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WEAK ISOMETRIES AND DIRECT DECOMPOSITIONS OF PARTIALLY ORDERED GROUPS

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Dedicated to Professor J. Jakubík on the occasion of his 70th birthday

ABSTRACT. In this paper the relations between weak isometries in a partially ordered group H and direct decompositions of H are investigated.

Isometries in an abelian lattice ordered group (l-group) have been introduced and investigated by Swamy [15], [16]. Jakubík [4] proved that for every stable isometry f in an l-group G there exists a direct decomposition $G = A \times B$ of G such that $f(x) = x(A) - x(B)$ for each $x \in G$. Isometries in non-abelian l-groups were also studied in [1] and [5]. Weak isometries in l-groups were introduced by Jakubík [6]. Rachůnek [14] generalized the concept of the isometry for any partially ordered group (po-group). Isometries and weak isometries in some types of po-groups have been investigated in [7], [8], [9], [10], [12], [13], [14]. In [11] it was proved that each stable weak isometry in a directed group is an involutory group automorphism.

First we recall some notions and notations used in the paper.

Let G be a po-group. The group operation will be written additively. We denote $G^+ = \{x \in G;\ x \geq O\}$. If $A \subseteq G$, then we denote by $U(A)$ and $L(A)$ the set of all upper bounds and the set of all lower bounds of the set A in G, respectively. For $A = \{a, b\}$ we shall write $U(a,b)$, $(L(a,b))$ instead of $U(\{a,b\})$, $(L(\{a,b\}))$. If for $a, b \in G$ there exists the least upper bound (greatest lower bound) of the set $\{a,b\}$ in G, then it will be denoted by $a \vee b$ $(a \wedge b)$. For each $a \in G$, $|a| = U(a, -a)$.

A po-group G is said to be the *direct product* of its posubgroups G_1 and G_2 (notation $G = G_1 \times G_2$) if the following conditions are fulfilled:

(1) If $a \in G_1$, $b \in G_2$, then $a + b = b + a$.

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(2) Each element $c \in G$ can be uniquely represented in the form $c = c_1 + c_2$, where $c_1 \in G_1$, $c_2 \in G_2$.

(3) If g, $h \in G$, $g = g_1 + g_2$, $h = h_1 + h_2$ where $g_1, h_1 \in G_1$, $g_2, h_2 \in G_2$, then $g \leq h$ if and only if $g_1 \leq h_1$, $g_2 \leq h_2$.

In this case it is also spoken about the *direct decomposition* of the po-group G. If $G = P \times Q$ is a direct decomposition of G, then for $x \in G$ we denote by $x(P)$ and $x(Q)$ the components of x in the direct factors P and Q, respectively.

If G is a po-group, then a mapping $f\colon G \to G$ is called a *weak isometry* if

$$\big|f(x) - f(y)\big| = |x - y| \qquad \text{for each} \quad x, y \in G .$$

A weak isometry f is called a *stable weak isometry* if $f(O) = O$. A weak isometry f is called an *isometry* if f is a bijection. Let $a \in G$, the mapping $g\colon G \to G$ defined by $g(x) = x + a$ for each $x \in G$, is called a *right translation* in G.

A po-group G is called *directed* if $U(x, y) \neq \emptyset$ and $L(x, y) \neq \emptyset$ for each $x, y \in G$.

A *Riesz group* is any po-group G which is directed and satisfies the Riesz interpolation property, i.e., for each $a_i, b_j \in G$ $(i, j = 1, 2)$ such that $a_i \leq b_j$ $(i, j = 1, 2)$ there exists $c \in G$ such that $a_i \leq c \leq b_j$ $(i, j = 1, 2)$.

A po-group G is called *isolated* if $a \in G$ and $a \geq O$ for some positive integer imply $a \geq O$.

The set of all positive integers will be denoted by $\mathbb{N}$.

1. The sets A_1, B_1, A and B

If g is a weak isometry in a po-group H and we put $h(x) = g(x) - g(O)$ for each $x \in H$, then h is a stable weak isometry in H. So, every weak isometry in a po-group can be uniquely represented as a composition of a stable weak isometry and a right translation. Thus it suffices to examine only stable weak isometries.

Throughout this section we suppose that f is a stable weak isometry in a po-group G.

