-1}\ \hat{R}(\varphi_{ab}) = 1\,, \tag{35}$$

where the transformation of identity can be taken as $\hat{R}(\varphi_{ab} = 0)$.

Other properties of the transformation $\hat{R}(\varphi_{ab})$ can be specified as it follows [43]

- P1. Generally $\hat{R}(\varphi_{ab})$ do not form a group, i.e. a general product of two transformations

  $$\hat{R}(\varphi_{ab}) \cdot \hat{R}(\varphi_{bc})$$

  does not represent a transformation.
- P2. Operators of $\hat{R}(\varphi_{ab})$ form only a cyclic group with elements $\hat{R}(\varphi_{ab})$, $\varphi_{ab} = n\frac{\pi}{2}$, $n = 0, 1, 2, \dots$, for which the transformation $\hat{R}$ is an ordinary rotation in Euclidean space (this result can be generalized: the number of the independent elements of the cyclic group is equal to the number of operators of the task which mutually commute).

Both mentioned properties of transformation of $\rho(\lambda)$ are not so unexpected, because it is known that whereas the transformations of a state vector in quantum mechanics generally form a group, it is not so for probabilities and a special role of the commutations (P2) in the LHV scheme can be expected, too.

It must be emphasized, however, that both properties P1 and P2 of $\hat{R}(\varphi_{ab})$ were deduced here as a consequence of the different symmetries which take place in the space of the LHV and in the ordinary macroscopic space defined by the measuring devices. In other words, we can use the measurement of quantum correlations as a mean for the study of the symmetry of the LHV space.

It appears that the description with the relative measure of probability demonstrates both classical and nonclassical behaviour.

The classical properties may be seen in the fact, that RM admits a concept of counterfactual definiteness [35] (= the possibility of considering unique results of the unperformed experiments). It can be seen from the following. Knowledge of λ_1^a of the first particle and relation between λ_1^a and λ_2^a makes it possible to predict definite results of experiment on the second particle, when its projection is measured on any direction of $\overrightarrow{b}$. It allows us also to use λ_2^b in the same way and to predict the definite result of the measurement of the projection of the first particle on any direction of $\overrightarrow{a}$. In this sense the unperformed experiments have the unique results. Due to the fact that the transformations of $\hat{R}(\varphi_{ab})$ do not form a continuous group of parameter φ_{ab}, however, this property cannot be used for the derivation of contradiction with quantum mechanics. In this respect our proposed model has a similar property as the model introduced before by Pitowsky [36]–[38].

Nonexistence of the group of transformations of $\rho(\lambda)$ has also peculiar consequences when the propositions based on different probability schemes are combined. In the pure quantum mechanical treatment it can lead to the different type paradoxes and confusions.

As an example, the "relativity of the quantum mechanical predictions" indicated by Pitowsky [39] can be introduced or the analysis of the Hardy's paper about relativistic invariance of the LHV picture [40]. In the language of quantum mechanics such paradoxes can be solved if the foreseeing prescription of von Neumann about necessity of the definition of the measuring procedure for each operator is used [41].

In the language of the RM a special care must be taken when the different probability schemes are combined.

5. Conclusion

We have investigated thoroughly implications following from the Bell inequalities for LHV and have indicated in which way these constraints can be overcome by the relative probability measure. It is not difficult to understand why the RM invalidates other no-go theorems as Braunstein and Cave's inequalities [23], Feynman's inequality [42, 43], and von Neumann's theorem [13, 41] as far as they use the concept of independent probability measure. It was shown [16], that the consistent use of the concept of RM succesfully solves paradoxes connected with Greenberger–Horne–Zeilinger correlations [44], too. It must be stressed that the main problem is neither in the commutativity of the measured operators (e.g. GHZ-like correlations deal with commuting operators) nor in the simultaneous measurability of them (as a rule the different setting of measuring devices are needed for their verification), but in the possibility to describe the experimental situation with single RM.

Similar theories are often called contextual. We are not sure if such a denotation is truthful. Because the contextuality may include also the nonlocal behaviour, we have preferred the term relative measure of probability which, in our opinion, corresponds better to its content.

