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POLAR OPERATORS IN LATTICE OF RADICAL CLASSES OF ℓ–GROUPS

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Dedicated to Professor J. Jakubík on the occasion of his 70th birthday

ABSTRACT. In this paper we prove that for most radical classes $\mathcal{R}_{12i_3\dots i_7}$ there exists a unique largest radical class $\mathcal{R}'_{12i_3\dots i_7}$ whose intersection with $\mathcal{R}_{12i_3\dots i_7}$ is trivial. $\mathcal{R}'_{12i_3\dots i_7}$ is called the polar of $\mathcal{R}_{12i_3\dots i_7}$. We give concrete construction for polars of radical classes generated by the integer group $\mathbb{Z}$.

1. Preliminaries

We use the common terminology and notation of [1, 2, 4]. Throughout the paper G is an ℓ-group. We use the additive group notation. Let $\{G_\lambda \mid \lambda \in \Lambda\}$ be a family of ℓ-groups and $\prod_{\lambda\in\Lambda} G_\lambda$ their direct product. If an ℓ-group G is a subdirect product of G_λ, we denote this by $G \subseteq' \prod_{\lambda\in\Lambda} G_\lambda$. We denote the ℓ-subgroup of $\prod_{\lambda\in\Lambda} G_\lambda$ consisting of the elements with only finitely many non-zero components by $\sum_{\lambda\in\Lambda} G_\lambda$. An ℓ-group G is said to be a *completely subdirect product* of G_λ, if G is an ℓ-subgroup of $\prod_{\lambda\in\Lambda} G_\lambda$ and $\sum_{\lambda\in\Lambda} G_\lambda \subseteq G$. An ℓ-group G is said to be an *ideal subdirect product* of G, denoted $G \subseteq^* \prod_{\lambda\in\Lambda} G_\lambda$, if $G \subseteq' \prod_{\lambda\in\Lambda} G_\lambda$ and G is an ℓ-ideal of $\prod_{\lambda\in\Lambda} G_\lambda$. For each $\lambda \in \Lambda$ let φ_λ be the projection from $\prod_{\lambda\in\Lambda} G_\lambda$ onto G_λ and

$$\overline{G}_\lambda = \Big\{g \in \prod_{\lambda\in\Lambda} G_\lambda \mid \lambda' \neq \lambda \Rightarrow g_{\lambda'} = 0\Big\}.$$

For any $\lambda \in \Lambda$ and $a_\lambda \in G_\lambda$ let $\overline{a}_\lambda = (0,\dots,0,\,a_\lambda,\,0,\dots,0)$.

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Let G be an ℓ-group. $\mathcal{C}(G)$ denotes the complete lattice of all convex ℓ-subgroups of G. For $g \in G$, $G(g)$ is the principal convex ℓ-subgroup generated by g. Let $X \subseteq G$. $X_G^{\perp} = \{g \in G \mid |g| \wedge |x| = 0 \text{ for all } x \in X\}$ is called the *polar* of X in G and $X^{\perp\perp}$ is called the *double polar* of X. If G is an ℓ-subgroup of an ℓ-group H, the order closure $\overline{G}_H$ of G in H is the smallest closed ℓ-subgroup of H containing G.

We can make new ℓ-groups from some original ℓ-groups. These operations include:

1. taking convex ℓ-subgroups,
2. forming joins of convex ℓ-subgroups,
3. forming completely subdirect products,

3'. forming direct products,

4. taking ℓ-homomorphic images,

4'. taking complete ℓ-homomorphic images,

4". taking ℓ-homomorphic images,

5. forming extensions, that is, G is an extension of A by using B if A is an ℓ-ideal of G and $B = G\,/\,A$,
6. taking order closures, that is, G is an order closure of A if A is a convex ℓ-subgroup of an ℓ-group H and $G = \overline{A}_H$,
7. taking double polars, that is, G is a double polar of A if A is a convex ℓ-subgroup of an ℓ-group H and $G = A_H^{\perp\perp}$.