We denote

$$A_1 = \big\{x \in G^+;\ f(x) = x\big\}, \quad B_1 = \big\{x \in G^+,\ f(x) = -x\big\},$$
$$A = A_1 - A_1, \quad B = B_1 - B_1 .$$

1.1. THEOREM.

(i) *If $x \in A_1$, then $f(-x) = -x$.*

(ii) *If $x \in B_1$, then $f(-x) = x$.*

Proof. i) From $U(x) = |x| = |-x| = \left|f(-x)\right|$, we get $x \geq f(-x)$. Then from

$$U(2x) = |2x| = \left|x - (-x)\right| = \left|f(x) - f(-x)\right| = \left|x - f(-x)\right| = U\big(x - f(-x)\big)$$

we obtain $f(-x) = -x$.

Analogously we can verify (ii). □

1.2. THEOREM. *Let x, $y \in A_1$. Then*

(i) $f(x+y) = x+y$,

(ii) $f(x-y) = x-y$,

(iii) $f(-x+y) = -x+y$.

Proof. (i) From the relation

$$|x| = |x+y-y| = \left|f(x+y) - f(y)\right| = \left|f(x+y) - y\right|$$

we get $x \geq y - f(x+y)$. Thus $x + f(x+y) \geq y \geq O$. Then we obtain $f(x+y) + x \geq O$.
By 1.1 (i),

$$|x+y+x| = \left|f(x+y) - f(-x)\right| = \left|f(x+y) + x\right|.$$

Since $x+y+x \geq O$, we have $f(x+y) = x+y$.

The other propositions of the theorem can be verified analogously. □

1.3. THEOREM. *Let x, $y \in B_1$. Then*

(i) $f(x+y) = -x-y$,

(ii) $f(x-y) = -x+y$.

Proof. (i) Since

$$|x| = |x+y-y| = \left|f(x+y) - f(y)\right| = \left|f(x+y) + y\right|,$$

we have $-f(x+y) + x \geq y \geq O$. Then $x - f(x+y) \geq O$. According to 1.1(ii),

$$|x+y+x| = \left|f(x+y) - f(-x)\right| = \left|f(x+y) - x\right|.$$

This implies $f(x+y) = -x-y$.

Analogously we can verify (ii). □

1.4. THEOREM.

(i) *A_1 is a semigroup.*

(ii) *A is a convex subgroup of G.*

(iii) *$A^+ = A_1$.*

P r o o f . (i) It is a consequence of 1.2.

(ii) Let $a, b \in A$. Hence $a = a_1 - a_2$, $b = b_1 - b_2$, where $a_1, a_2, b_1, b_2 \in A_1$. Then we have

$$a + b = (a_1 + b_1) - (b_2 - b_1 + a_2 + b_1) .$$

From (i) we get $a_1 + b_1$, $a_2 + b_1 \in A_1$.

By 1.2, $f(-b_1+a_2+b_1) = -b_1+a_2+b_1$. Since $-b_1+a_2+b_1 \geq O$, $-b_1+a_2+b_1 \in A_1$. Then from 1.2 we obtain

$$f(b_2 - b_1 + a_2 + b_1) = b_2 - b_1 + a_2 + b_1 .$$

Clearly $b_2 - b_1 + a_2 + b_1 \geq O$. Hence $b_2 - b_1 + a_2 + b_1 \in A_1$.

Therefore $a+b \in A$. Then clearly A is a group. We now verify the convexity of A. Let $x \in A^+$ and let $O \leq y \leq x$ for some $y \in G$. Clearly $y \geq f(y)$.

By 1.2, $f(x) = x$. Then from the relation

$$|x - y| = \big|f(x) - f(y)\big| = \big|x - f(y)\big|$$

it follows that $f(y) \geq y$. Hence $f(y) = y$. Therefore $y \in A_1 \subseteq A$.

(iii) It follows from 1.2. □

1.5. THEOREM.

(i) *B_1 is a commutative semigroup.*

(ii) *B is an abelian convex subgroup of G.*

(iii) $B^+ = B_1$.

P r o o f . (i) Let x, $y \in B_1$. From 1.3 it follows that $f(x+y) = -x-y \leq O$. Then from $|x + y| = \big|f(x + y)\big|$ we get $f(x + y) = -y - x$. Thus B_1 is a semigroup and $x + y = y + x$.

(ii) From (i) it follows that B is an abelian group. Thus it remains to prove the convexity of B. Let $z \in B^+$ and let $O \leq t \leq z$ for some $t \in G$.