It is possible to say that from the point of view of the historical controversy between Einstein and Bohr about the role and content of quantum mechanics the concept of RM lies somewhere in the middle between these contradictory positions. In other words, it partially gives truth both of them [45].

It remains to comment three important questions concerning RM. Is it really a local model? Does it correspond to the concept of realistic description of Nature? Does the use of RM lead to a more reasonable explanation of the quantum phenomena?

As to locality, it can be certainly asserted that the description with the RM is local. It is constructed and interpreted in this way.

More difficult is the problem of its reality (here we mean the independence of the description on the phenomenon of the observer). In any case, the RM contradicts the Bell scheme of LHV, which is very often taken as a synonym of "local realism" and it seems not to fulfil definitions of the realistic description, given, e.g., by Santos [7] or Gudder [46]. A wider concept of reality could be a way out of this situation, but the author does not feel himself strong in this rather philosophical field.

The analogy with the special theory of relativity can help in claryfying the whole problem. Can we speak about masses, distances and the time intervals without specifying the frame of reference which we use for description of the physical phenomena? We think, that we do, although for the concrete consideration we must use a definite reference frame. The description of a concrete phenomenon will depend on the frame reference used, the concrete variables will acquire different values, the geometrical picture will change, the space-like events may be reversed, the different frequencies due to the Thomas rotation can be registered, etc. All this will not bother us, because we know, that it is a direct consequence of the transformation law of the special theory of relativity. Moreover, we know, that there exists a special reference frame with some extremal properties, when masses and time intervals are minimal and distances maximal.

A very analogous reasoning may be done in our case for the relative probability measure, too, when it is considered from the different "frame of reference" defined by the measuring devices. In this case the special reference frame also exists, which has some extremal properties. It is that one, where the measurements actually occur. It differs from the other frames by the fact that in it the information entropy of the measured part of the considered system can reach a minimum (equal to zero) by the proper evaluating of hidden variables.

The use of the RM as an interpretational language of quantum mechanics can be useful when we feel the necessity to explain what does happen in a quantum phenomenon. This scheme, by our opinion, is as close to the classical statistics as possible. It allows us to explain the correlations of the space-like events in local way – as a consequence of the symmetry of the system considered. The price which we must pay for it [47], is the abandoning of the concept of the absolute independent probability measure, which is an inherent feature of classical statistics, and, therefore, the loss of the possibility to describe the quantum phenomena per se.

Whether this price will be compensated by the better understanding of the quantum world is an open question. It would be desirable to generalize the proposed concept in such a way so that the rational heart of the matter would not be lost, to consider possibility of application of the RM in other quantum phenomena, to make clear the question of the possible relativization of this model, to reveal a dynamics of hidden variables,etc.