A family $\mathcal{U}$ of ℓ-groups is called a *class* if it is closed under some operations. If a class $\mathcal{U}$ is closed under the operations $1, 2, i_3, \ldots, i_7$, we call $\mathcal{U}$ a $12i_{3\ldots i_7}$–*class* where $i_3 \in \{3, 3', 4''\}$, $i_4 \in \{4, 4', 4''\}$ $i_5 \in \{4'', 5\}$ $i_6 \in \{4'', 6\}$, $i_7 \in \{4'', 7\}$. All our classes always assumed to contain along with a given ℓ-group all its ℓ-isomorphic images, so we omit the index 4". Thus, a *radical class* [5] is a 12-class, a *quasi-torsion class* [6] is a 124'–class, a *torsion class* [7] is a 124–class, a *closed-kernel radical class* [3, 10] is a 126–class, a *polar kernel radical class* [3] is a 127–class. A 125–class is called a *complete radical class.* We call a 123'–class (a 123–class) a *product radical class* (a *subproduct radical class*) [8, 9]. Let $T_{12i_3\ldots i_7}$ be the complete lattice of all $12i_3\ldots i_7$–classes. Let $\mathcal{R}_{12i_3\ldots,i_7}$ be a $12i_3\ldots i_7$-class and G be an ℓ-group. Then there exists a larger convex ℓ-subgroup of G belonging to $\mathcal{R}_{12i_3\ldots i_7}$. We denote it by $\mathcal{R}_{12i_3\ldots i_7}(G)$ and call it the $\mathcal{R}_{12i_3\ldots i_7}$-radical of G.

In [10] we discussed the relationship between several radical classes and gave the characteristic properties for several radical mappings. This paper is a continuation of [10].

2. Polar operators

In this section we will show that for most of $12i_3\dots i_7$-classes $\mathcal{R}_{12i_3\dots i_7}$, there exists a unique largest radical class $\mathcal{R}'_{12i_3\dots i_7}$ in $T_{12i_3\dots i_7}$, whose intersection with $\mathcal{R}_{12i_3\dots i_7}$ is trivial. We call the mapping $\mathcal{R}_{12i_3\dots i_7} \to \mathcal{R}'_{12i_3\dots i_7}$ a *polar operator* in $T_{12i_3\dots i_7}$, denoted by $PO_{12i_3\dots i_7}$. $\mathcal{R}'_{12i_3\dots i_7}$ is said to be the *polar* of $\mathcal{R}_{12i_3\dots i_7}$ in $T_{12i_3\dots i_7}$.

LEMMA 2.1. ([3], [5]). *For any radical class $\mathcal{R}_{12}$, $\mathcal{R}'_{12} = \{G \mid \mathcal{R}_{12}(G) = 0\}$ is the unique largest radical class such that $\mathcal{R}_{12} \cap \mathcal{R}'_{12} = 0$. For each ℓ-group G, $\mathcal{R}'_{12}(G) = \mathcal{R}_{12}(G)^{\perp}_{G}$.*

The following lemma is clear.

LEMMA 2.2. *For any $\mathcal{R}_{12} \in T_{12}$, $\mathcal{R}'_{12} \in T_{126} \cap T_{127}$.*

COROLLARY 2.3. *For any $\mathcal{R}_{12} \in T_{12}$, $\mathcal{R}'_{12} \in T_{125}$.*

P r o o f. $T_{127} \subseteq T_{125}$ (see Theorem 2.7 in [10] and also see Proposition 4.4 in [3]). □

From Lemma 2.1, Lemma 2.2 and Corollary 2.3 we have

THEOREM 2.4. *For any closed-kernel radical class (polar radical class, complete radical class) $\mathcal{R}$, there exists a unique largest closed-kernel radical class (polar radical class, complete radical class, respectively) $\mathcal{R}'$ such that $\mathcal{R} \cap \mathcal{R}' = 0$ and $\mathcal{R}'(G) = \mathcal{R}(G)^{\perp}$ for each ℓ-group G.*

In [7] J. M a r t i n e z proved that for any torsion class $\mathcal{R}_{124}$ the torsion class $\mathcal{R}'_{124} = \{G \mid$ if $C \in \mathcal{C}(G)$ and H is an ℓ-homomorphic image of C, then $\mathcal{R}_{124}(H) = 0\}$ is the unique largest torsion class whose intersection with $\mathcal{R}_{124}$ is trivial. Similarly, we can show that for any quasi-torsion class $\mathcal{R}_{124'}$, the quasi-torsion class $\mathcal{R}'_{124'} = \{G \mid$ if $C \in \mathcal{C}(G)$ and H is a complete ℓ-homomorphic image of C, then $\mathcal{R}_{124'}(H) = 0\}$ is the unique largest quasi-torsion class whose intersection with $\mathcal{R}_{124'}$ is trivial.

