

COMPATIBLE ORDERS OF SEMILATTICES

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Dedicated to Professor J. Jakubík on the occasion of his 70th birthday

ABSTRACT. Two discrete semimodular semilattices S and S_1 have isomorphic graphs if and only if S is of the form $A \times B$ and S_1 is of the form $A^\partial \times B$ for a lattice A and a semilattice B . We prove that for discrete semilattices S and S_1 this latter condition holds if and only if S and S_1 have isomorphic graphs and the isomorphism preserves the order on some special types of cells and proper cells.

G. Birkhoff ([1, Problem 8]) asked for necessary and sufficient conditions on a lattice $L = (L; \vee, \wedge)$ in order that every lattice $M = (M; \vee^*, \wedge^*)$ whose (unoriented) graph is isomorphic with the graph of L be lattice-isomorphic to L . For the case when the lattices L and M are supposed to be distributive or modular, the problem was solved by Jakubík and Kolibiar (see [2, 4, 8, 9, 11, 12, 13]). In [8] Jakubík also showed that if one of L or M is modular (distributive), then so is the other. Duffus and Rival [3] solved the problem for those graded lattices which are determined by the ordered subset of their atoms and coatoms.

In [12] Jakubík proved that for discrete modular lattices L and M on the same underlying set L , the graphs $G(L)$ and $G(M)$ are isomorphic if and only if the following condition holds:

- (a) there exist lattices $A = (A; \leq)$, $B = (B; \leq)$ and a direct product representation $\psi: L \rightarrow A \times B$ via which L is isomorphic with $A \times B$ and M is isomorphic with $A^\partial \times B$ where A^∂ stands for the dual of A .

Note that this yields a solution to Birkhoff's problem within the class of discrete modular lattices, since a modular lattice L will be uniquely determined by its graph if and only if every direct factor of L is self-dual.

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Jakubík proved in [5] that for discrete lattices (with no assumption of modularity) Condition (a) is equivalent to

- (b) L and M have isomorphic graphs and all proper cells of L and all proper cells of M are either preserved or reversed (see below for the definitions).

In [15] Kolibiar proved that for discrete semimodular semilattices S and S_1 on the same underlying set S , the graphs $G(S)$ and $G(S_1)$ are isomorphic if and only if the following condition holds:

- (c) there exist a lattice $A = (A; +, \cdot)$, a semilattice $B = (B; \cup)$ and a map $\psi: S \rightarrow A \times B$ via which ψ is a subdirect embedding of S into $A \times B$ and S_1 into $A^\theta \times B$.

In this paper we give new characterizations of (c) and derive Kolibiar's result as a corollary.

An order \leq_1 is said to be a *compatible order* of a semilattice $S = (S; \leq)$ if \leq_1 is a subsemilattice of S^2 .

In [14], it is proved that if \leq_1 is a compatible order of a semilattice $(S; \vee, \leq)$, then the relations θ_1, θ_2 on S defined by (*) are congruence relations on $(S; \vee)$:

$$\left. \begin{array}{l} a \theta_1 b \text{ if and only if } a \leq u \geq b \text{ and } a \leq_1 u \geq_1 b \\ a \theta_2 b \text{ if and only if } a \leq v \geq b \text{ and } a \geq_1 v \leq_1 b \end{array} \right\} \text{ for some } u, v \in S. \quad (*)$$

LEMMA 1 [14]. *Let $\psi: S \rightarrow S' \times S''$ be a subdirect representation of a semilattice S . Denote $\psi(x)$ by (x_1, x_2) . Given $a, b \in S$, set $a \leq_1 b$ if $a_1 \geq b_1$ and $a_2 \leq b_2$. Then \leq_1 is a compatible order of S .*

The order \leq_1 of the lemma above is said to have *stemmed from a subdirect representation* of S . If $S = (S; \vee, \leq)$ and $S_1 = (S; \vee_1, \leq_1)$ are semilattices and \leq_1 stems from a subdirect representation of S , we write $S_1 \# S$.

THEOREM 1 [14]. *Let \leq_1 be a compatible order of a semilattice $S = (S; \vee, \leq)$. The following conditions are equivalent:*

- (i) \leq_1 stems from a subdirect representation of S ;
- (ii) each interval $\{x \in S \mid a \leq_1 x \leq_1 b\}$ is a convex subset of S ;
- (iii) if $a \leq b \leq c$, then $a \leq_1 c$ implies $a \leq_1 b \leq_1 c$, and $c \leq_1 a$ implies $c \leq_1 b \leq_1 a$.

