Tatra Mountains Math. Publ. **6** (1995), 75–81

TATRA MOUNTAINS
Mathematical Publications

TRIANGULAR NORM–BASED ADDITION OF LINEAR FUZZY NUMBERS

ANNA KOLESÁROVÁ

ABSTRACT. The addition of linear fuzzy numbers based on triangular norms is studied. In the case of continuous Archimedean t-norms with strictly convex generators a necessary and sufficient condition for linearity of the t-norm-based sum is given.

1. Basic notions

Let us introduce the definitions and basic properties of finite fuzzy numbers and t-norms which will be used in the next part of the paper.

A *finite fuzzy number* is a convex normal fuzzy set p in the universum of real numbers $\mathbb{R}$ which has a continuous membership function μ_p and for which there exist numbers $a, b \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}^+$ such that

(i) $\mu_p(x) = 1$ if $x \in [a,b]$ and $\mu_p(x) = 0$ if $x \leq a - \alpha$ or $x \geq b + \beta$.

(ii) μ_p is increasing in the interval $[a-\alpha, a]$ and decreasing in $[b, b+\beta]$.

The interval $[a,b]$ is the peak of the fuzzy number p and the interval $[a-\alpha, b+\beta]$ is its support.

The membership function of a finite fuzzy number p can be expressed in the following form [2]

$$\mu_p(x) = \begin{cases} 1, & \text{for } x \in [a,b], \\ L\left(\frac{a-x}{\alpha}\right), & \text{for } x \in [a-\alpha, a], \\ R\left(\frac{x-b}{\beta}\right), & \text{for } x \in [b, b+\beta], \\ 0, & \text{otherwise}, \end{cases}$$

where $L, R \colon [0,1] \to [0,1]$ are shape functions which are non-decreasing, continuous and $L(0) = R(0) = 1$, $L(1) = R(1) = 0$.

AMS Subject Classification (1991): 04A72.
Key words: fuzzy number, linear fuzzy number, triangular norm, extension principle.

For a finite fuzzy number p we will use the notation $p = (a, b, \alpha, \beta)_{LR}$. A finite fuzzy number is said to be a *linear fuzzy number* if $L(x) = R(x) = 1 - x$, $x \in [0,1]$. A linear fuzzy number p will be denoted by $p = (a, b, \alpha, \beta)$.

Let us note that for $a \neq b$ we will get trapezoidal fuzzy numbers, and for $a = b$ triangular fuzzy numbers.

The original sum of fuzzy numbers p, q has been defined by Zadeh's extension principle :

$$\mu_{p \oplus q}(z) = \sup_{z=x+y} \big(\mu_p(x) \wedge \mu_q(y)\big), \quad z \in \mathbb{R}. \tag{1}$$

If we use in (1) instead of the operation $\wedge = \min$, which is only a special kind of a t-norm, some t-norm T, we get the sum of fuzzy numbers based on the t-norm T:

$$\mu_{p \underset{T}{\oplus} q}(z) = \sup_{z=x+y} T\big(\mu_p(x), \mu_q(y)\big), \quad z \in \mathbb{R}, \tag{2}$$

or, in the modified form,

$$\mu_{p \underset{T}{\oplus} q}(z) = \sup_{x \in \mathbb{R}} T\big(\mu_p(x), \mu_q(z-x)\big), \quad z \in \mathbb{R}. \tag{3}$$

Recall that a t-norm T is a binary operation, $T\colon [0,1] \times [0,1] \to [0,1]$, which is commutative, associative, non-decreasing in each argument and $T(x,1) = x$ for each $x \in [0,1]$. For the t-norm $\wedge = \min$ we will use the notation T_M.

Any continuous Archimedean t-norm T (i.e., a continuous t-norm for which $T(x,x) < x$, $x \in (0,1)$) can be represented by means of its additive generator f. Namely,

$$T(x,y) = f^{(-1)}\big(f(x) + f(y)\big) \quad \text{for each} \quad x, y \in [0,1],$$

where $f^{(-1)}$ is a pseudo-inverse of f given by $f^{(-1)}(u) = f^{-1}\big(\min\{u, f(0)\}\big)$. Therefore for a continuous Archimedean t-norm T the T–sum of p and q can be expressed in the form :

$$\mu_{p \underset{T}{\oplus} q}(z) = \sup_{x \in \mathbb{R}} f^{(-1)}\Big(f\big(\mu_p(x)\big) + f\big(\mu_q(z-x)\big)\Big), \quad z \in \mathbb{R}. \tag{4}$$

Let us note that the additive generator f of a t-norm T is a continuous, strictly decreasing function, $f\colon [0,1] \to [0,\infty]$ with $f(1) = 0$.

