Tatra Mountains Math. Publ. **6** (1995), 13–18

TATRA MOUNTAINS
Mathematical Publications

INTEGRATION BY PARTS IN VECTOR LATTICES

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ABSTRACT. A formula for integration by parts for the Stieltjes integral in the context of vector lattices is established.

1. Introduction

Defining a Riemann-Stieltjes integral $\int_a^b f(t)\, dg(t)$, we normally start with the assumptions that f is a continuous function and g is a function of bounded variation. A similar approach is used, e.g., for introducing the integral, where the function f, the integrand, takes its values in a vector lattice X, and a function g, the integrator, takes its values in the linear space of order-continuous operators on the space X into another vector lattice Y. It is the purpose of this paper to show that the assumptions on the integrand and the integrator are reversible in some sense. This is shown by establishing a formula for integration by parts for the Stieltjes integral in the context of vector lattices. We close the paper by an example where this formula can be applied.

2. Riemann-Stieltjes integral in vector lattices

For vector lattices X and Y, we denote by $\mathcal{L}_o(X,Y)$ the linear space of order-continuous linear mappings from X to Y [1], [2].

Let $T = [a,b]$ ($a, b \in \mathbb{R}$ with $a < b$), and functions $f\colon T \to X$ and $g\colon T \to \mathcal{L}_o(X,Y)$. Consider a partition D of T

$$s_0 = a \le s_1 \le s_2 \le \cdots \le s_n = b\,,$$

and choose intermediate points (t_j) with

$$s_{j-1} \le t_j \le s_j\,, \qquad 1 \le j \le n\,.$$

AMS Subject Classification (1991): 28B15.

Key words: vector lattice, function of bounded variation, integration by parts.

For such a partition $D = (s_{j-1}, s_j)$, $j = 1, \dots, n$, we will define $p(D)$ by

$$p(D) = \max_{1 \le j \le n} (s_j - s_{j-1}) .$$

We may consider the two Riemann-Stieltjes sums

$$S_{1,D}(g, f) = \sum_{j=1}^{n} g(t_j)\left[f(s_j) - f(s_{j-1})\right] ,$$

and

$$S_{2,D}(g, f) = \sum_{j=1}^{n} \left[g(s_j) - g(s_{j-1})\right] \left(f(t_j)\right) .$$

DEFINITION. We will say that g has *a Riemann-Stieltjes integral with respect to* f if $S_{1,D}(f, g)$ has a limit in the (o)-convergence in Y as $p(D) \to 0$. This limit will be called the *Riemann-Stieltjes integral of the operator-valued function* g *with respect to the function* f and will be denoted by

$$\int_T g(s)\, d f(s) .$$

Similarly, we will say that f *has a Riemann-Stieltjes integral with respect to* g if $S_{2,D}(g, f)$ has a limit in the (o)-convergence in Y as $p(D) \to 0$. This limit will be called the *Riemann–Stieltjes integral of the function* f *with respect to the operator-valued function* g and will be denoted by

$$\int_T f(s)\, dg(s) .$$

Remark. If Y is a Banach lattice, then $Z = \mathcal{L}_o(X, Y)$ is also a Banach lattice and it is clear that X may be embedded in $\mathcal{L}_o(Z, Y)$. In that case, the Riemann-Stieltjes integral of the operator-valued function f with respect to the function g is the Riemann-Stieltjes integral of the function f (with values in Z) with respect to the operator-valued function $\tilde{g}$ (with values in $\mathcal{L}_o(Z, Y)$.

The following theorem relates the two kinds of Riemann-Stieltjes integrals [4, 7.4].

