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LATTICE EXTENSIONS OF ALGEBRAS AND MALCEV PRODUCTS

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Dedicated to Professor J. Jakubík on the occasion of his 70th birthday

ABSTRACT. It is known that all normal extensions of an algebra $\mathfrak{A} = (A, F)$ from a variety $\mathcal{V}$ belong to the Malcev product $\mathcal{V} \circ \mathcal{S}$ where $\mathcal{S}$ is the variety of all F–semilattices. Now we show that all quasi-boolean extensions of $\mathfrak{A}$ are contained in the Malcev product $\mathcal{S} \circ \mathcal{V}$.

It is known that all normal extensions of an algebra $\mathfrak{A} = (A, F)$ from a variety $\mathcal{V}$ belong to the Malcev product $\mathcal{V} \circ \mathcal{S}$ where $\mathcal{S}$ is the variety of all F–semilattices. Theorem 3 below states that all quasi-boolean extensions of $\mathfrak{A}$ are contained in the Malcev product $\mathcal{S} \circ \mathcal{V}$.

Let L be a complete lattice with the least element O and the greatest element I and let A be a non-empty set. Denote by $A^0[L]$ the set of all mappings $\nu : A \to L$ such that if $a, b \in A$, $a \neq b$, then $\nu(a) \wedge \nu(b) = O$. For $\nu \in A^0[L]$ define its weight as $[\nu] := \bigvee_{a \in A} \nu(a)$, and denote by $A[L]$ the subset of $A^0[L]$ consisting of all those mappings ν which have weight I.

If $\mathfrak{A} = (A, F)$ is an algebra with finitary operations, then for every n-ary $(n > 0)$ operation $f \in F$ and for all $\nu_1, \dots, \nu_n \in A^0[L]$, $a \in A$ put

$$f(\nu_1, \dots \nu_n)(a) := \bigvee_{a = f(a_1, \dots, a_n)} (\nu_1(a_1) \wedge \dots \wedge \nu_n(a_n)) .$$

Thus there appear two partial algebras $\mathfrak{A}^0[L] = (A^0[L], F)$ and $\mathfrak{A}[L] = (A[L], F)$. When L is a complete Boolean lattice, all operations in both $\mathfrak{A}^0[L]$ and $\mathfrak{A}[L]$ are everywhere defined. In this case $\mathfrak{A}^0[L]$ is the so called *normal extension* of the algebra $\mathfrak{A}$ and $\mathfrak{A}[L]$ is its *Boolean extension* (or *Boolean power*).

If algebra $\mathfrak{A}$ belongs to some variety $\mathcal{V}$, then all its Boolean extensions are also $\mathcal{V}$–algebras. Normal extensions in general do not preserve equational theories. It was shown in [2] that an identity $p = q$, which holds in algebra $\mathfrak{A}$ is also

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true in all normal extensions of $\mathfrak{A}$ iff this identity is normal (or regular), which means that the same variables occur on either side. Varieties defined by normal identities are called *normal* (or *regular*) *varieties*. The least normal variety $N(\mathcal{V})$ containing a given variety $\mathcal{V}$ is generated by all normal extensions of $\mathcal{V}$–algebras [4]. Assume that F does not include nullary operations and that at least one member of F is at least binary operation. If $(A, \wedge)$ is a meet-semilattice, then it may be treated as an F–algebra when defining $f(x) := x$ for unary and $f(x_1, \ldots, x_n) := x_1 \wedge \cdots \wedge x_n$ for n-ary $(n > 1)$ operations $f \in F$. Such F–semilattices form a subvariety $\mathcal{S}$ of the variety $\mathcal{F}$ of all F–algebras. It is the least normal variety of F–algebras and for every $\mathcal{V} \subseteq \mathcal{F}$ we have $N(\mathcal{V}) = \mathcal{V} \vee \mathcal{S}$ (join in the lattice of all subvarieties of $\mathcal{F}$).