2. Results

In [4] a membership function of a T–sum $\underset{T}{\overset{n}{\underset{i=1}{\oplus}}} p_i$ of finite fuzzy numbers $p_i = (a_i, b_i, \alpha, \beta)_{LR}$, $i = 1, 2, \ldots, n$, in the case of an Archimedean t-norm T having a strictly convex twice differentiable additive generator f and fuzzy numbers with concave, twice differentiable shape functions L, R is determined.

According to [4] :

$$\mu_{\overset{n}{\underset{\substack{i=1\\T}}{\oplus}} p_i}(z) = \begin{cases} 1, & \text{for } A \leq z \leq B, \\ f^{(-1)}\Big(nf\big(L(\frac{A-z}{n\alpha})\big)\Big), & \text{for } A - n\alpha \leq z \leq A, \\ f^{(-1)}\Big(nf\big(R(\frac{z-B}{n\beta})\big)\Big), & \text{for } B \leq z \leq B + n\beta, \\ 0, & \text{otherwise}, \end{cases} \tag{5}$$

where $A = \sum_{i=1}^{n} a_i$, $B = \sum_{i=1}^{n} b_i$.

In this paper we will be interested in such t-norms T with strictly convex generators for which T–sums of linear fuzzy numbers are again linear fuzzy numbers. The following assertion gives a necessary condition for linearity of T–sums in such a case.

PROPOSITION 1. *Let T be a t-norm with a strictly convex twice differentiable additive generator f. If the T–sum $p \underset{T}{\oplus} q$ of any linear fuzzy numbers p, q is a linear fuzzy number, then*

$$f(x) = a(1-x)^s, \qquad \text{for some} \quad a \in \mathbb{R}^+, \quad s \in (1, \infty).$$

P r o o f. Let T be a t-norm with a strictly convex additive generator f and let $p = q = (1,1,1,1)$. If we take into account that the shape function L of p and q is given by $L(x) = 1 - x$, $x \in [0,1]$, then according to (5) for each $z \in [0,2]$ it holds:

$$\mu_{p \underset{T}{\oplus} q}(z) = f^{(-1)}\left(2f\left(1 - \frac{2-z}{2}\right)\right) = f^{(-1)}\left(2f\left(\frac{z}{2}\right)\right). \tag{6}$$

Since $p \underset{T}{\oplus} q$ is a linear fuzzy number, there exists a number $c \in [0,2]$ such that:

$$\mu_{p \underset{T}{\oplus} q}(z) = \frac{1}{2-c}(z-c), \qquad z \in [c, 2]. \tag{7}$$

We exclude $c = 0$ from our considerations, since for $c = 0$ we have $T = T_M$. The t-norm T_M has no additive generator and does not satisfy the assumption given in the proposition.

Comparing (6) and (7) we get

$$f^{-1}\left(2f\left(\frac{z}{2}\right)\right) = \frac{z-c}{2-c}, \qquad z \in (c, 2). \tag{8}$$

Let us consider a one to one correspondence between intervals $(0,\infty)$ and $(0,2)$ given by the mapping h, $h(s)=2-2^{\frac{s-1}{s}}$, $s\in(0,\infty)$.

Since for any $c\in(0,2)$ there exists a unique element $s\in(0,\infty)$ such that $c=2-2^{\frac{s-1}{s}}$, we can rewrite (8) into the form:

$$2f\Big(\frac{z}{2}\Big)=f\left(\frac{z-2+2^{\frac{s-1}{s}}}{2^{\frac{s-1}{s}}}\right),$$

or

$$2f\Big(\frac{z}{2}\Big)=f\left(1-\Big(1-\frac{z}{2}\Big)2^{\frac{1}{s}}\right).$$

Let us denote $g(x)=f(1-x)$. Then the previous formula can be written in the form:

$$2g\Big(1-\frac{z}{2}\Big)=g\left(\Big(1-\frac{z}{2}\Big)2^{\frac{1}{s}}\right).$$

Putting $1-\frac{z}{2}=u$ and $2^{\frac{1}{s}}=\lambda$, we get

$$\lambda^s g(u)=g(u\lambda). \tag{9}$$

For a given $s\in(0,\infty)$ the only continuous, strictly increasing, non-negative solutions of the functional equation (9) in the interval $(0,\infty)$ are functions g given by $g(u)=au^s$ for some $a\in\mathbb{R}^+$, see, e.g., [1].