By 1.3, $f(z) = -z$. Then from the relation

$$|z - t| = \big|f(z) - f(t)\big| = \big|-z - f(t)\big|$$

we obtain $O \geq z - t - z \geq f(t)$. Then $|t| = \big|f(t)\big|$ implies $f(t) = -t$. Therefore $t \in B_1 \subseteq B$.

(iii) This is obvious from (ii) and 1.3. □

1.6. THEOREM. *Let $x \in A_1$, $y \in B_1$.*

(i) $f(x + y) = x - y$,

(ii) $f(-x - y) = -x + y$,

(iii) $f(x - y) = x + y$,

(iv) $f(-x + y) = -x - y$.

Proof. (i) Since

$$|x| = |x + y - y| = \big|f(x + y) - f(y)\big| = \big|f(x + y) + y\big|,$$

we obtain $x - y \geq f(x + y)$. From the relation

$$|x + y - x| = \big|f(x + y) - f(x)\big| = \big|f(x + y) - x\big|$$

we get $f(x + y) \geq x - y$. Thus $f(x + y) = x - y$.

The other propositions of the theorem can be proved analogously. □

1.7. THEOREM. *Let $x \in A$, $y \in B_1$. Then $f(x - y) = x + y$.*

Proof. If $x \in A$, then $x = a_1 - a_2$ where $a_1, a_2 \in A_1$. According to 1.6 (iii), we have

$$|a_1 + a_2 - a_1| = \big|(a_1 - a_2 - y) - (a_1 - y)\big| = \big|f(a_1 - a_2 - y) - f(a_1 - y)\big| =$$
$$= \big|f(a_1 - a_2 - y) - (a_1 + y)\big|.$$

This implies $f(a_1 - a_2 - y) \geq a_1 - a_2 + y$.

By 1.6 (ii),

$$|a_1| = \big|(a_1 - a_2 - y) - (-a_2 - y)\big| = \big|f(a_1 - a_2 - y) - f(-a_2 - y)\big| =$$
$$= \big|f(a_1 - a_2 - y) - y + a_2\big|.$$

From this we get $a_1 - a_2 + y \geq f(a_1 - a_2 - y)$. Therefore $f(x - y) = x + y$. □

1.8. THEOREM. *Let $x \in A_1$, $y \in B$. Then $f(-x + y) = -x - y$.*

Proof. If $y \in B$, then $y = b_1 - b_2$ where $b_1, b_2 \in B_1$. In view of 1.6 (ii) from the relation

$$\begin{aligned} |-x + b_1 + x| &= \big|(-x + b_1 - b_2) - (-x - b_2)\big| \\ &= \big|f(-x + b_1 - b_2) - f(-x - b_2)\big| \\ &= \big|f(-x + b_1 - b_2) - (-x + b_2)\big| \end{aligned}$$

we obtain $f(-x + b_1 - b_2) \geq -x - b_1 + b_2$.

According to 1.5 and 1.6 (iv),

$$\begin{aligned} |-x - b_1 + b_2 + b_1 + x| &= \big|(-x + b_1 - b_2) - (x + b_1)\big| \\ &= \big|f(-x + b_1 - b_2) - f(-x + b_1)\big| \\ &= \big|f(-x + b_1 - b_2) + b_1 + x\big|. \end{aligned}$$

From this we get $-x - b_1 + b_2 \geq f(-x + b_1 - b_2)$. Finally, in view of 1.5 we have $f(-x + b_1 - b_2) = -x + b_2 - b_1$. □

1.9. Theorem. *Let $x \in A$, $y \in B$. Then $f(x+y) = x - y$.*

Proof. If $x \in A$, $y \in B$, then $x = a_1 - a_2$, $y = b_1 - b_2$, where $a_1, a_2 \in A_1$, $b_1, b_2 \in B_1$.
By 1.8, $f(-a_2 + b_1 - b_2) = -a_2 + b_2 - b_1$. Then from the relation

$$\begin{aligned} |a_1| &= \big|(a_1 - a_2 + b_1 - b_2) - (-a_2 + b_1 - b_2)\big| \\ &= \big|f(a_1 - a_2 + b_1 - b_2) - f(-a_2 + b_1 - b_2)\big| \\ &= \big|f(a_1 - a_2 + b_1 - b_2) + b_1 - b_2 + a_2\big| \end{aligned}$$

