Tatra Mountains Math. Publ. **6** (1995), 149–156

TATRA MOUNTAINS
Mathematical Publications

ON THE CONVERGENCE OF OBSERVABLES IN FUZZY QUANTUM LOGICS

BELOSLAV RIEČAN

ABSTRACT. Almost everywhere convergence of sequences of observables in fuzzy quantum logics is defined. A probability space is constructed in such a way that the a.e. convergence of random variables in this space is related to the convergence of observables.

As an alternative to the quantum logics model, some fuzzy models have been constructed ([1], [5], [6], [7]). Here a set $F \subset \langle 0,1\rangle^X$ of fuzzy subsets $f\colon X \to \langle 0,1\rangle$ of a set X is considered instead of a space of linear subspaces of a Hilbert space. A state is a mapping $m\colon F \to \langle 0,1\rangle$, an observable is a mapping $x\colon \mathcal{B}(\mathbb{R}) \to F$. The state as well as the observable preserve the corresponding operations.

Of course, there are various sets of operations with fuzzy sets and consequently, various fuzzy quantum models. Using the Zadeh operations we obtain the union of two elements $f,g\colon X \to \langle 0,1\rangle$ by the formula

$$(f \vee g)(t) = \max\bigl(f(t), g(t)\bigr),$$

using the Giles operations we obtain

$$(f \vee g)(t) = \min\bigl(f(t) + g(t), 1\bigr)$$

(particularly $(f \vee g)(t) = f(t) + g(t)$, if $f(t) + g(t) \leqq 1$). The first possibility has been used in [7] and [1] , the second one in [5], [6] (see also [3], [8], [9]). While in the first case it is natural to imitate the following characterization of a.e. convergence

$$P\biggl(\bigcup_{p=1}^{\infty}\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\Bigl\{t;\ |\zeta_n(t) - \zeta(t)| \geqq \tfrac{1}{p}\Bigr\}\biggr) = 0,$$

in the second case this concept is problematic, since using min and max operations is not so natural.

The aim of the present note is an exposition of some definitions of a.e. convergence of sequences of observables in the fuzzy quantum logic (using the Giles connectives). We suggest two ways.

AMS Subject Classification (1991): 28E10, 81P10.
Key words: observables, almost everywhere convergence.

1. States and observables in fuzzy quantum logic

We assume that a set $F \subset \langle 0,1\rangle^X$ of functions $f\colon X \to \langle 0,1\rangle$ is given containing the function 1_X ($1_X(t) = 1$ for every $t \in X$).

A *state on* F is a mapping $m\colon F \to \langle 0,1\rangle$ satisfying the following properties:

(i) $m(1_X) = 1$.

(ii) If $f, g, h \in F$ and $f = g + h$, then $m(f) = m(g) + m(h)$.

(iii) If $f_n \in F$ $(n = 1, 2, \ldots)$, $f \in F$ and $f_n \nearrow f$ (i.e., $f_n(t) \leqq f_{n+1}(t)$ for every n and t and $f(t) = \lim\limits_{n\to\infty} f_n(t)$ for every $t \in X$), then $m(f) = \lim\limits_{n\to\infty} m(f_n)$.

An *observable in* F is a mapping $x\colon \mathcal{B}(\mathbb{R}) \to F$ ($\mathcal{B}(\mathbb{R})$ is the family of all Borel subsets of $\mathbb{R}$) satisfying the following conditions:

(i) $x(\mathbb{R}) = 1_X$.

(ii) If $A, B \in \mathcal{B}(\mathbb{R})$, $A \cap B = \emptyset$, then $x(A \cup B) = x(A) + x(B)$.

(iii) If $A_n \in \mathcal{B}(\mathbb{R})$ $(n = 1, 2, \ldots)$, $A \in \mathcal{B}(\mathbb{R})$ and $A_n \nearrow A$ (i.e., $A_n \subset A_{n+1}$ for every n and $A = \bigcup\limits_{n=1}^{\infty} A_n$), then $x(A_n) \nearrow x(A)$.