THEOREM. *If either integral exists then both exist and we have*

$$\int_T f(s)\, dg(s) = g(b)\left(f(b)\right) - g(a)\left(f(a)\right) - \int_T g(s)\, df(s) .$$

Proof. As in the classical case, we have

$$\sum_{j=1}^{n} \left[g(s_j) - g(s_{j-1})\right] \left(f(t_j)\right) = g(b)\left(f(b)\right) - g(a)\left(f(a)\right) - \sum_{j=0}^{n} g(s_j)\left[f(t_{j+1}) - f(t_j)\right] ,$$

where $t_0 = a$ and $t_{n+1} = b$. In the second sum, we consider the t_i's as subdivision points and the s_i's as intermediate points. If we denote by D the partition given by (s_j, t_j) and by D_1 the partition given by (t_j, s_j), it is clear that $p(D_1) \le 2p(D)$ and $p(D) \le 2p(D_1)$, which completes the proof. □

3. Riemann-Stieltjes integral of (o)-uniformly continuous functions

In this section we prove that the Riemann-Stieltjes integral of a (o)-uniformly continuous function f with respect to a function g with (o)-bounded semi-variation exists. From the theorem of the previous section, it will follow that the integral of a function g with (o)-bounded semi-variation with respect to a (o)-uniformly continuous function f also exists.

Let X and Y be vector lattices and $T = [a, b]$ an interval of the real line.

Definition 1. We will say that a function $f\colon T \to X$ is *(o)-uniformly continuous* if there exists a sequence $(x_n)_{n\in\mathbb{N}}$ decreasing to 0 in X and a sequence $(\delta_n)_{n\in\mathbb{N}}$ of real numbers decreasing to 0 such that $|t_1 - t_2| \le \delta_n \Longrightarrow \big|f(t_1) - f(t_2)\big| \le x_n$ for each $n \in \mathbb{N}$.

Similarly, we will say that a function $g\colon T \to \mathcal{L}_o(X, Y)$ is *(o)-uniformly continuous* if there exists a sequence $(U_n)_{n\in\mathbb{N}}$ decreasing to 0 in $\mathcal{L}_o(X, Y)$ and a sequence $(\delta_n)_{n\in\mathbb{N}}$ of real numbers decreasing to 0 such that $|t_1 - t_2| \le \delta_n \Longrightarrow \big|(g(t_1) - g(t_2))(x)\big| \le U_n\big(|x|\big)$ for each $n \in \mathbb{N}$ and $x \in X$.

If Y is a Banach lattice, $Z = \mathcal{L}_o(X, Y)$ is also a Banach lattice. In that case, it is easy to see that this last definition is equivalent to the previous one with Z instead of X.

Definition 2. A function $f\colon T \to X$ is *of (o)-bounded variation* if there exists $u \in X$ such that for any partition $(t_0, t_1, \dots, t_n)$ of $[a, b]$ the following inequality holds

$$\sum_{i=1}^{n} \big|f(t_{i+1}) - f(t_i)\big| \le u\,.$$

A function $g\colon T \to \mathcal{L}_o(X, Y)$ is *of (o)-bounded semi-variation* if there exists $U \in \mathcal{L}_o(X, Y)$ such that for any partition $(t_0, t_1, \dots, t_n)$ of $[a, b]$ the following inequality holds

$$\Big|\sum_{i=1}^{n} \big(g(t_{i+1}) - g(t_i)\big)(x_i)\Big| \le U(x)\,,$$

whenever $x_i, x \in X$ are such that $|x_i| \le x$.

It is easy to see that if a function $f\colon T \to X$ is of (o)-bounded variation, then the function $\hat{f}\colon T \to \mathcal{L}_o\big(\mathcal{L}_o(X, Y), Y\big)$ defined by $\hat{f}(t)(u) = u\big(f(t)\big)$ is of (o)-bounded semi-variation.

As we only defined the concept of (o)-bounded semi-variation in the case where the spaces are lattices, we have to suppose that $\mathcal{L}_o(X, Y)$ is a lattice. This will be the case if Y is a complete vector lattice.

We want to show that if f is (o)-uniformly continuous and g is of (o)-bounded semi-variation, then the Riemann-Stieltjes integral of f with respect to the operator-valued function g exists in Y provided Y is a σ-complete lattice.