Note that condition (i) can be reformulated in the following way (as follows from the proof of Theorem 1): for the congruence relations θ_1, θ_2 corresponding to \leq_1 , see (*), we have $\theta_1 \cap \theta_2 = \omega$, where ω is the least congruence relation, and \leq_1 stems from the subdirect representation of S given by θ_1 and θ_2 .

LEMMA 2. *If \leq_1 is a compatible order of a semilattice $S = (S; \vee, \leq)$, and θ_1, θ_2 are the corresponding congruence relations, then $\leq_1 \subseteq (\theta_1 \cap \leq) \circ (\theta_2 \cap \geq)$.*

Moreover, if \leq_1 fulfils the conditions of Theorem 1, then $\leq_1 = (\theta_1 \cap \leq) \circ (\theta_2 \cap \geq)$.

COROLLARY 1. Let \leq_1 be a compatible order of a semilattice $S = (S; \vee, \leq)$, let θ_1, θ_2 be the corresponding congruence relations and let \leq_1 fulfil the conditions of Theorem 1. For $a, b \in S$,

- (i) if $a \leq b$ and $a \theta_1 b$, then $a \leq_1 b$, and
- (ii) if $a \leq b$ and $a \theta_2 b$, then $b \leq_1 a$.

THEOREM 2. Let $S = (S; \vee, \leq)$ and $S_1 = (S; \vee_1, \leq_1)$ be semilattices. Then the following are equivalent:

- (i) $S \# S_1$ and $S_1 \# S$;
- (ii) there are $\theta_1, \theta_2 \in \text{Con } S \cap \text{Con } S_1$ such that $\leq = (\theta_1 \cap \leq_1) \circ (\theta_2 \cap \geq_1)$ and $\leq_1 = (\theta_1 \cap \leq) \circ (\theta_2 \cap \geq)$;
- (iii) there is a lattice $(X; +, \cdot)$, a semilattice $(Y; \cup)$ and a map $\psi : S \rightarrow X \times Y$ such that ψ is a semilattice embedding of S into $(X; +) \times (Y; \cup)$ and ψ is a semilattice embedding of S_1 into $(X; \cdot) \times (Y; \cup)$.

Proof. (i) \Rightarrow (ii) Assume that $S \# S_1$ and $S_1 \# S$. Then, by Lemma 1 and Lemma 2, the congruence relations θ_1, θ_2 of S and $\bar{\theta}_1, \bar{\theta}_2$ of S_1 defined as in (*) fulfil

$$\leq = (\bar{\theta}_1 \cap \leq_1) \circ (\bar{\theta}_2 \cap \geq_1) \quad \text{and} \quad \leq_1 = (\theta_1 \cap \leq) \circ (\theta_2 \cap \geq).$$

By the definitions of θ_1 and $\bar{\theta}_1$, we have $\theta_1 = \bar{\theta}_1$. It remains to show that $\theta_2 = \bar{\theta}_2$.

Let $a \theta_2 b$. Then $a \leq u \geq b$ and $a \geq_1 u \leq_1 b$ for some $u \in S$. It follows that $a \vee_1 b \leq u \vee_1 b = b$ since \leq is compatible with \vee_1 . Similarly we have $a \vee_1 b \leq a$. Hence $a \leq_1 a \vee_1 b \geq_1 b$ and $a \geq a \vee_1 b \leq b$ imply $a \bar{\theta}_2 b$. Thus $\theta_2 \subseteq \bar{\theta}_2$. Analogously, $\bar{\theta}_2 \subseteq \theta_2$.

(ii) \Rightarrow (iii) Assume that (ii) holds. Then \leq is compatible with \vee_1 and \leq_1 is compatible with \vee . Let $(a, b) \in \theta_1 \cap \theta_2$. Then $a \leq a \vee b \geq b$ and $\leq_1 = (\theta_1 \cap \leq) \circ (\theta_2 \cap \geq)$ imply that $a \leq_1 a \vee b \leq_1 b$ and $b \leq_1 a \vee b \leq_1 a$ which yields $a = b$; i.e., $\theta_1 \cap \theta_2 = \omega$.