PROPOSITION 1. *If $m\colon F \to \langle 0,1\rangle$ is a state, $x\colon \mathcal{B}(\mathbb{R}) \to F$ is an observable and $m_x\colon \mathcal{B}(\mathbb{R}) \to \langle 0,1\rangle$ is defined by the formula $m_x(A) = m\big(x(A)\big)$, $A \in \mathcal{B}(\mathbb{R})$, then m_x is a probability measure.*

The proof of Proposition 1 is straightforward. Observe that (under some assumptions about F concerning closedness of F under certain operations) the properties (ii) of an observable and a state can be substituted by the following properties:

If $f, g, h \in F$, $g \le h$ and $f = h - g$, then $m(f) = m(h) - m(g)$.

If $A, B \in \mathcal{B}(\mathbb{R})$ and $A \subset B$, then $x(B \setminus A) = x(B) - x(A)$.

This characterization is a starting point for the definition of so-called D–posets (see [2]) including quantum logics as well as fuzzy quantum logics.

2. Finite compatibility of a sequence of observables

A sequence $(x_n)_n$ of observables $x_n\colon \mathcal{B}(\mathbb{R}) \to F$ is called to be finitely compatible, if, for every n, there is a mapping $h_n\colon \mathcal{B}(\mathbb{R}^n) \to F$ satisfying the following conditions:

(i) $h_n(\mathbb{R}^n) = 1_X$.

(ii) If $A, B \in \mathcal{B}(\mathbb{R}^n)$, $A \cap B = \emptyset$, then $h_n(A \cup B) = h_n(A) + h_n(B)$.

(iii) If $A_i \in \mathcal{B}(\mathbb{R}^n)$ $(i = 1, 2, \dots)$, $A \in \mathcal{B}(\mathbb{R}^n)$, $A_i \nearrow A$, then $h_n(A_i) \nearrow h_n(A)$.

(iv) $h_n(A_1 \times \cdots \times A_n) = x_1(A_1) \cdot x_2(A_2) \cdot \ldots \cdot x_n(A_n)$ for every $A_1, A_2, \dots, A_n \in \mathcal{B}(\mathbb{R})$.

Recently it was proved ([4]) that the mapping h_n exists, whenever F consists of all measurable fuzzy subsets of X.

If $J \subset \mathbb{N}$, $J = \{t_1, \dots, t_k\}$, $t_1 < t_2 < \cdots < t_k$, then we can define $h_J \colon \mathcal{B}(\mathbb{R}^k) \to F$ by the formula

$$h_J(A) = h_{t_k}\left(\pi^{-1}_{J_2,J}(A)\right),$$

where $J_2 = \{1, 2, \dots, t_k\}$ and $\pi_{J_2,J} \colon \mathbb{R}^{t_k} \to \mathbb{R}^{|J|}$ is the projection.

Proposition 2. *The mapping $h_J \colon \mathcal{B}(\mathbb{R}^{|J|}) \to F$ has the following properties:*

(i) $h_J(\mathbb{R}^{|J|}) = 1_X$.

(ii) *If $A, B \in \mathcal{B}(\mathbb{R}^{|J|})$, $A \cap B = \emptyset$, then $h_J(A \cup B) = h_J(A) + h_J(B)$.*

(iii) *If $A_i \in \mathcal{B}(\mathbb{R}^{|J|})$ $(i = 1, 2, \dots)$, $A \in \mathcal{B}(\mathbb{R}^{|J|})$, $A_i \nearrow A$, then $h_J(A_i) \nearrow h_J(A)$.*

(iv) *If $A \in \mathcal{B}(\mathbb{R})$, then $h_J(\{(t_1, \dots, t_k) \in J;\ t_i \in A\}) = x_{t_i}(A)$.*

Proposition 3. *Let F be closed under the proper difference (i.e., $f, g \in F$, $f \le g \Rightarrow g - f \in F$). If $J_1 \subset J_2$, then*

$$h_{J_2}\left(\pi^{-1}_{J_2,J_1}(A)\right) = h_{J_1}(A)$$

for every $A \in \mathcal{B}(\mathbb{R}^{|j_1|})$.

