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EQUIVALENTIONS OVER FUZZY QUANTITIES

MILAN MAREŠ

ABSTRACT. Arithmetic operations over fuzzy quantities or fuzzy numbers do not generally fulfil some of important group properties, namely those concerning the opposite (or inverse) elements. This seriously complicates the solution of equations in which some fuzzy elements appear — either as coefficients, or as variables and right-hand-sides. This lack of group properties can be overcome if some kind of equivalence between fuzzy quantities is considered instead of the strong equality. This fact can be used for solving equations with fuzzy elements. The equality can be substituted by an equivalence, which turns the equation into equivalention, and some of the arithmetic operations can be effectively applied.

There exist several types of equivalences adequate to different operations (cf. [5]). In this brief paper we remember only one of them, applicable to one — in fact the simplest one — type of equivalentions. As mentioned in the conclusive remarks there exist theoretical tools which make our expectation concerning other types of equivalentions rather optimistic.

1. Fuzzy Quantity

By $\mathbb{R}$ we denote the set of all real numbers. Due to [3] and a few other papers, any fuzzy subset a of $\mathbb{R}$ with membership function $f_a\colon \mathbb{R} \to [0,1]$ such that

$$\sup\{f_a(x)\colon\ x \in \mathbb{R}\} = 1\,, \tag{1}$$

$$\exists\, x_1, x_2 \in \mathbb{R},\ x_1 < x_2, \quad \text{such that} \quad f_a(x) = 0 \quad \text{for} \quad x \notin [x_1, x_2]\,, \tag{2}$$

is called a *fuzzy quantity*. The set of all fuzzy quantities is denoted by $\mathscr{R}$.

If $a, b \in \mathscr{R}$, then the fuzzy quantity $a \oplus b \in \mathscr{R}$ with

$$f_{a\oplus b}(x) = \sup_{y\in\mathbb{R}}\bigl(\min(f_a(y),\, f_b(x-y))\bigr) \qquad \text{for all} \quad x \in \mathbb{R} \tag{3}$$

is called a *sum of a and b*.

To simplify the formalism we denote by $\langle y\rangle$, (where $y \in \mathbb{R}$), the fuzzy quantity for which

$$\begin{aligned} f_{\langle y\rangle}(y) &= 1\,,\\ f_{\langle y\rangle}(x) &= 0\,, \quad \text{for} \quad x \neq y,\ x \in \mathbb{R}\,. \end{aligned} \tag{4}$$

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If $a \in \mathscr{R}$ then $-a \in \mathscr{R}$ is the fuzzy quantity for which

$$f_{-a}(x) = f_a(-x) \qquad \text{for all } x \in \mathbb{R}. \tag{5}$$

As shown in [3], $\mathscr{R}$ with $\oplus$ is a commutative monoid, i.e.,

$$a \oplus b = b \oplus a\,, \tag{6}$$

$$(a \oplus b) \oplus c = a \oplus (b \oplus c)\,, \tag{7}$$

$$a \oplus \langle 0 \rangle = a\,, \tag{8}$$

for any a, b, $c \in \mathscr{R}$. The equality between fuzzy quantities $a = b$, $a, b \in \mathscr{R}$, here (and everywhere in this paper) means the pointwise equality of membership functions, $f_a(x) = f_b(x)$ for all $x \in \mathbb{R}$. This is also the main reason why $(\mathscr{R}, \oplus)$ is not a group as generally $a \oplus (-a)$ is not equal to $\langle 0 \rangle$. This discrepancy can be overcome if the crisp zero $\langle 0 \rangle$ is substituted by some "fuzzy zero" and/or the strict equality between fuzzy quantities is substituted by some weaker relation (cf. [3, 5, 6]).