LEMMA. *Let $T = [a,b]$, X and Y vector lattices, $f\colon T \to X$, $g\colon T \to \mathcal{L}_o(X,Y)$ of (o)-bounded semi-variation, $\delta \in \mathbb{R}$, $\delta > 0$ and $v \in X$, $v > 0$ such that*

$$|t_1 - t_2| \le \delta \Longrightarrow \big|f(t_1) - f(t_2)\big| \le v\,.$$

If D_1 and D_2 are partitions such that D_2 is finer than D_1 and $p(D_1) \le \delta$, then $|S_{2,D_1} - S_{2,D_2}| \le U(v)$ where U is the element of $\mathcal{L}_o(X,Y)$ given by the fact that g is of (o)-bounded semi-variation.

Proof. Let $D_1 = (t_0, t_1, \dots, t_k)$ with intermediate points s_i $(1 \le i \le k)$ and $D_{2,i}$ the subpartition of $[t_{i-1}, t_i]$ given by D_2 with points $t_{i,j}$ and intermediate points $s_{i,j}$ $(1 \le j \le k_i)$.

Then we have

$$S_{2,D_1} - S_{2,D_2} = \sum_{i=1}^{k}\Big[\big(g(t_i) - g(t_{i-1})\big)\big(f(s_i)\big) - \sum_{j=1}^{k_i}\big(g(t_{i,j}) - g(t_{i,j-1})\big)\big(f(s_{i,j})\big)\Big]$$

As $t_{i,0} = t_{i-1}$ and $t_{i,k_i} = t_i$, we have

$$S_{2,D_1} - S_{2,D_2} = \sum_{i=1}^{k}\sum_{j=1}^{k_i}\big(g(t_{i,j}) - g(t_{i,j-1})\big)\big(f(s_i) - f(s_{i,j})\big)\,.$$

Using the fact that $\big|f(s_i) - f(s_{i,j}))\big| \le v$, the last inequality shows that $|S_{2,D_1} - S_{2,D_2}| \le U(v)$ which completes the proof. □

THEOREM. *Let $T = [a,b]$, X be a vector lattice and Y a σ-complete vector lattice.*

If $f\colon T \to X$ is uniformly (o)-continuous and $g\colon T \to \mathcal{L}_o(X,Y)$ is of (o)-bounded semi-variation, then $\int f\,dg$ exists in Y.

Proof. As f is uniformly (o)-continuous, there exists a sequence $(v_n)_{n\in\mathrm{N}}$ in X and a sequence $(\delta_n)_{n\in\mathrm{N}}$ of nonnegative real numbers such that $v_n \downarrow 0$, $\delta_n \downarrow 0$ with the property

$$|t_1 - t_2| \le \delta_n \Longrightarrow \big|f(t_1) - f(t_2)\big| \le v_n\,.$$

As g is of (o)-bounded semi-variation, there exists U in $\mathcal{L}_o(X,Y)$ such that for any partition $(t_0, t_1, \dots, t_n)$ of $[a,b]$ we have

$$\Big|\sum_{i=1}^{n}\big(g(t_{i+1}) - g(t_i)\big)(x_i)\Big| \le U(x)$$

whenever $x_i, x \in X$ are such that $|x_i| \le x$.

If D_n be a sequence of partitions of T such that $p(D_n) \le \delta_n$ and D_{n+1} is a refinement of D_n, let $S_n = \sum\limits_{n=1}^{k_n}\big(g(t_{i+1}) - g(t_i)\big)\big(f(s_i)\big)$.

By the lemma, we have $|S_{2,n} - S_{2,n+m}| \le U(v_n)$. As $U(v_n) \downarrow 0$, the sequence $(S_{2,n})_{n\in\mathrm{N}}$ is (o)-Cauchy in Y.

Let y in Y be the (o)-limit of the sequence.

To show that $y = \int f\,dg$, we only have to prove that the Riemann sums $(S'_{2,n})$ for another sequence (D'_n) of partitions of T such that $p(D'_n) \to 0$ have limit y. We may assume that $p(D'_n) \leq \delta_n$. Using a common refinement of D_n and D'_n and the lemma, we have $|S_n - S'_n| \leq 2U(v_n)$ so that $\lim S_{2,n} = \lim S'_{2,n}$. □

COROLLARY. *Let $T = [a,b]$, X be a vector lattice and Y a σ-complete vector lattice.*

If $f\colon T \to \mathcal{L}_o(X,Y)$ is uniformly (o)-continuous and $g\colon T \to X$ is of (o)-bounded variation then $\int f\,dg$ exists in Y.