A $123'4'$-class is called a product quasi-torsion class.

THEOREM 2.5. *For any product quasi-torsion class $\mathcal{R}_{123'4'}$, there exists a unique largest product quasi-torsion class $\mathcal{R}'_{123'4'}$ whose intersection with $\mathcal{R}_{123'4'}$ is trivial.*

P r o o f. Let

$$\mathcal{R}'_{123'4'} = \{G \mid \text{if } C \in \mathcal{C}(G) \text{ and } H \text{ is a complete } \ell\text{-homomorphic image of } C, \quad \text{then} \quad \mathcal{R}_{123'4'}(H) = 0\}.$$

Similarly to Theorem 2.3 of [7] we see that $\mathcal{R}'_{123'4} \in \mathcal{T}_{124}$. Now suppose that $\{G_\lambda \mid \lambda \in \Lambda\} \subseteq \mathcal{R}'_{123'4'}$, $C \in \mathcal{C}\Big(\prod\limits_{\lambda\in\Lambda} G_\lambda\Big)$ and f is a complete ℓ-homomorphism from C onto an ℓ-group H. For each $\lambda \in \Lambda$ let

$$\overline{\varphi_\lambda(C)} = \{g \in C \mid g_\lambda \in \varphi_\lambda(C) \quad \text{and} \quad g_{\lambda'} = 0 \quad \text{for} \quad \lambda' \neq \lambda\}.$$

Then $\overline{\varphi_\lambda(C)} \in \mathcal{C}(C) \cap \mathcal{C}(\overline{G}_\lambda)$ for $\lambda \in \Lambda$. Hence

$$\mathcal{R}_{123'4}\Big(\big[\overline{\varphi_\lambda(C)}\big]\Big) = 0 \quad (\lambda \in \Lambda),$$

$$\mathcal{R}_{123'4}\Big(\prod_{\lambda\in\Lambda} f\big[\overline{\varphi_\lambda(C)}\big]\Big) = \prod_{\lambda\in\Lambda} \mathcal{R}_{123'4}\Big(f\,\big[\overline{\varphi_\lambda(C)}\big]\Big) = 0\,. \tag{2.1}$$

The mapping $f'\colon c = (\dots, c_\lambda, \dots) \longrightarrow (\dots, f(\overline{c}_\lambda), \dots)$ defines an ℓ-homomorphism from $\prod\limits_{\lambda\in\Lambda} \varphi_\lambda(C)$ onto $\prod\limits_{\lambda\in\Lambda} f\big[\overline{\varphi_\lambda(C)}\big]$. If $a, b \in C$ such that $f(a) = f(b)$ in H, then $|\overline{a}_\lambda - \overline{b}_\lambda| \le |a - b|$, and so $f(\overline{a}_\lambda) = f(\overline{b}_\lambda)$ for each $\lambda \in \Lambda$. This means that the mapping

$$g\colon h \longrightarrow \big(\dots, f\big(\overline{f^{-1}(h)_\lambda}\big), \dots\big)$$

is an ℓ-homomorphism from H into $\prod\limits_{\lambda\in\Lambda} f\big[\overline{\varphi_\lambda(C)}\big]$ and $g \circ f = f'|_C$. The fact that $\operatorname{Ker} f$ is closed implies that $f^{-1}(\operatorname{Ker} g) = \operatorname{Ker} f$, so $\operatorname{Ker} g = 0$ and g is an embedding. Since C is a convex ℓ-subgroup of $\prod\limits_{\lambda\in\Lambda} \varphi_\lambda(C)$, H is also a convex ℓ-subgroup of $\prod\limits_{\lambda\in\Lambda} f\big[\overline{\varphi_\lambda(C)}\big]$. It follows from the formula (2.1) that

$$\mathcal{R}_{123'4'}(H) = H \cap \mathcal{R}_{123'4'}\Big(\prod_{\lambda\in\Lambda} f\big[\overline{\varphi_\lambda(C)}\big]\Big) = 0\,.$$

Therefore $\prod\limits_{\lambda\in\Lambda} G_\lambda \in \mathcal{R}'_{123'4'}$, and $\mathcal{R}/_{123'4'} \in \mathcal{T}_{123'4'}$.