Therefore we have

$$f(x)=g(1-x)=a(1-x)^s \qquad \text{for some} \quad a\in\mathbb{R}^+ \quad \text{and} \quad s\in(0,\infty).$$

Since f is by the assumption a strictly convex, twice differentiable function, only the values $s>1$ are satisfactory. Note, that the values $s\in(1,\infty)$ correspond to the values $c\in(0,1)$. □

In other words, we have just proved that if a t-norm T has a strictly convex, twice differentiable additive generator and the T–sum of any linear fuzzy numbers is a linear fuzzy number, then T must necessarily be a member of Yager's family of t-norms for some $s\in(1,\infty)$.

Recall that the family of t-norms $\{T_s^Y\}_{s\in(0,\infty)}$, where

$$T_s^Y(x,y)=\max\left\{0,1-\left[(1-x)^s+(1-y)^s\right]^{\frac{1}{s}}\right\} \quad \text{for} \quad x,y\in[0,1] \quad \text{and} \quad s\in(0,\infty)$$

was introduced by Yager in 1980 for modeling fuzzy intersection. The corresponding normed additive generators f_s^Y are given by $f_s^Y(x)=(1-x)^s$.

Now we show that the addition based on each Yager's t-norm T_s^Y, $s>1$ preserves linearity. Before proving this fact, let us make the following remark.

Remark 1. It is easy to see that the support of the sum $p\underset{T}{\oplus}q$, where $p=(a_p,b_p,\alpha_p,\beta_p)_{L_pR_p}$ and $q=(a_q,b_q,\alpha_q,\beta_q)_{L_qR_q}$ is a subinterval of the interval

$[a_p+a_q-\alpha_p-\alpha_q,\ b_p+b_q+\beta_p+\beta_q]$ and the peak of the sum for each t-norm T is the interval $[a_p+a_q,\ b_p+b_q]$. Moreover, according to the decomposition rule of finite fuzzy numbers into two separate parts [3], the left part of the membership function of the sum, i.e., its values for $z\in[a_p+a_q-\alpha_p-\alpha_q,\ a_p+a_q]$, depend only on the left parts of μ_p and μ_q. It means that in (2) to determine $\mu_{p\underset{T}{\oplus}q}(z)$ for such z, it is enough to use values $x\in[a_p-\alpha_p,\ a_p]$ and $y\in[a_q-\alpha_q,\ a_q]$. The analogous assertion holds for the right side of $\mu_{p\underset{T}{\oplus}q}$.

PROPOSITION 2. *Let T_s^Y, $s>1$ be a Yager's t-norm. Then the T_s^Y-sum of arbitrary linear fuzzy numbers p and q is a linear fuzzy number.*

Proof. By Remark 1 it is sufficient to prove the assertion for triangular fuzzy numbers.

Fix $s>1$. Let $p=(a_p,a_p,\alpha_p,\beta_p)$, $q=(a_q,a_q,\alpha_q,\beta_q)$ and $p\underset{T_s^Y}{\oplus}q=r$, where T_s^Y is the Yager t-norm. For $z\le a_p+a_q$ it holds

$$\begin{aligned}\mu_r(z)&=\sup_{x\in\mathbb{R}}T_s^Y\big(\mu_p(x),\ \mu_q(z-x)\big)\\&=\sup_{x\in[a_p-\alpha_p,a_p]}\Big(1-\min\Big\{1,\ \big[(1-\mu_p(x))^s+(1-\mu_q(z-x)^s\big]^{\frac1s}\Big\}\Big)\\&=1-\min\Big\{1,\ \inf\big[(1-\mu_p(x))^s+(1-\mu_q(z-x)^s\big]^{\frac1s}\Big\}.\end{aligned}$$

Using linearity of fuzzy numbers p, q, i.e., the fact that $\mu_p(x)=\frac{1}{\alpha_p}(x-a_p+\alpha_p)$ and $\mu_q(z-x)=\frac{1}{\alpha_q}(z-x-a_q+\alpha_q)$ for $x\in[a_p-\alpha_p,a_p]$, we get

$$\mu_r(z)=1-\min\Big\{1,\inf\Big[\big(1-\tfrac{1}{\alpha_p}(x-a_p+\alpha_p)\big)^s+\big(1-\tfrac{1}{\alpha_q}(z-x-a_q+\alpha_q)\big)^s\Big]^{\frac1s}\Big\}$$

or

$$\mu_r(z)=1-\min\Big\{1,\inf\Big[\Big(\frac{a_p-x}{\alpha_p}\Big)^s+\Big(\frac{a_q-z+x}{\alpha_q}\Big)^s\Big]^{\frac1s}\Big\}. \tag{10}$$

Let us denote $h(x)=\left(\frac{a_p-x}{\alpha_p}\right)^s+\left(\frac{a_q-z+x}{\alpha_q}\right)^s$ and let us look on the minimal value of h for $x\in[a_p-\alpha_p,a_p]$.