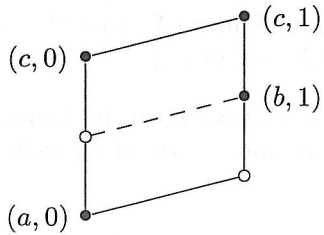
Now we will show that the operation join of $(S/\theta_1; \leq_1)$ is the meet operation of $(S/\theta_1; \leq)$; or equivalently, it is enough to show that for any $a, b \in S$, $[a \vee b] \theta_1 = [a] \theta_1$ if and only if $[a \vee_1 b] \theta_1 = [b] \theta_1$. Let $[a \vee b] \theta_1 = [a] \theta_1$. Then $[b] \theta_1 \leq [a \vee_1 b] \theta_1$ since $[b] \theta_1 \leq [a] \theta_1$ and \leq is compatible with \vee_1 . Since $b \leq_1 a \vee_1 b$ and $\leq_1 = (\theta_1 \cap \leq) \circ (\theta_2 \cap \geq)$, we have $u \in S$ such that $b \theta_1 u \theta_2 a \vee_1 b$ and $b \leq u \geq a \vee_1 b$; so $[b] \theta_1 = [u] \theta_1 \geq [a \vee_1 b] \theta_1$. Hence $[b] \theta_1 = [a \vee_1 b] \theta_1$. We can prove the converse analogously. Therefore $(S/\theta_1; \vee, \vee_1)$ is a lattice.

Analogously, we can prove that $(S/\theta_2; \leq)$ is isomorphic to $(S/\theta_2; \leq_1)$; or equivalently, the join operation of $(S/\theta_2; \leq)$ is the join operation of $(S/\theta_2; \leq_1)$.

Since the natural map is an embedding of S into $(S/\theta_1; \vee) \times (S/\theta_2; \vee)$ and S_1 into $(S/\theta_1; \vee_1) \times (S/\theta_2; \vee_1)$, we have $(S/\theta_1; \vee, \vee_1)$ and $(S/\theta_2; \vee)$ are the required lattice and semilattice, respectively.

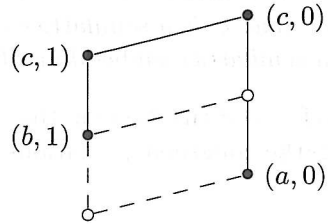
(iii) \Rightarrow (i) Let \leq and \leq_1 denote the order relations on $(X; +) \times (Y; \cup)$ and $(X; \cdot) \times (Y; \cup)$ respectively, and let \leq_2 denote the order relation on $(X; +, \cdot)$ and \leq_3 denote the order relation on $(Y; \cup)$.

Let $T = (T; \leq)$ and $T_1 = (T; \leq_1)$ be images of the subdirect representation ψ and let $(a_1, a_2), (b_1, b_2), (c_1, c_2)$ be elements in T with $(a_1, a_2) \leq (b_1, b_2)$. Then $a_1 \leq_2 b_1$ and $a_2 \leq_3 b_2$. So $a_1 \cdot c_1 \leq_2 b_1 \cdot c_1$ and $a_2 \cup c_2 \leq_3 b_2 \cup c_2$; i.e., $(a_1 \cdot c_1, a_2 \cup c_2) \leq (b_1 \cdot c_1, b_2 \cup c_2)$, which shows that \leq is compatible with the operation of $(X; \cdot) \times (Y; \cup)$. By analogy, \leq_1 is compatible with the operation of $(X; +) \times (Y; \cup)$. Hence $S\#S_1$ and $S_1\#S$ follow from the subdirect representation. \square



$(S; \vee, \leq)$

Figure 1.



$(S; \vee_1, \leq_1)$

Figure 2

Figure 1 and Figure 2 show that if S_1 above is a compatible ordered set of S which stems from a 2-factor subdirect representation of S , then it does not necessarily follow that the graph $G(S)$ and $G(S_1)$ are isomorphic. In this paper, we shall prove the following theorem for a pair of semilattices.