It is clear that $\mathcal{R}_{123'4'} \cap \mathcal{R}'_{123'4'} = 0$. If $\mathcal{U}$ is a product quasi-torsion class so that $\mathcal{U} \cap \mathcal{R}_{123'4'} = 0$, $G \in \mathcal{U}$, $C \in \mathcal{C}(G)$ and $\varphi\colon C \longrightarrow H$ is a complete ℓ-homomorphism, then $C \in \mathcal{U}$ and $H \in \mathcal{U}$. Hence $\mathcal{R}_{123'4'}(H) = 0$, that is $G \in \mathcal{R}'_{123'4'}$ and $\mathcal{U} \subseteq \mathcal{R}'_{123'4'}$. We have proved that $\mathcal{R}'_{123'4'}$ is the unique largest product quasi-torsion class whose intersection with $\mathcal{R}_{123'4'}$ is trivial. □

LEMMA 2.6. *For any* $\mathcal{R}_{124'} \in \mathcal{T}_{124'}$, $\mathcal{R}'_{124'} \in \mathcal{T}_{124'6}$.

P r o o f. Suppose that G is a convex ℓ-subgroup of H and $G \in \mathcal{R}'_{124'}$. We want to show that $\overline{G}_H \in \mathcal{R}'_{124'}$. Let $C \in \mathcal{C}(\overline{G}_H)$ and C' be a complete ℓ-homomorphic image of C with ℓ-homomorphism φ. Put $C_1 = C \cap G$ and $C'_1 = \varphi(C_1)$. Then $\overline{C'_1} = C'$ and C'_1 is dense in C'. Since $G \in \mathcal{R}'_{124'}$, $\mathcal{R}_{124'}$ $(C'_1) = 0$. But

$$\mathcal{R}_{124'}(C'_1) = \mathcal{R}_{124'}(C') \cap C'_1\,,$$

hence $\mathcal{R}_{124'}(C'_1) = 0$. We have proved that $\mathcal{R}'_{124'} \in \mathcal{T}_{124,6}$. □

From Lemma 2.6 we get

THEOREM 2.7. *For any closed-kernel quasi-torsion class $\mathcal{R}$ there exists a unique largest closed-kernel quasi-torsion class $\mathcal{R}'$ whose intersection with $\mathcal{R}$ is trivial.*

A $1234'$-class is called a subproduct quasi-torsion class.

THEOREM 2.8. *For any subproduct quasi-torsion class $\mathcal{R}_{1234'}$ there exists a unique largest subproduct quasi-torsion class $\mathcal{R}'_{1234'}$ whose intersection with $\mathcal{R}_{1234'}$ is trivial.*

Proof. Let

$$\mathcal{R}'_{1234'} = \{G \mid \text{if } C \in \mathcal{C}(G) \text{ and } H \text{ is a complete } \ell\text{-homomorphic image of } C, \quad \text{then} \quad \mathcal{R}_{1234'}(H) = 0\}.$$

We need only to show that $\mathcal{R}'_{1234'}$ is closed under forming completely subdirect products. Suppose that $\{G_\lambda \mid \lambda \in \Lambda\} \subseteq \mathcal{R}'_{1234'}$ and G is a completely subdirect product of G_λ. Let C be a convex ℓ-subgroup of G and f be a complete ℓ-homomorphism from C onto H. Since $\sum_{\lambda\in\Lambda} G_\lambda \subseteq G$ and $C \in \mathcal{C}(G)$, $\overline{\varphi_\lambda(C)} \in \mathcal{C}(C) \cap \mathcal{C}(\overline{G}_\lambda)$ for each $\lambda \in \Lambda$. Hence

$$\mathcal{R}_{1234'}\Big(f\big[\overline{\varphi_\lambda(C)}\big]\Big) = 0, \quad \text{and}$$

$$\mathcal{R}_{1234'}\Big(\prod_{\lambda\in\Lambda} f\big[\overline{\varphi_\lambda(C)}\big]\Big) = \prod_{\lambda\in\Lambda} \mathcal{R}_{1234'}\Big(f\big[\overline{\varphi_\lambda(C)}\big]\Big) = 0.$$