The derivative of the function h is :

$$h'(x)=-\frac{s}{\alpha_p}\Big(\frac{a_p-x}{\alpha_p}\Big)^{s-1}+\frac{s}{\alpha_q}\Big(\frac{x+a_q-z}{\alpha_q}\Big)^{s-1}.$$

The only point x for which $h'(x)=0$ is

$$x_0=\frac{a_p\lambda-a_q+z}{1+\lambda},\qquad\text{where}\quad\lambda=\Big(\frac{\alpha_q}{\alpha_p}\Big)^{\frac{s}{s-1}}. \tag{11}$$

It can be shown that the function h acquires its minimum in the interval $[a_p - \alpha_p, a_p]$ in the point x_0 and the minimal value of h is

$$h(x_0) = \left[\frac{a_p + a_q - z}{\alpha_p \alpha_q (1+\lambda)}\right]^s \left(\alpha_q^s + \lambda^s \alpha_p^s\right)^{\frac{1}{s}}.$$

The function $h^{1/s}$ acquires its minimal value in the point x_0, too, and it holds:

$$h^{\frac{1}{s}}(x_0) = \frac{a_p + a_q - z}{\alpha_p \alpha_q (1+\lambda)} \left(\alpha_q^s + \lambda^s \alpha_p^s\right)^{\frac{1}{s}}. \tag{12}$$

If we denote

$$k_s = \frac{\alpha_p \alpha_q (1+\lambda)}{\left(\alpha_q^s + \lambda^s \alpha_p^s\right)^{\frac{1}{s}}}, \tag{13}$$

then using (12), (10) can be expressed in the form :

$$\mu_r(z) = 1 - \min\left\{1, \frac{1}{k_s}(a_p + a_q - z)\right\}. \tag{14}$$

It means that for $z \in [a_p + a_q - \alpha_p - \alpha_q,\ a_p + a_q]$ for which $\frac{1}{k_s}(a_p + a_q - z) \geq 1$ we get $\mu_r(z) = 0$. Further, if $\frac{1}{k_s}(a_p + a_q - z) < 1$ it holds

$$\mu_r(z) = 1 - \frac{1}{k_s}(a_p + a_q - z),$$

or, in the modified form,

$$\mu_r(z) = \frac{1}{k_s}(z - a_p - a_q + k_s).$$

If we express the number k given by (13) only by means of α_p and α_q, we get

$$k_s = \alpha_p \left[1 + \left(\frac{\alpha_q}{\alpha_p}\right)^{\frac{s}{s-1}}\right]^{\frac{s-1}{s}}. \tag{15}$$

It can be easily shown that $\lim_{s\to\infty} k_s = \alpha_p + \alpha_q$, $\lim_{s\to 1^+} k_s = \max\{\alpha_p, \alpha_q\}$ and $k_s \in (\max\{\alpha_p, \alpha_q\}, \alpha_p + \alpha_q)$ for all $s \in (1, \infty)$.

If we sum up the previous results, we can write

$$\mu_r(z) = \begin{cases} 0, & \text{for } z \leq a_p + a_q - k_s, \\ \frac{1}{k_s}(z - a_p - a_q + k_s), & \text{for } a_p + a_q - k_s \leq z \leq a_p + a_q, \end{cases}$$

where $k_s \in (\max\{\alpha_p, \alpha_q\},\ \alpha_p + \alpha_q)$ is given by (15).

It means that the left part of the membership function of the sum $p \underset{T_s^Y}{\oplus} q$ is linear. The same procedure can be used for proving linearity of the right part. □

Remark 2. The number k_s defined by (15) expresses the "uncertainty" of the left side of the sum. As we have seen, for given α_p, α_q, the value k_s depends only on the parameter s of the used t-norm T_s^Y. So, choosing the parameter s ($s > 1$), we can change the uncertainty of the sum from $\max\{\alpha_p, \alpha_q\}$ to $\alpha_p + \alpha_q$.

Summarizing, we get our main result.

THEOREM 1. *Let T be a continuous Archimedean t-norm with strictly convex, twice differentiable additive generator. Then the T–sum of any linear fuzzy numbers is a linear fuzzy number if and only if the t-norm T is a Yager's t-norm T_s^Y for some $s > 1$.*

The proof of this assertion follows from Propositions 1 and 2. □

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Received March 30, 1994

Department of Mathematics
Faculty of Electrical Engineering
Slovak Technical University
Ilkovičova 3
SK–812 19 Bratislava
SLOVAKIA