We say that $a \in \mathscr{R}$ is 0-symmetric iff $a = -a$, i.e.,

$$f_a(x) = f_a(-x) \quad \text{for all } x \in \mathbb{R}. \tag{9}$$

The set of all 0-symmetric fuzzy quantities is denoted by $\mathscr{S}_0$. Further, we say that a, $b \in \mathscr{R}$ are equivalent and write $a \sim_{\oplus} b$ iff there exist s_1, $s_2 \in \mathscr{S}_0$ such that

$$a \oplus s_1 = b \oplus s_2\,. \tag{10}$$

The 0-symmetric fuzzy quantities represent the "fuzzy zero" mentioned above, and the equivalence $\sim_{\oplus}$ can be naturally considered for the "weaker equality". It is easily seen that

$$a \oplus (-a) \in \mathscr{S}_0, \qquad \text{i.e.} \qquad a \oplus (-a) \sim_{\oplus} \langle 0 \rangle\,, \tag{11}$$

as $s \sim_{\oplus} \langle 0 \rangle$ for any $s \in \mathscr{S}_0$. The fact that $a = b$ implies $a \sim_{\oplus} b$, together with (11), means that $(\mathscr{R}, \oplus)$ is a group according to the equivalence relation $\sim_{\oplus}$. Particular equivalence classes of $\sim_{\oplus}$ represent equivalent elements of that group, the class of elements equivalent to $\langle 0 \rangle$ represents the "zero" and for any $a \in \mathscr{R}$, the set

$$\{b \in \mathscr{R} :\ b \sim_{\oplus} (-a)\}$$

represents the opposite elements to a.

If $a \in \mathscr{R}$, $r \in \mathbb{R}$ then the fuzzy quantity $r \cdot a \in \mathscr{R}$ with

$$f_{r \cdot a} \begin{cases} f_a(x/r)\,, & \text{iff } r \neq 0\,, \\ f_{\langle 0 \rangle}(x)\,, & \text{iff } r = 0\,, \end{cases} \tag{12}$$

for any $x \in \mathbb{R}$, is called the *product of* r *and* a. Evidently,

$$r \cdot (a \oplus b) = r \cdot a \oplus r \cdot b, \quad a, b \in \mathscr{R},\ r \in \mathbb{R}, \tag{13}$$

but, on the other hand, for r_1, $r_2 \in \mathbb{R}$, $a \in \mathscr{R}$, the equality between $(r_1 + r_2) \cdot a$ and $(r_1 \cdot a) \oplus (r_2 \cdot a)$ is not generally fulfilled.

2. Linear equivalentions

The algebraic properties of fuzzy quantities briefly mentioned above mean that it is practically impossible to use even the simplest solving algorithms for usual linear equations with the strict equality and some fuzzy elements. Nevertheless, the 0-symmetry and related equivalence concept allow to use the analogies of those algorithms for at least some types of "equations" if we substitute the strict equality by the equivalence $\sim_{\oplus}$. Analogously to the term equations we call them *equivalentions.*

In this brief contribution we focus our attention on a simple special type of such equivalentions — the linear ones with fuzzy right-hand-side, fuzzy variables, and crisp coefficients. It means the equivalentions having the form

$$a_1 \cdot x_1 \oplus \cdots \oplus a_n \cdot x_n \sim_{\oplus} r, \tag{14}$$

where $r \in \mathscr{R}$, $x_1, \ldots, x_n \in \mathscr{R}$ are fuzzy variables, and $a_1, \ldots, a_n \in \mathbb{R}$, $a_i \neq 0$, $i = 1, \ldots, n$, are crisp coefficients.

2.1 Single variable.

The simplest special kind of (14) is the equivalention

$$a \cdot x \oplus (-r) \sim_{\oplus} \langle 0 \rangle, \qquad \text{or} \qquad a \cdot x \sim_{\oplus} r, \tag{15}$$

with $x,\, r \in \mathscr{R}$, $a \in \mathbb{R}$, $a \neq 0$. The properties of $\sim_{\oplus}$ described above evidently imply that

$$a \cdot x \oplus (-r) \sim_{\oplus} \langle 0 \rangle \quad \text{iff} \quad a \cdot x \oplus (-r) \oplus r \sim_{\oplus} r \quad \text{iff} \quad a \cdot x \oplus s \sim_{\oplus} r \quad \text{iff} \quad a \cdot x \sim_{\oplus} r,$$

for some $s = (-r) \oplus r \in \mathscr{S}_0$. Analogously

$$a \cdot x \oplus (-r) \sim_{\oplus} b \quad \text{iff} \quad a \cdot x \sim_{\oplus} b \oplus r,$$

for any $b \in \mathscr{R}$. If b is crisp, i.e., $b \in \mathbb{R}$, then evidently $b \oplus r$ denotes $\langle b \rangle \oplus r$.