we obtain $a_1 - a_2 + b_2 - b_1 \geq f(a_1 - a_2 + b_1 - b_2)$.
In view of 1.7 we have $f(a_1 - a_2 - b_2) = a_1 - a_2 + b_2$.
Then from

$$\begin{aligned} |a_1 - a_2 + b_1 + a_2 - a_1| &= \big|(a_1 - a_2 + b_1 - b_2) - (a_1 - a_2 - b_2)\big| \\ &= \big|f(a_1 - a_2 + b_1 - b_2) - f(a_1 - a_2 - b_2)\big| \\ &= \big|f(a_1 - a_2 + b_1 - b_2) - b_2 + a_2 - a_1\big| \end{aligned}$$

it follows that $f(a_1 - a_2 + b_1 - b_2) \geq a_1 - a_2 - b_1 + b_2$.
Thus according to 1.5, we have $f(a_1 - a_2 + b_1 - b_2) = a_1 - a_2 - (b_1 - b_2)$.
□

2. Stable weak isometries and direct decompositions

In [12] the above mentioned Jakubík's result concerning stable isometries and direct decompositions of l-groups was extended to distributive multilattice groups. But this result cannot be extended to all directed groups [13, p. 3].

In [13] the following Theorem 2 was proved which establishes necessary and sufficient conditions under which to a stable weak isometry f in a directed group G there exists a direct decomposition.

Theorem 2 [13]. *Let f be a stable weak isometry in a directed group G. Let*

$$A_1 = \{x \in G^+,\ f(x) = x\}, \quad B_1 = \{x \in G^+,\ f(x) = -x\},$$
$$A = A_1 - A_1, \quad B = B_1 - B_1 .$$

Then the following conditions are equivalent:

(i) *For each $x \in G^+$ there exists the least upper bound of $\{O, f(x)\}$ in G^+.*

(ii) *For each $x \in G^+$ there exists $x_1 \in G^+$ such that*

$$O \leq x_1 \leq x, \qquad f(x) \leq x_1 \leq f(x) + x .$$

(iii) *G is the direct product of the po-group A and the abelian po-group B and $f(z) = z(A) - z(B)$ for each $z \in G$.*

□

Now we shall show that the relation concerning weak isometries and direct decompositions is also valid in Riesz groups.

2.1. THEOREM. *Let G be a Riesz group and f be a stable weak isometry in G. Then there exists a direct decomposition $G = A \times B$ of G with B abelian such that $f(x) = x(A) - x(B)$ for each $x \in G$.*

Proof. Let $x \in G^+$. Then from $|x| = |f(x)|$ it follows that $x \geq f(x)$, $x \geq -f(x)$. Then

$$f(x) + x \geq O, \qquad f(x) + x \geq f(x).$$

Since G is a Riesz group, there exists $x_1 \in G$ such that

$$O \leq x_1 \leq x, \qquad f(x) \leq x_1 \leq f(x) + x.$$

Applying Theorem 2 [13] we obtain the desired result. □

The notation from Theorem 2.1 will be also adopted in the following two theorems.

2.2. THEOREM. *Let G be a Riesz group and let f be a stable weak isometry in G.*

(i) *If x, $y \in G$, $y \leq x$, then $f([y,x]) = [y(A) - x(B),\ x(A) - y(B)]$.*

(ii) *If x, $y \in G$, $f(y) \leq f(x)$, then*

$$[f(y),\ f(x)] = f([y(A) + x(B),\ x(A) + y(B)]).$$

(iii) *A non-void subset H of G is a directed convex subset of G if and only if $f(H)$ is a directed convex subset of G.*

Proof. (i) It is a routine computation to verify that

$$x(A) - y(B) = f(x) \vee f(y), \quad y(A) - x(B) = f(x) \wedge f(y),$$

and

$$y(A) - x(B) \leq f(a) \leq x(A) - y(B) \quad \text{for each} \quad a \in [y,x].$$

Thus

$$f([y,x]) \subseteq [y(A) - x(B),\ x(A) - y(B)].$$

Let $b \in G$, $y(A) - x(B) \leq b \leq x(A) - y(B)$. Then $x \leq b(A) - b(B) \leq y$, $f(b(A) - b(B)) = b$.

Therefore $\left[y(A)-x(B)\,,\ x(A)-y(B)\right] \subseteq f([y,x])$.

Analogously we can verify (ii).