Proof. Let $\widehat{Y}$ be the completion of Y and $Z = \mathcal{L}_o(X,\widehat{Y})$. It is clear that the Riemann-Stieltjes integral of the operator-valued function f with respect to the function g is the Riemann-Stieltjes integral of the function $f\colon T \to Z$ with respect to the operator-values function $\hat{g}\colon T \to \mathcal{L}_o(Z,\widehat{Y})$ which is of (o)-bounded semi-variation. This shows that the integral exists in $\widehat{Y}$. As the integral is the limit of a sequence of elements in Y (the Riemann sums are in Y), the integral exists in Y. □

4. Fourier and Fourier-Stieltjes series in vector lattices

In the following let T denote the quotient group $\mathbb{R}/2\pi\mathbb{Z}$ ($\mathbb{R}$ and $\mathbb{Z}$ denoting the additive group of reals and integers respectively). As a model, we may think of the interval $[0,2\pi[$. It is well known [2] that every (o)-bounded linear operator $U\colon C([0,2\pi]) \to Y$ and hence every (o)-bounded linear operator $U\colon C(T) \to Y$ is expressible in the form

$$U(f) = \int_T f(t)\,dg(s)\,,$$

where $g\colon T \to Y$ is a suitable U-dependant function of (o)-bounded variation on $[0,2\pi]$. On computing the Fourier coefficients $\widehat{U}(n)$ of U, i.e.,

$$\widehat{U}(n) = U(e^{-ins}) = \frac{1}{2\pi}\int_T e^{-ins}\,dg(s)\,,$$

we have also (by definition) the Fourier-Stieltjes coefficients $\hat{dg}(u)$ of g [3, 12.5.10].

An application of the formula for integration by parts shows that

$$\hat{dg}(u) = \widehat{U}(n) = \frac{1}{2\pi}\big[g(2\pi) - g(0)\big](-1)^n + (in)\frac{1}{2\pi}\int_T g(s)e^{-ins}\,ds\,,$$

that is

$$\hat{dg}(u) = \widehat{U}(n) = \frac{1}{2\pi}\big[g(2\pi) - g(0)\big](-1)^n + (in)\hat{g}(n)\,,$$

where $\hat{g}(n) = \frac{1}{2\pi}\int_T g(s)e^{-ins}\,ds$ is the Fourier coefficient of g.

Since

$$\sum_{n\neq 0} \frac{(-1)^n e^{inx}}{in} = \sum_{n\neq 0} \frac{e^{in(x+\pi)}}{in}$$

is the Fourier-Lebesgue series of a scalar function of bounded variation, hence $\big[g(2\pi) - g(0)\big]\frac{(-1)^n}{in}$, $n \neq 0$, are the Fourier coefficients of some function $h\colon [0, 2\pi] \to Y$ of (o)-bounded variation. So we may say that the trigonometric series

$$\sum_{n\in\mathbb{Z}} y_n e^{inx}, \qquad y_n \in Y\,,$$

is a Fourier-Stieltjes series of some $g\colon T \to Y$ of (o)-bounded variation if and only if the formally integrated series (shorn of its constant term)

$$\sum_{n\neq 0} (in)^{-1} y_n e^{inx}, \qquad y_n \in Y\,,$$

is the Fourier-Lebesgue series of some function of (o)-bounded variation with values in Y.

REFERENCES

[1] CRISTESCU, R.: *Ordered Vector Spaces and Linear Operators*, Abacus Press, Kent, 1976.
[2] CRISTESCU, R.: *Representation of linear operators on spaces of vector valued functions*, Math. Slovaca **34** (1984), 405–409.
[3] EDWARDS, R. E.: *Fourier Series II. 2nd ed.*, Springer–Verlag, Berlin, 1982.
[4] HILLE, E.: *Methods in Classical and Functional Analysis*, Addison–Wesley Publishing Company, Reading, 1972.

Received September 6, 1994

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