THEOREM 3. Let $S = (S; \vee, \leq)$ and $S_1 = (S; \vee_1, \leq_1)$ be discrete semilattices. Then $S\#S_1$ and $S_1\#S$ if and only if the following conditions hold:

- (A) $G(S) = G(S_1)$,
- (B) if either S or S_1 contains a cell of type $\diamond(1, n)$, say $C = \{u \prec x \prec v \succ y_n \cdots \succ y_1 \succ u\}$, then the other contains one of the following four cells of type $\diamond(1, n)$: $C, C^\theta, D = \{y_1 \prec_1 y_2 \prec_1 \cdots \prec_1 y_n \prec_1 v \prec_1 x \succ_1 u \succ_1 y_1\}$ or D^θ , and
- (C) in both S and S_1 , all proper cells of type $\vee(m, n)$ with $m > 1$ and $n > 1$ are preserved or reversed.

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A semilattice $S = (S; \leq)$ is called *discrete* if each bounded chain in S is finite.

Let $S = (S; \leq)$ be a semilattice. For $a, b \in S$ with $a \leq b$ an interval $[a, b]$ is the set of all elements $x \in S$ satisfying $a \leq x \leq b$. We call $[a, b]$ a *prime interval* (or equivalently, b covers a , in symbols $a \prec b$) if $|[a, b]| = 2$.

By the graph $G(S)$ we mean the (undirected) graph whose vertex set is S and whose edges are those pairs $\{a, b\}$, which satisfy either $a \prec b$ or $b \prec a$.

Let $S = (S; \leq)$ and $S_1 = (S_1; \leq_1)$ be semilattices. It is said that $G(S)$ is isomorphic with $G(S_1)$ if there is a bijection $f: S \rightarrow S_1$ such that for all $a, b \in S$, $\{a, b\}$ is an edge of $G(S)$ if and only if $\{f(a), f(b)\}$ is an edge of $G(S_1)$. Throughout this paper we assume, without loss of generality, that $S = S_1$ and f is the identity map whenever $G(S)$ is isomorphic to $G(S_1)$, whence $G(S) = G(S_1)$.

If $G(S) = G(S_1)$, then a set $C \subseteq S$ is said to be *preserved* if, whenever $a, b \in C$ and $a \prec b$, then $a \prec_1 b$.

Let $u, v, x_1, \dots, x_m, y_1, \dots, y_n$ be distinct elements in S such that

- (i) $u \prec x_1 \prec \dots \prec x_m \prec v, u \prec y_1 \prec \dots \prec y_n \prec v$, and
- (ii) either v is the least upper bound of x_1 and y_1 (denoted by $v = x_1 \vee y_1$) or u is the greatest lower bound of x_m and y_n (denoted by $u = x_m \wedge y_n$).

Then the set $C = \{u, v, x_1, \dots, x_m, y_1, \dots, y_n\}$ is said to be a *cell* of S . If $x_1 \vee y_1 = v$, we call C a *cell of type* $\vee(m, n)$. Dually, if $x_m \wedge y_n = u$, we call C a *cell of type* $\wedge(m, n)$. If $x_1 \vee y_1 = v$ and $x_m \wedge y_n = u$, we call C a *cell of type* $\diamond(m, n)$. If $m = n$, then C is a *cell of length* $n + 1$. A cell C is called *proper* if either $m > 1$ or $n > 1$.

A semilattice S is said to be *upper semimodular* if S satisfies the following *Upper Covering Condition* (UCC):

(UCC): if a and b cover c with $a \neq b$, then both a and b are covered by $a \vee b$.

Let S and S_1 be discrete semimodular semilattices. Then S and S_1 contain no cells of type $\vee(m, n)$ with $m, n \geq 1$ & $m + n > 2$; i.e., Conditions (B) and (C) of Theorem 3 always hold. Therefore we obtain one of Kolibiar's results [15] as a corollary.

COROLLARY 2. [15] *Let S and S_1 be semimodular semilattices. Then S and S_1 satisfy Condition (c) if and only if $G(S) = G(S_1)$.*

We now prove Theorem 3 via the following lemmata.

LEMMA 3. *Let $S = (S; \vee, \leq)$ and $S_1 = (S; \vee_1, \leq_1)$ be discrete semilattices satisfying $S \# S_1$ and $S_1 \# S$. Then Conditions (A), (B) and (C) hold.*

Proof. Assume that $a \prec b$. Then $a \leq b$ implies $a \leq a \vee_1 b \leq b$ which yields $a = a \vee_1 b$ or $b = a \vee_1 b$; i.e., $b \leq_1 a$ or $a \leq_1 b$. If $a \leq_1 c \leq_1 b$ for some

$c \in S$, then it follows by Theorem 1 with $a \leq b$ that $a \leq c \leq b$; hence $a = c$ or $b = c$, which shows that $a \prec_1 b$. Similarly if $b \leq_1 c \leq_1 a$ for some $c \in S$, then $b \prec_1 a$.