(iii) Let H be a directed convex subset of G. First we show that $f(H)$ is a convex subset of G. Let $z' \in G$ such that

$$f(y) \le z' \le f(x) \qquad \text{for some} \quad x, y \in H\,.$$

It is easy to show that

$$x(A)+y(B) = x \vee y\,, \qquad y(A)+x(B) = x \wedge y\,.$$

According to (ii) there exists

$$z \in \left[y(A)+x(B)\,,\ x(A)+y(B)\right]$$

such that $f(z) = z'$. Since H is a directed convex subset of G, we obtain $y(A)+x(B)$, $x(A)+y(B) \in H$. Then from the convexity of H it follows that $z \in H$. Thus $z' \in f(H)$.

Now we show that $f(H)$ is a directed subset of G. Let x_1', $y_1' \in f(H)$. Let x_1, y_1 be elements of H such that

$$f(x_1) = x_1'\,, \qquad f(y_1) = y_1'\,.$$

Then there exist elements $u, v \in H$ such that

$$u \in L(x_1, y_1)\,, \qquad v \in U(x_1, y_1)\,.$$

Since

$$u \le v(A)+u(B) \le v\,, \qquad u \le u(A)+v(B) \le v\,,$$

from the convexity of H it follows that

$$v(A)+u(B), \qquad u(A)+v(B) \in H\,.$$

By (i),

$$f([u,v]) = \left[f\big(u(A)+v(B)\big)\,, f\big(v(A)+u(B)\big)\right].$$

Since x_1, $y_1 \in [u,v]$, we obtain

$$f\big(v(A)+u(B)\big) \in U\big(f(x_1)\,,\ f(y_1)\big)\,, \quad f\big(u(A)+v(B)\big) \in L\big(f(x_1)\,, f(y_1)\big)\,.$$

Since f is a bijection, we can consider stable weak isometry f^{-1} and analogously prove the sufficiency of the condition. □

In [14] R a c h ů n e k proved that for every isometry f in a 2-isolated abelian Riesz group H the relation

$$f\Big(U\big(L(x,y)\big) \cap L\big(U(x,y)\big)\Big) = U\Big(L\big(f(x),\ f(y)\big)\Big) \cap L\Big(U\big(f(x)\,,\ f(y)\big)\Big)$$

is valid for each $x, y \in H$ and raised the question whether the assumption of commutativity of H can be cancelled in this assertion. The following Theorem generalizes R a c h ů n e k 's result and gives the positive answer to his question.

2.3. THEOREM. *Let f be a weak isometry in a Riesz group G. Then*

$$f\Big(U\big(L(x,y)\big)\cap L\big(U(x,y)\big)\Big) = U\Big(L\big(f(x),\ f(y)\big)\Big)\cap L\Big(U\big(f(x),\ f(y)\big)\Big)$$

for each $x, y \in G$.

Proof. If f is a translation, the assertion obviously holds. Since each weak isometry in G is a composition of a stable weak isometry and a right translation, it suffices to consider the case when f is a stable weak isometry in G.

Let

$$x,\ y\in G, \qquad a\in U\big(L(x,y)\big)\cap L\big(U(x,y)\big)\,,$$

let

$$v\in U\big(f(x)\,,f(y)\big)\,, \qquad u\in L\big(f(x)\,,f(y)\big)\,.$$

By 2.2 (i),

$$f\big([u(A)-v(B),\ v(A)-u(B)]\big)=[u,v]\,.$$

Since f is a bijection and $f(x)$, $f(y)\in[u,v]$, we obtain

$$u(A)-v(B)\in L(x,y)\,, \qquad v(A)-u(B)\in U(x,y)\,.$$

Then

$$u(A)\le a(A)\le v(A)\,, \qquad -v(B)\le a(B)\le -u(B)\,.$$

From this we get $u\le a(A)-a(B)=f(a)\le v$. Thus

$$f(a)\in U\Big(L\big(f(x)\,,f(y)\big)\Big)\cap L\Big(U\big(f(x)\,,f(y)\big)\Big)\,.$$

Therefore

$$f\Big(U\big(L(x,y)\big)\cap L\big(U(x,y)\big)\Big) \subseteq U\Big(L\big(f(x),\ f(y)\big)\Big)\cap L\Big(U\big(f(x),\ f(y)\big)\Big)\,.$$