The solution of (15) is evidently a fuzzy quantity $x \in \mathscr{R}$ fulfilling

$$x \sim_{\oplus} (1/a) \cdot r. \tag{16}$$

2.2 Systems of linear equivalentions.

Here we consider the system of n linear equivalentions with n variables

$$\begin{array}{c} a_{11}x_1 \oplus \cdots \oplus a_{1n}x_n \sim_{\oplus} r_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1 \oplus \cdots \oplus a_{nn}x_n \sim_{\oplus} r_n \end{array} \tag{17}$$

with $a_{11}, \ldots, a_{1n}, \ldots, a_{n1}, \ldots, a_{nn} \in \mathbb{R}$, $r_1, \ldots, r_n \in \mathscr{R}$, $x_1, \ldots, x_n \in \mathscr{R}$. There are many algorithms for solving the linear equations similar to (17), but they cannot be applied to the case of fuzzy equivalentions. We illustrate this fact by the substitution method. Let us consider two-variables system

$$a_{11} \cdot x_1 \oplus a_{12} \cdot x_2 \sim_\oplus r_1$$
$$a_{21} \cdot x_1 \oplus a_{22} \cdot x_2 \sim_\oplus r_2 .$$

It is correct to calculate

$$a_{21}\, x_1 \sim_\oplus r_2 \oplus (-a_{22}) \cdot x_2 ,$$
$$x_1 \sim_\oplus (1/a_{21}) \cdot r_2 \oplus (-a_{22}/a_{21}) \cdot x_2 ,$$

and to substitute

$$(a_{11}/a_{21}) \cdot r_2 \oplus (-a_{22}/a_{21}) \cdot x_2 \oplus a_{12} \cdot x_2 \sim_\oplus r_1 ,$$

which is equivalent to

$$a_{12} \cdot x_2 \oplus (-a_{22}/a_{21}) \cdot x_2 \sim_\oplus r_1 \oplus (-a_{11}/a_{21}) \cdot r_2 , \tag{18}$$

but the absence of the demanded distributivity law does not allow to continue, and to rearrange (18) into something like

$$\big(a_{12} - (a_{22}/a_{21})\big) \cdot x_2 \sim_\oplus r_1 \oplus (-a_{11}/a_{21}) \cdot r_2 .$$

In case of other methods, like the triangulization of the left-hand-side matrix, the same lack of distributivity makes the solution more difficult. Only the incorrect (distributive) manipulation with fuzzy variables is less evident. The same incorrect distributivity is latently present even in the determinant method or other methods derived for the solution of systems of linear equivalentions.

Rather different situation occurs if a single linear equivalention of n variables is considered. The results regarding linear combinations of fuzzy quantities in [6] as well as their general properties (cf. [3]) enable to re-arrange equivalention (14) into

$$x_1 \sim_\oplus (1/a_1) \cdot r \oplus (-a_2/a_1) \cdot x_1 \oplus \cdots \oplus (-a_n/a_1) \cdot x_n , \tag{19}$$

where x_1 is equivalent to the right-hand-side linear combination.

3. A few remarks on other types

The general results derived for fuzzy quantities allow to consider the possible modifications of the concepts given above for some other, more complex, types of equivalentions.

So, the properties of multiplication over fuzzy quantities can be sufficient for solving some simple equivalentions with both, variables and coefficients, being fuzzy. Only the equivalence relation should be evidently substituted by its multiplicative modification. The results regarding combined general equivalence (see [5]) together with those given in [3] and [4] can probably offer some solution algorithms for even rather more complex equivalentions with some types of fuzzy variable and fuzzy constants.

The results regarding universal distributivity of special class of symmetric fuzzy quantities, given in [7], enable to solve systems like (17) with symmetric right-hand-sides and crisp coefficients $a_{ij},\ i,\ j = 1, \dots, n$, quite analogously to the deterministic equations.

The results regarding powers and roots of positive fuzzy quantities (such that $f_a(x) = 0$ for $x \leq 0$) given in [4] probably offer effective tools for solution of at least simple equivalentions (and also very simple equations) with exponents.

The limited extent of this paper does not allow to study these possibilities in a adequate way. Anyhow, their detailed investigation and discussion can be considered for a challenging possibility.

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Academy of Sciences of the Czech Republic
Institute of Information Theory and Automation
Pod vodárenskou věží 4
CZ–182 08 Praha 8
CZECH REPUBLIC

E-mail: mares@utia.cas.cz