Similarly to Theorem 2.5 H can be considered as a ℓ-subgroup of $\prod_{\lambda\in\Lambda} f\big[\overline{\varphi_\lambda(C)}\big]$. Since $C \in \mathcal{C}(G)$ and $\sum_{\lambda\in\Lambda} G_\lambda \subseteq G$, $\sum_{\lambda\in\Lambda} \varphi_\lambda(C) \subseteq C$. That is, C is a completely subdirect product of $\{\varphi_\lambda(C) \mid \lambda \in \Lambda\}$. Therefore H is a completely subdirect product of $\Big\{f\big[\overline{\varphi_\lambda(C)}\big] \mid \lambda \in \Lambda\Big\}$. It follows from Theorem 3.2 of [10] that

$$\mathcal{R}_{1234'}(H) = \mathcal{R}_{1234'}\Big(\prod_{\lambda\in\Lambda} f\big[\overline{\varphi_\lambda(C)}\big]\Big) \cap H = 0.$$

and so $G \in \mathcal{R}'_{1234'}$. □

For a proper variety $\mathcal{R}$ there is no variety $\mathcal{R}'$ such that $\mathcal{R} \cap \mathcal{R}' = 0$, because every variety contains the abelian variety $\mathcal{A}$.

Let $\mathcal{R}$ and $\mathcal{I}$ be two radical classes in $T_{12i_3\dots i_7}$. Define $\mathcal{R}'' = (\mathcal{R}')'$. Then

$$\begin{aligned}
&(1)\ \mathcal{R} \subseteq \mathcal{R}'', \\
&(2)\ \text{if } \mathcal{R} \subseteq \mathcal{I}, \text{ then } \mathcal{R}' \supseteq \mathcal{I}', \\
&(3)\ \mathcal{R}' = \mathcal{R}''', \\
&(4)\ (\mathcal{R} \vee \mathcal{I})' = \mathcal{R}' \wedge \mathcal{I}'.
\end{aligned} \tag{2.2}$$

Thus the polar operator defines a Galois connection which has the above properties. We call the mapping $\mathcal{R} \longrightarrow \mathcal{R}''$ a polar closure operator in $T_{12i_3\ldots i_7}$, denoted by $PCO_{12i_3\ldots,i_7}$. $\mathcal{R}''$ is called the polar closure of $\mathcal{R}$. A polar closure operator has the following properties:

$$\begin{aligned}&(1)\ \mathcal{R}'' = (\mathcal{R}'')'', \\ &(2)\ \text{if } \mathcal{R} \subseteq \mathcal{I}, \text{ then } \mathcal{R}'' \subseteq \mathcal{I}''.\end{aligned} \tag{2.3}$$

3. Examples

Let G be an ℓ-group. We denote by $\mathcal{R}_{12i_3\ldots i_7G}$ the intersection of all $12i_3\ldots i_7$-classes containing G. It is said to be the $12i_3\ldots i_7$-class generated by G. In [3] [10] the construction for $12i_3\ldots i_7$-classes generated by the integer group Z were given.

$$\mathcal{R}_{12Z} = \Big\{ \sum_{\alpha\in A} Z_\alpha \mid Z_\alpha = Z \quad \text{for all } \alpha \in A \Big\}, \tag{3.1}$$

$$\mathcal{R}_{123Z} = \Big\{ G \mid \sum_{\alpha\in A} Z_\alpha \subseteq G \subseteq' \prod_{\alpha\in A} Z_\alpha, \quad Z_\alpha = Z \quad \text{for all } \alpha \in A \Big\}, \tag{3.2}$$

$$\mathcal{R}_{123'Z} = \Big\{ G \mid G \subseteq^* \prod_{\alpha\in A} Z_\alpha = Z \quad \text{for all } \alpha \in A \Big\}, \tag{3.3}$$

$$\begin{aligned}\mathcal{R}_{126Z} = \Big\{ G \mid\ & G \text{ is an order closure of a convex } \ell\text{-subgroup} \\ & \sum_{\alpha\in A} Z_\alpha\ (Z_\alpha = Z) \text{ of an } \ell\text{-group } H \Big\},\end{aligned} \tag{3.4}$$

$$\begin{aligned}\mathcal{R}_{127Z} = \Big\{ G \mid\ & G \text{ is a double polar of a convex } \ell\text{-subgroup} \\ & \sum_{\alpha\in A} Z_\alpha\ (Z_\alpha = Z) \text{ of an } \ell\text{-group } H \Big\}.\end{aligned} \tag{3.5}$$

In this section we will determine the polars of the above $12i_3\ldots i_7$-classes generated by Z. An ℓ-group G is called locally dense if for any element $0 < g \in G$ there exists $x \in G$ such that $0 < x < g$.