Proof. First let $A = A_{t_1} \times \cdots \times A_{t_k} \in \mathcal{B}(\mathbb{R}^{|J_1|})$. Then

$$\pi^{-1}_{J_2,J_1}(A) = \mathbb{R} \times \cdots \times \mathbb{R} \times A_{t_1} \times \mathbb{R} \times \cdots \times \mathbb{R} \times A_{t_k} \times \mathbb{R} \times \cdots \times \mathbb{R},$$

hence

$$\begin{aligned} h_{J_2}\left(\pi^{-1}_{J_2,J_1}(A)\right) &= x_{u_1}(\mathbb{R}) \cdots x_{u_s}(\mathbb{R}) x_{t_1}(A_{t_1}) \cdots \\ &= 1_X \cdots 1_X \cdot x_{t_1}(A_{t_1}) \cdots \\ &= x_{t_1}(A_{t_1}) \cdots x_{t_k}(A_{t_k}) = h_{J_1}(A). \end{aligned}$$

Put now

$$\mathcal{K} = \left\{A \in \mathcal{B}(\mathbb{R}^{|J|});\ h_{J_2}\left(\pi^{-1}_{J_2,J_1}(A)\right) = h_{J_1}(A)\right\}.$$

By preceding $\mathcal{K} \supset \mathcal{D}$ where $\mathcal{D}$ is the family of all rectangles $A_{t_1} \times \cdots \times A_{t_k}$ $(A_{t_1}, \dots, A_{t_k} \in \mathcal{B}(\mathbb{R}))$. By (ii) we obtain that $\mathcal{K} \supset s(\mathcal{D})$, where $s(\mathcal{D})$ is the ring generated by $\mathcal{D}$. By (iii) and (ii) we obtain that $\mathcal{K}$ is a q-σ-algebra (i.e., $A \in \mathcal{K} \Rightarrow A^c \in \mathcal{K}$ and $A_i \in \mathcal{K}$ $(i = 1,2,\ldots)$, $A_i \cap A_j = \emptyset$ $(i \neq j) \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{K})$.

Therefore

$$\mathcal{K} \supset q\text{-}\sigma\big(s(\mathcal{D})\big) = \sigma\big(s(\mathcal{D})\big) = \mathcal{B}\big(\mathbb{R}^{|J|}\big)$$

and

$$h_{J_2}\big(\pi^{-1}_{J_2,J_1}(A)\big) = h_{J_1}(A)$$

whenever $A \in \mathcal{B}\big(\mathbb{R}^{|J_1|}\big)$. □

PROPOSITION 4. *Let for every finite $J \subset \mathbb{N}$, $J \neq \emptyset$, a function $P_J \colon \mathcal{B}\big(\mathbb{R}^{|J|}\big) \to \langle 0,1\rangle$ be defined by $P_J(A) = m\big(h_J(A)\big)$, $A \in \mathcal{B}\big(\mathbb{R}^{|J|}\big)$. Then P_J is a probability measure. Moreover, if $J_1 \subset J_2$, then*

$$P_{J_2}\big(\pi^{-1}_{J_2,J_1}(A)\big) = P_{J_1}(A)$$

for every $A \in \mathcal{B}\big(\mathbb{R}^{|J_1|}\big)$.

P r o o f . By Proposition 3, for any $A \in \mathcal{B}\big(\mathbb{R}^{|J|}\big)$,

$$P_{J_2}\big(\pi^{-1}_{J_2,J_1}(A)\big) = m\Big(h_{J_2}\big(\pi^{-1}_{J_2,J_1}(A)\big)\Big) =$$
$$= m\big((h_{J_1}(A)\big) = P_{J_1}(A)\,. \qquad \square$$

Now the Kolmogorov consistency theorem is applicable. Let $\mathbb{R}^{\mathbb{N}}$ be the set of all sequences of real numbers, $\pi_J \colon \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{|J|}$ be the projection, i.e., $\pi_J\big((T_n)_n\big) = (t_{i_1},\ldots,t_{i_k})$ for $J = \{i_1,\ldots,i_k\}$. If $\mathcal{V}$ is the family of all cylinders, i.e., the set of all sets of the form $\pi_J^{-1}(A)$, $J \subset \mathbb{N}$, $A \in \mathcal{B}\big(\mathbb{R}^{|J|}\big)$, then there exists exactly one probability measure $P \colon \sigma(\mathcal{V}) \to \langle 0,1\rangle$ such that

$$P\big(\pi_J^{-1}(A)\big) = P_J(A)$$

for every finite $J \subset \mathbb{N}$ and every $A \in \mathcal{B}\big(\mathbb{R}^{|J|}\big)$.