Now we can consider stable weak isometry f^{-1} instead of f. Let

$$x'=f(x)\,, \qquad y'=f(y)\,.$$

Then for x', y' we have

$$f^{-1}\Big(U\big(L(x',\ y')\big)\cap L\big(U(x',y')\big)\Big)\subseteq U\big(L(x,y)\big)\cap L\big(U(x,y)\big)\,.$$

Therefore

$$U\Big(L\big(f(x),\ f(y)\big)\Big)\cap L\Big(U\big(f(x),\ f(y)\big)\Big)\subseteq f\Big(U\big(L(x,y)\big)\cap L\big(U(x,y)\big)\Big)\,.$$

□

By Theorem 3 [13], to every stable weak isometry in a divisible directed group G there exists the above mentioned direct decomposition of G.

Now we shall show that an analogous result may be obtained for isolated abelian directed groups.

It is easy to verify (cf. e.g., [3, Section 1.3]) that if G is an isolated abelian po-group, then there exists an isolated divisible abelian po-group $Z(G)$ such that G is a po-subgroup of $Z(G)$ and if $z \in Z(G)$, then there exist $x \in G$ and $m \in \mathbb{N}$ such that $mz = x$.

In the present paper $Z(G)$ has the same meaning as in [3, Section 1.3], i.e., $Z(G)$ is the set of all expressions of the form $\frac{x}{n}$, where $x \in G$, $n \in \mathbb{N}$, subject to the rules of:

a) equality: $\frac{x}{n} = \frac{t}{k}$ if and only if $kx = nt$,

b) addition: $\frac{x}{n} + \frac{t}{k} = \frac{kx+nt}{nk}$,

c) partial order: for $z \in Z(G)$ we have $z > O$ if and only if there exists $x \in G$, $x > O$ such that $z = \frac{x}{n}$ for some $n \in \mathbb{N}$.

From this it follows that

$$-\left(\frac{x}{n}\right) = \frac{-x}{n}, \; k\left(\frac{x}{n}\right) = \frac{kx}{n} \quad \text{for each} \quad \frac{x}{n} \in Z(G), \; k \in \mathbb{N}.$$

2.4. THEOREM. *Let H be an isolated abelian directed group and g a stable weak isometry in H. Let*

$$\bar{g}\left(\frac{x}{n}\right) = \frac{g(x)}{n} \quad \textit{for each} \quad \frac{x}{n} \in Z(H).$$

Then $\bar{g}$ is a stable weak isometry in $Z(H)$.

Proof. Let $\frac{x}{n}, \frac{t}{k} \in Z(H)$. Let

$$a \in \left|\bar{g}\left(\frac{x}{n}\right) - \bar{g}\left(\frac{t}{k}\right)\right| = \left|\frac{g(x)}{n} - \frac{g(t)}{k}\right| = \left|\frac{kg(x)-ng(t)}{nk}\right|.$$

By Theorem 3 [11], g is a group homomorphism. Then we have $nka \in |g(kx) - g(nt)| = |kx - nt|$. Since H is isolated, we get $a \in \left|\frac{x}{n} - \frac{t}{k}\right|$. The converse inclusion can be proved analogously. □

2.5 THEOREM. *Let g be a stable weak isometry in an isolated abelian directed group H. Then there exists a direct decomposition $Z(H) = A \times B$ of $Z(H)$ with B abelian such that $g(x) = x(A) - x(B)$ for each $x \in H$.*

Proof. It follows from 2.4 and Theorem 3 [13]. □

On the other hand, the following Theorem shows that certain direct decompositions of a po-group give rise to stable isometries.

2.6. THEOREM. *Let H be a po-group and $H = P \times Q$ a direct decomposition of H with Q abelian. For each $x \in H$ define $g(x) = x(P) - x(Q)$. Then g is a stable isometry in H.*

P r o o f. Clearly g is a bijection and $g(O) = O$. Further, it is easy to verify that $|z| = |z(P)| + |z(Q)|$ for each $z \in H$. Obviously

$$(x-y)(P) = x(P) - y(P), \quad (x-y)(Q) = x(Q) - y(Q).$$

Then we have

$$\begin{aligned} |g(x) - g(y)| &= |x(P) - y(P) + [-(x(Q) - y(Q))]| \\ &= |x(P) - y(P)| + |x(Q) - y(Q)| \\ &= |(x-y)(P)| + |(x-y)(Q)| = |x-y|. \end{aligned}$$

□

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