Proposition 3.1. *$\mathcal{R}'_{12Z} = \{G \mid G$ is a locally dense ℓ-group$\}$.*

Proof. $\mathcal{R}'_{12Z} = \{G \mid \mathcal{R}_{12Z}(G) = 0\}$. By formula (3.1) we see that $G \in \mathcal{R}'_{12Z}$ if and only if G does not contain the integer group $\mathbb{Z}$ as a convex ℓ-subgroup, if and only if for any element $0 < g \in G$ the principal convex ℓ-subgroup $G(g)$ is not ℓ-isomorphic to $\mathbb{Z}$. Since $G(g) = \{x \in G \mid 0 \le |x| \le n|g|$ for some $n \in \mathbb{N}\}$, $G \in \mathcal{R}'_{12Z}$ if and only if G is locally dense. □

By Lemma 2.2 and Corollary 2.3 we have

COROLLARY 3.2. *$\mathcal{R}'_{12Z}$ is a closed-kernel, polar and complete radical class.*

PROPOSITION 3.3. $\mathcal{R}'_{126Z} = \{G \mid G$ *is a locally dense ℓ-group*$\}$.

Proof. Since $\mathcal{R}'_{12Z}$ is a closed-kernel radical class, $\mathcal{R}'_{12Z} \subseteq \mathcal{R}'_{126Z}$. But since $\mathcal{R}'_{126Z} \cap \mathcal{R}_{126Z} = 0$, $\mathcal{R}'_{126Z} = \mathcal{R}'_{12Z}$. □

By Theorem 2.4 and the formula (3.5), similarly to the proof of Proposition 3.3 we have

PROPOSITION 3.4. $\mathcal{R}'_{127Z} = \{G \mid G$ *is a locally dense ℓ-group*$\}$.

PROPOSITION 3.5. $\mathcal{R}'_{123'Z} = \{G \mid G$ *is a locally dense ℓ-group*$\}$.

Proof. By Theorem 4.1 of [8], $\mathcal{R}'_{123'Z} = \{G \mid \mathcal{R}_{123'Z}(G) = 0\}$. It follows from the formula (3.3) that if $G \in \mathcal{R}'_{123'Z}$ then G does not contain Z as a convex ℓ-subgroup. Conversely, if G does not contain Z as a convex ℓ-subgroup and $\mathcal{R}'_{123'Z}(G) \neq 0$, then $\mathcal{R}'_{123'Z}(G)$ is an ideal subdirect product of $\{Z_\alpha \mid \alpha \in A\}$ ($Z_\alpha = Z$ for all $\alpha \in A$). Since $\sum\limits_{\alpha\in A} Z_\alpha$ is a convex ℓ-subgroup of $\mathcal{R}_{123'}(G)$ and $\mathcal{R}_{123'Z}(G)$ is a convex ℓ-subgroup of G, so $\sum\limits_{\alpha\in A} Z_\alpha$ is a convex ℓ-subgroup of G, a contradiction. Hence $G \in \mathcal{R}'_{123'Z}$ if and only if G is locally dense. □

By Theorem 4.1 of [9] and the formula (3.2), using a proof similar to that of Proposition 3.5 we can get

PROPOSITION 3.6. $\mathcal{R}'_{123Z} = \{G \mid G$ *is a locally dense ℓ-group*$\}$.

Thus, the radical class $\mathcal{R}'_{12Z}$ of all locally dense ℓ-groups is simultaneously the polar of $\mathcal{R}_{12Z}$, $\mathcal{R}_{126Z}$, $\mathcal{R}_{127Z}$, $\mathcal{R}_{123Z}$ and $\mathcal{R}_{123'Z}$.

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