If $\mathcal{K}$ is some class of F–algebras and $\mathcal{A}, \mathcal{B}$ are its subclasses, then the *Malcev product* $\mathcal{A} \circ_{\mathcal{K}} \mathcal{B}$ is defined [1] as the class of all $\mathcal{K}$–algebras $\mathfrak{A} = (A, F)$, which admit a congruence θ with properties (i) $(\forall a \in A)\big(\theta(a) \in \mathcal{K} \Rightarrow \theta(a) \in \mathcal{A}\big)$ and (ii) $\mathfrak{A}/\theta \in \mathcal{B}$.

Theorem 6.7 from [4] implies the following proposition.

THEOREM 1. *Let L be a complete Boolean lattice and let $\mathcal{V}$ be any variety of F–algebras. Then $\mathfrak{A}^0[L] \in \mathcal{V} \circ_{\mathcal{F}} \mathcal{S}$ for every algebra $\mathfrak{A} \in \mathcal{V}$.*

A Malcev congruence on $\mathfrak{A}^0[L]$ is, e.g., $\theta := \big\{(\mu, \nu) \in A^0[L] \times A^0[L] \mid [\mu] = [\nu]\big\}$.

So the Boolean extension $\mathfrak{A}[L]$ is one of θ-classes of the normal extension $\mathfrak{A}^0[L]$.

An *orthogonal system* in a complete lattice L is a subset $\lambda = \{l_i \,|\, i \in I\} \subseteq L$ such that $l_i \wedge l_j = O$ whenever $i \neq j$. An orthogonal system λ is said to be *independent* if $\bigvee_{j \in J} l_j \wedge \bigvee_{k \in K} l_k = O$ for every partition $I = J \cup K$, $J \cap K = \emptyset$. *Quasi-boolean lattice* is a complete complemented lattice in which all orthogonal systems are independent. Certainly every complete Boolean lattice is quasi-boolean. Note that the pentagon N_5 is also a quasi-boolean lattice.

It is known [5, Theorem 1] that a complete complemented lattice is quasi-boolean iff it admits a meet-homomorphism onto a complete Boolean lattice which is one-to-one in O and I and preserves l.u.b.'s of all orthogonal systems. Such meet-homomorphism is called *canonical.*

THEOREM 2. (*see* [3], *Theorem 1*). *Let L be a complete complemented lattice. Then in the L–extension $\mathfrak{A}[L]$ of every algebra $\mathfrak{A}$ all operations are everywhere defined iff L is a quasi-boolean lattice.*

If $\mathfrak{A}$ is an algebra and L runs over the class of all quasi-boolean lattices, then algebras of the form $\mathfrak{A}[L]$ are called *quasi-boolean extensions* of $\mathfrak{A}$. The following result together with the above Theorem 1 reveals a certain duality between normal extensions and quasi-boolean extensions of algebras.

THEOREM 3. *Let L be a quasi-boolean lattice and let $\mathcal{V}$ be any variety of F–algebras. Then $\mathfrak{A}[L] \in \mathcal{S} \circ_{\mathcal{F}} \mathcal{V}$ for every algebra $\mathfrak{A} \in \mathcal{V}$.*

Proof. Let $\mathfrak{A} = (A, F)$ be any algebra from the variety $\mathcal{V}$ and let ϕ_0 be a canonical meet-homomorphism from L onto a complete Boolean lattice L^*. For $\nu \in A[L]$ and $a \in A$ define $(\phi(\nu))(a) := \phi_0(\nu(a))$. Then clearly $\phi(\nu) \in A[L^*]$. Since ϕ_0 preserves l.u.b.'s of orthogonal systems, we get by direct computation $\phi(f(\nu_1, \dots, \nu_n)) = f(\phi(\nu_1), \dots, \phi(\nu_n))$ for every n-ary operation $f \in F$ and for all $\nu_1, \dots, \nu_n \in A[L]$. Thus ϕ is a homomorphism from the quasi-boolean extension $\mathfrak{A}[L]$ onto the Boolean extension $\mathfrak{A}[L^*]$. Hence

$$\theta := \operatorname{Ker} \phi = \{(\mu, \nu) \in A[L] \times A[L] | \phi(\mu) = \phi(\nu)\}$$

is a congruence on $\mathfrak{A}[L]$. We will show that θ is a Malcev congruence corresponding to the product $\mathcal{S} \circ_{\mathcal{F}} \mathcal{V}$.