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FIGURES

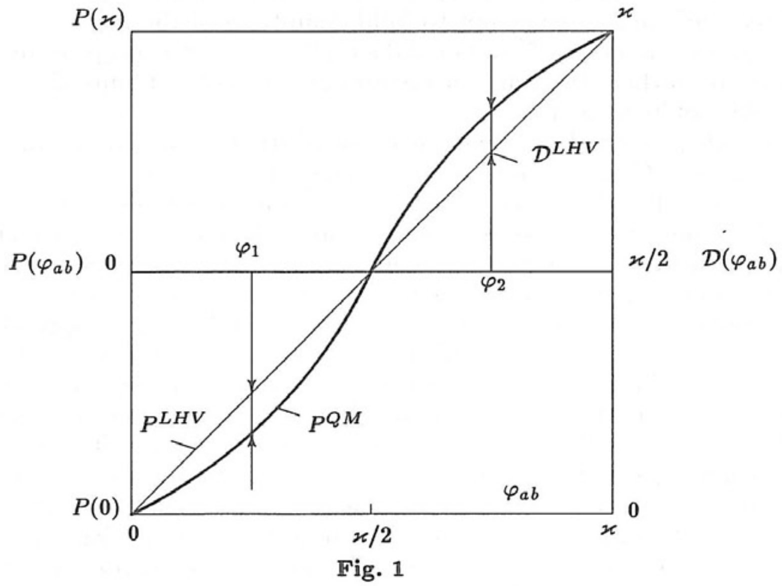

Fig. 1

The dependence of quantum mechanical function P^{QM} and of P^{LHV} and

$\mathcal{D}^{LHV}$ in Bell's scheme of the LHV on φ_{ab}. In this Fig. 1 P^{LHV} and $\mathcal{D}^{LHV}$ are presented by the same straight line. The relation between P^{QM} and P^{LHV} is demonstrated for values of φ_1 and φ_2 (See text).

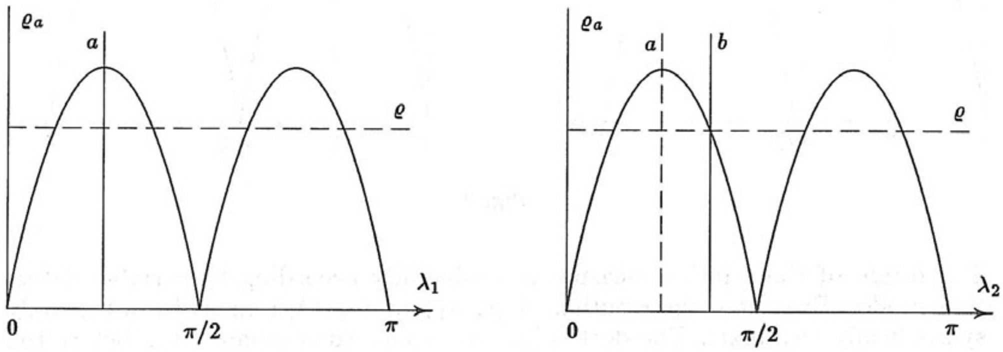

Fig. 2

Distribution of $\rho_a(\lambda_1)$ and $(\rho_a(\lambda_2)$ corresponding to the first and second particle, respectively. It is supposed that the measurements of the projection of the first particle immediately changes distribution of λ_2 of the second one. The projections are evaluated according to

$$A(a,\lambda_1) = \begin{cases} +1, & \text{for } |a-\lambda_1| \le \frac{\pi}{4}, \\ -1, & \text{otherwise}, \end{cases}$$

$$B(b,\lambda_2) = \begin{cases} +1, & \text{for } |b-\lambda_2| \le \frac{\pi}{4}, \\ -1, & \text{otherwise}, \end{cases}$$

which corresponds to the singlet state with even parity. The dotted line $\rho = \text{const}$ corresponds to ρ before the measurement. In the Fig. 2 distribution of λ only up to π is presented for brevity.

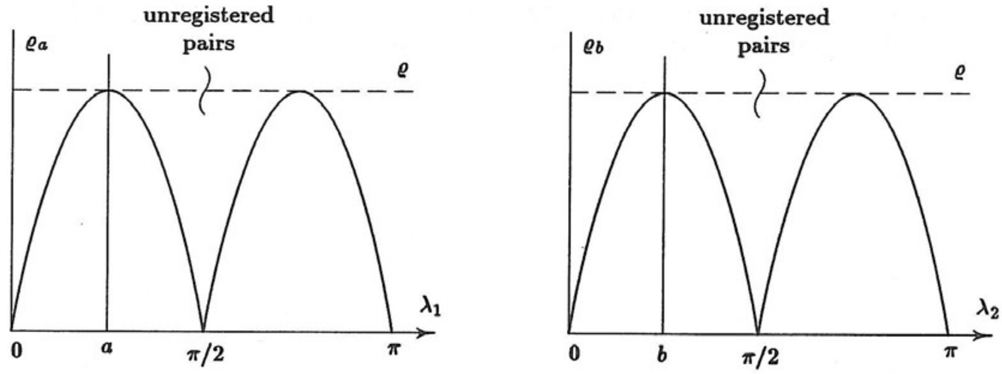

Fig. 3

The origin of the relative measure of probability according to variable detection model. Presented distribution of $\rho_a(\lambda_1)$ and $\rho_b(\lambda_2)$ must be taken only symbolically (see text). The dotted line $\rho = \text{const}$ corresponds to ρ before the measurement.