So, given a finitely compatible sequence $(x_n)_n$ of observables, we obtain a probability space $\big(\mathbb{R}^{\mathbb{N}}, \sigma(\mathcal{V}), P\big)$ such that

$$P\big(\pi_J^{-1}(A)\big) = m\big(h_J(A)\big)\,, \qquad A \in \mathcal{B}\big(\mathbb{R}^{|J|}\big)\,.$$

Particularly, if we define

$$\zeta_i \colon = \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$$

by the formula

$$\zeta_i\big((t_n)_n\big) = t_i\,,$$

$i = 1,2,\ldots$, then ζ_i is a random variable such that, for any $A \in \mathcal{B}\big(\mathbb{R}^{|J|}\big)$,

$$P_{\zeta_i}(A) = P\big(\zeta_i^{-1}(A)\big) = P\big(\pi^{-1}_{\{i\}}(A)\big) =$$
$$= P_{\{i\}}(A) = m\big(h_{\{i\}}(A)\big) = m\big(x_i(A)\big) = m_{x_i}(A)\,,$$

hence the random variable ζ_i and the observable x_i have the same probability distribution.

3. Almost everywhere convergence

A sequence $(\eta_i)_i$ of random variables on a probability space (Ω, S, P) *converges P–almost everywhere* to η if and only if

$$P\Big(\bigcap_{p=1}^{\infty}\bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty}\big\{t_i, |\eta_n(t)-\eta(t)|<\tfrac{1}{p}\big\}\Big)=1\,.$$

If we put $\eta = 0$, then (under some reformulations) $(\eta_i)_i$ converges to 0 if and only if

$$\lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} P\Big(\bigcap_{n=k}^{k+i}\eta_n^{-1}\big((-\tfrac{1}{p},\tfrac{1}{p})\big)\Big)=1\,.$$

This formulation has served us as a motivation for the a.e.–convergence in the fuzzy set case, mainly.

First we define the zero observable $0\colon \mathcal{B}(\mathbb{R}) \to F$ by the following formula

$$0(A)=\begin{cases}1_X\,, & \text{if } 0\in A\,,\\ 0_X\,, & \text{if } 0\notin A\,.\end{cases}$$

Further we shall assume that F is closed under the product of functions. A sequence $(x_n)_n$ of observables, $x_n\colon \mathcal{B}(\mathbb{R}) \to F$ converges m–a.e., to the zero observable, if

$$\lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} m\Big(\prod_{n=k}^{k+i} x_n\big((-\tfrac{1}{p},\tfrac{1}{p})\big)\Big)=1\,.$$

If moreover F is closed under countable supremas (infimas) of non-decreasing (non-increasing) sequences, then the above definition can be rewritten in the following way:

$$m\Big(\inf_p\sup_k\prod_{n=k}^{\infty} x_n\big((-\tfrac{1}{p},\tfrac{1}{p})\big)\Big)=1\,.$$

Let us return now to our basic situation. We have a family $F\subset\langle 0,1\rangle^X$ of functions containing 1_X and closed under the product of functions, a state $M\colon F \to \langle 0,1\rangle$, and a finitely countable sequence $(x_n)_n$ of observables. Let $\big(\mathbb{R}^{\mathbb{N}},\ \sigma(V),\ P\big)$ be the corresponding probability space guaranteed by the Kolmogorov consistency theorem. The following theorem holds.

THEOREM 1. *Let $(x_n)_n$ be a sequence of finitely compatible observables, $x_n\colon \mathcal{B}(\mathbb{R}) \to F$, $(\zeta_n)_n$ be the sequence of corresponding projections defined on $\big(\mathbb{R}^{\mathbb{N}},\ \sigma(\mathcal{V}), P\big)$. Then $(x_n)_n$ converges m–a.e., to 0 if and only if $(\xi_n)_n$ converges to 0.*