LEMMA. $(\mu, \nu) \in \theta \iff (\forall a, b \in A; a \neq b)(\mu(a) \wedge \nu(b) = O)$.

(Indeed, if $(\mu, \nu) \in \theta$ and $\mu(a) \wedge \nu(b) \neq O$ for some distinct elements $a, b \in A$ then for $\mu^* = \phi(\mu) = \phi(\nu)$ we have $\mu^*(a) \wedge \mu^*(b) \neq O$ which is impossible since $\mu^* \in A[L^*]$.

On the other hand, if $(\mu, \nu) \notin \theta$, i.e., $\phi(\mu) = \mu^* \neq \nu^* = \phi(\nu)$, then $\mu^*(a) \neq \nu^*(a)$ for some $a \in A$. Hence $\mu^*(a) \wedge [\nu^*(a)]\prime \neq O$ or $[\mu^*(a)]\prime \wedge \nu^*(a) \neq O$ (strokes mark complements). Assume for example that the former situation takes place. Then we have

$$O \neq \mu^*(a) \wedge [\nu^*(a)]\prime = \mu^*(a) \wedge \bigvee_{x \neq a} \nu^*(x) = \bigvee_{x \neq a} (\mu^*(a) \wedge \nu^*(x)) .$$

Thus $\mu^*(a) \wedge \nu^*(b) \neq O$ for some $b \neq a$ and so $\mu(a) \wedge \nu(b) \neq O$ completing the proof of Lemma).

Now let $(\mu, \nu) \in \theta$. Under the point-wise order $\mu \leq \nu :\iff (\forall a \in A)(\mu(a) \leq \nu(a))$ each θ-class is a meet-semilattice (a lattice in fact). Take $\nu_1, \dots, \nu_n$ from some θ-class which is a subalgebra of $\mathfrak{A}[L]$. Using Lemma, we have

$$f(\nu_1, \dots, \nu_n)(a) = \bigvee_{a = f(a_1, \dots, a_n)} (\nu_1(a_1) \wedge \dots \wedge \nu_n(a_n)) = \bigvee_{a = f(x, \dots, x)} (\nu_1(x) \wedge \dots \wedge \nu_n(x))$$

for an arbitrary n-ary operation $f \in F$ and for every $a \in A$.

First assume that $f(\nu_1, \dots, \nu_n)(a) \neq O$. It follows from Lemma that $f(a, \dots, a) = a$ and $\nu_1(x) = \dots = \nu_n(x) = O$ for $x \neq a$, $f(x, \dots, x) = a$. So $f(\nu_1, \dots, \nu_n)(a) = \nu_1(a) \wedge \dots \wedge \nu_n(a) = (\nu_1 \wedge \dots \wedge \nu_n)(a)$. If, otherwise, $f(\nu_1, \dots, \nu_n)(a) = O$ then clearly $\nu_1(a) = \dots = \nu_n(a) = O$ and we can write formally $f(\nu_1, \dots, \nu_n)(a) = (\nu_1 \wedge \dots \wedge \nu_n)(a)$. Hence, $f(\nu_1, \dots, \nu_n) = \nu_1 \wedge \dots \wedge \nu_n$ and so $\theta(a)$ is a F–semilattice.

Finally, as $\mathfrak{A}/\theta \cong \mathfrak{A}[L^*]$ we have $\mathfrak{A}/\theta \in \mathcal{V}$ (recall that varieties are closed under Boolean extensions).

This completes the proof of Theorem 3. □

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