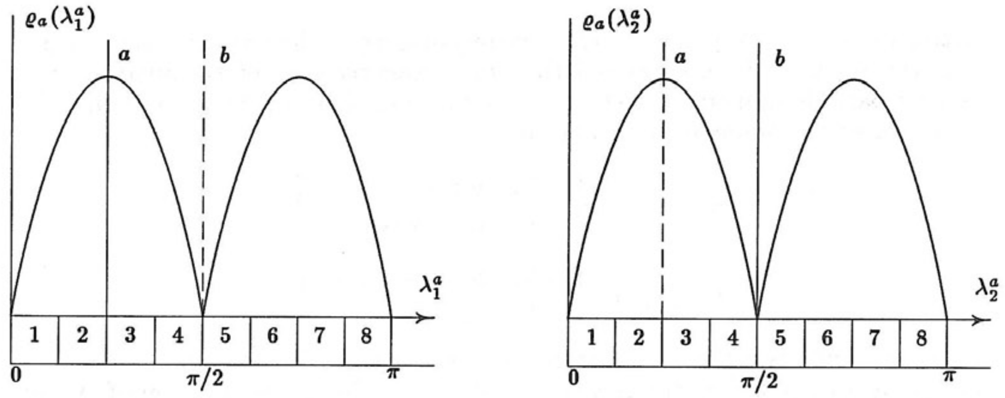

Fig. 4

Distribution of λ_1^a and λ_2^a as it perceived by first experimenter measuring the projection of the linear polarization of the first particle along $\vec{a}$.

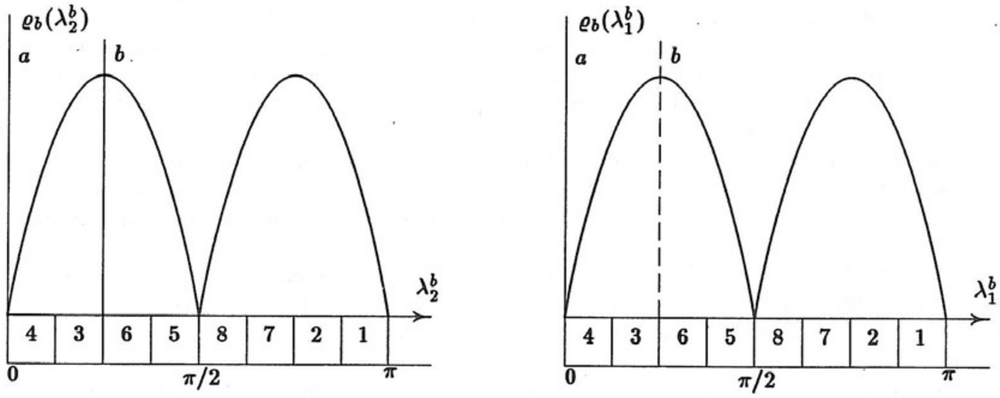

Fig. 5

Distribution of λ_2^b and λ_1^b as it is perceived by second experimenter measuring the linear polarization of the second particle along $\overrightarrow{b}$. The projections on both Figs. 4 and 5 are evaluated according to

$$A(a,\lambda_1) = \begin{cases} +1, & \text{for } |a-\lambda_1| \le \frac{\pi}{4}, \\ -1, & \text{otherwise}, \end{cases}$$

$$B(b,\lambda_2) = \begin{cases} +1, & \text{for } |b-\lambda_2| \le \frac{\pi}{4}, \\ -1, & \text{otherwise}, \end{cases}$$

which corresponds to the singlet state with even parity. The denumeration of boxes of the discrete values of λ in Fig. 5 is obtained from the transformation law of $\rho_a(\lambda_1^a)$ to $\rho_b(\lambda_2^b)$ as it is performed in the frame reference related to $\overrightarrow{a}$. In both Figs.4 and 5 the distribution of λ only up to π is presented for brevity.

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