Proof. Evidently, for any integer $p > 0$,

$$P\Big(\bigcap_{n=k}^{k+i}\big\{t; |\xi_n(t)| < \tfrac{1}{p}\big\}\Big) = P_J\Big(\prod_{n=k}^{k+i}\big\{(t_k,\dots,t_{k+i})\,;\tfrac{1}{p} < t_n < \tfrac{1}{p}\big\}\Big)$$

$$= m\Big(h_J\Big(\prod_{n=k}^{k+i}(-\tfrac{1}{p},\tfrac{1}{p})\Big)\Big) = m\Big(\prod_{n=k}^{k+i} x_n\big((\tfrac{1}{p},\tfrac{1}{p})\big)\Big).$$

□

4. Upper and lower limits

We shall start with the same assumptions as before and we shall use the following elementary properties of random variables:

$$P(\limsup_{n\to\infty}\eta_n < c) = P(\inf_{k\ge 1}\sup_{n\ge k}\eta_n < c) =$$

$$= \lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} P\Big(\bigcap_{n=k}^{k+1}\eta_n^{-1}\big((-\infty, c-\tfrac{1}{p})\big)\Big),$$

where $c \in \mathbb{R}$.

Proposition 5. *If (x_n) is a sequence of finitely compatible observables, $x_n\colon \mathcal{B}(\mathbb{R}) \to F$ and $\xi_n\colon \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ be the corresponding projections, $c \in \mathbb{R}$, then*

$$P(\limsup_{n\to\infty}\xi_n < c) = \lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} m\Big(\prod_{n=k}^{k+i}\big(x_n(-\infty, c-\tfrac{1}{p})\big)\Big).$$

Proof. It follows by the equality

$$P\Big(\bigcap_{n=k}^{k+i}\big\{t \in \mathbb{R}^{\mathbb{N}};\ \xi_n(t) < c-\tfrac{1}{p}\big\}\Big) = P_J\Big(\prod_{n=k}^{k+i}(-\infty,\ c-\tfrac{1}{p})\Big) =$$

$$= m\Big(h_J\Big(\prod_{n=k}^{k+i}(-\infty,\ c-\tfrac{1}{p})\Big)\Big) = m\Big(\prod_{n=k}^{k+i} x_n\big((-\infty,\ c-\tfrac{1}{p})\big)\Big).$$

□

Analogous considerations can be made in the case of $\liminf \xi_n$. Of course, if F is closed under complements $f^{\perp} = 1 - f$.

Proposition 6. $P(\liminf_{n\to\infty}\xi < c) = 1 - \lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} m\Big(\prod_{n=k}^{k+i}\big(\langle c-\tfrac{1}{p},\infty)\big)\Big)$, $c \in \mathbb{R}$.

Proof. Evidently

$$P\Big(\liminf_{n\to\infty} \xi_n < c\Big) = P\bigg(\bigcup_{p=1}^{\infty}\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} \xi_n^{-1}\Big(\big(-\infty,\, c-\tfrac{1}{p}\big)\Big)\bigg)$$

$$= \lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} m\bigg(h_J\Big(\bigcup_{n=k}^{k+i} \xi_n^{-1}\Big(\big(-\infty,\, c-\tfrac{1}{p}\big)\Big)\Big)\bigg).$$

On the other hand

$$h_J\bigg(\bigcup_{n=k}^{k+i} \xi_n^{-1}\Big(\big(-\infty,\, c-\tfrac{1}{p}\big)\Big)\bigg) = 1_X - \bigg[h_J\bigg(\bigcup_{n=k}^{k+i} \xi_n^{-1}\Big(\big(-\infty,\, c-\tfrac{1}{p}\big)\Big)\bigg)\bigg]^{\perp}$$

$$= 1_X - h_J\bigg(\bigg(\bigcup_{n=k}^{k+i} \xi_n^{-1}\Big(\big(-\infty,\, c-\tfrac{1}{p}\big)\Big)\bigg)'\bigg)$$

$$= 1_X - \bigg(\bigcap_{n=k}^{k+i} \xi_n^{-1}\Big(\big\langle c-\tfrac{1}{p}, \infty\big)\Big)\bigg)$$

$$= 1_X - \prod_{n=k}^{k+i} x_n\Big(\big\langle c-\tfrac{1}{p}, \infty\big)\Big)\,.$$

□

Remark. If we for arbitrary $a, b \in \langle 0,1\rangle$ define $a \oplus b = 1-(1-a)*(1-b) = a+b-ab$, then de Morgan's rules hold: $1-(a\oplus b) = (1-a)\cdot(1-b)$, $1-a\cdot b = (1-a)\oplus(1-b)$. With respect to the operation we can reformulate the preceding result in the following way:

$$P\Big(\liminf_{n\to\infty} \xi_n < c\Big) = \lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} m\bigg(\bigoplus_{n=k}^{k+i} x_n\Big(\big\langle c-\tfrac{1}{p}, \infty\big)\Big)\bigg).$$

We have presented two concepts of the m–a.e. of a sequence of observables. The first was introduced in Definition 2. The second concept concerns the upper and lower limits:

$$m\Big(\limsup_{n\to\infty} x_n\big((-\infty, c)\big)\Big) = m\Big(\liminf_{n\to\infty} x_n\big((-\infty, c)\big)\Big)$$

for every $x \in \mathbb{R}$. Now we shall show that these two concepts coincide:

THEOREM 2. *If F is closed under complements, then $(x_n)_n$ converges to the zero observable if and only if*

$$\lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty}\bigg(\prod_{n=k}^{k+i}\Big(x_n\big(-\infty,\, c-\tfrac{1}{p}\big)\Big)\bigg) =$$

$$= \lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty}\bigg(\bigoplus_{n=k}^{k+i} x_n\Big(\big\langle c-\tfrac{1}{p}, \infty\big)\Big)\bigg) = m\Big(0\big((-\infty, c)\big)\Big).$$

Proof. If the equalities stated in theorem hold, then, by Proposition 5 and 6, also $P\big(\limsup_{n\to\infty} \xi_n < c\big) = P\big(\liminf_{n\to\infty} \xi_n < c\big)$ for every $c \in \mathbb{R}$, hence

$$\limsup_{n\to\infty} \xi_n(\omega) = \liminf_{n\to\infty} \xi_n(\omega) = \xi(\omega)$$

for P-almost $\omega \in \mathbb{R}^{\mathrm{N}}$, hence $(\xi_n)_n$ converges P-a.e. Moreover,

$$P(\xi < c) = m\Big(0\big((-\infty, c)\big)\Big) = \begin{cases} 1, & \text{if } 0 < c, \\ 0, & \text{if } c \le 0. \end{cases}$$

It follows $\xi(\omega) = 0$ for almost every $\omega \in \mathbb{R}^{\mathrm{N}}$. By Theorem 1, we obtain that $(x_n)_n$ converges m–a.e. to 0. The opposite implication can be proved similarly. □

REFERENCES

[1] DVUREČENSKIJ, A.—RIEČAN, B.: *Fuzzy quantum models*, Int. J. General Systems **20** (1991), 39–54.

[2] KOLESÁROVÁ, A.—RIEČAN, B.: *T-fuzzy observables*, Tatra Mountains Math. Publ. **1** (1992), 73–82.

[3] KÔPKA, F.—CHOVANEC, F.: *D-posets*, Math. Slovaca **44** (1994), 21–34.

[4] MESIAR, R.—RIEČAN, B.: *On the joint observable in some quantum structures*, Tatra Mountains Math. Publ. **3** (1993), 183–190.

[5] PYKACZ, J.: *Fuzzy sets in foundations of quantum mechanics*, in: Computer management and system science (D. Dubois, R. Lowen, M. Roubens, eds.), IFSA'91, Brussels, 1991, pp. 205–208.

[6] PYKACZ, J.: *Generalized fuzzy quantum logics*. Submitted to Foundations of Physics.

[7] RIEČAN, B.: *A new approach to some notions of statistical quantum mechanics*, BUSEFAL **35** (1988), 4–6.

[8] RIEČAN, B.: *Fuzzy connectives and quantum models*, in: Cybernetics and Systems Research '92, Vol. 1 (R. Trappl, ed.), World Scientific, Singapore, 1992, pp. 335–338.

[9] RIEČAN, B.: *Law of large numbers in certain ordered structures*, in: PROBASTAT'91 (P. Pázman, J. Volaufová, eds.), Math. Inst. SAS, Bratislava, 1992, pp. 288–292.

Received March 24, 1994

Mathematical Institute
Slovak Academy of Sciences
Štefánikova 49
SK–814 73 Bratislava
SLOVAKIA

E-mail: riecan@mau.savba.sk