Analogously $a \prec_1 b$ implies $a \prec b$ or $b \prec a$. Hence $G(S) = G(S_1)$.

To show that S and S_1 satisfy Condition (B), let $C = \{u \prec x \prec v \succ y_n \succ \dots \succ y_1 \succ u\}$ be a cell of type $\diamond(1, n)$ in S (see Figure 3(1)). By the assumption, Condition (A) and the definitions of θ_1 and θ_2 (defined as in (*)), we have either $x \theta_1 u \theta_1 y_1$, $x \theta_2 u \theta_2 y_1$, $x \theta_1 u \theta_2 y_1$ or $x \theta_2 u \theta_1 y_1$.

Case 1: $x \theta_1 u \theta_1 y_1$. Then $x \geq_1 u \leq_1 y_1$. It follows by Corollary 1 that $x \theta_1 v$ and $x \leq v$ imply $x \leq_1 v$. Since $u \theta_1 x$ implies $y_i \theta_1 v$ for all $i = 1, 2, \dots, n$, the transitivity of θ_1 yields $y_i \theta_1 y_j$ for all i, j . Thus $y_i \theta_1 y_{i+1}$ and $y_i \prec y_{i+1}$ imply $y_i \prec_1 y_{i+1}$ for all $i = 1, 2, \dots, n$; i.e., $y_1 \prec_1 y_2 \prec_1 \dots \prec_1 y_n \prec_1 v$. Therefore C is a cell of S_1 of type $\diamond(1, n)$.

Case 2: $x \theta_2 u \theta_2 y_1$. We can prove analogously to Case 1 that C^∂ is a cell of S_1 of type $\diamond(1, n)$.

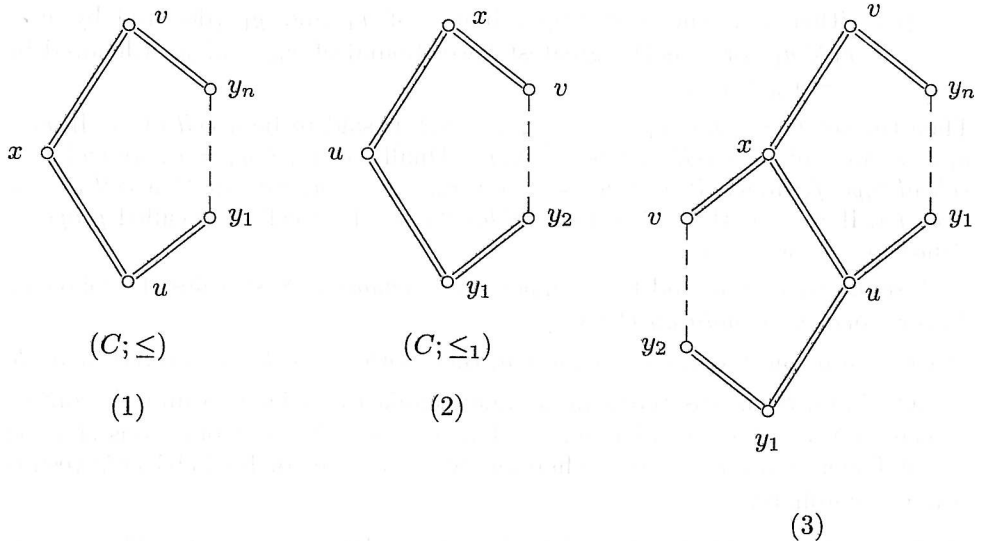


FIGURE 3.

Case 3: $x \theta_1 u \theta_2 y_1$. Then $x \theta_2 v$ and $v \theta_1 y_i$ for all $i = 1, 2, \dots, n$. It follows from Corollary 1 and the transitivity of θ_1 that $y_1 \prec_1 y_2 \prec_1 \dots \prec_1 y_n \prec_1 v \prec_1 x \succ_1 u \succ_1 y_1$; i.e., D is a cell of S_1 of type $\diamond(1, n)$ (see Figure 3).

Case 4: $x \theta_2 u \theta_1 y_1$. We can prove analogously to Case 3 that D^∂ is a cell of S_1 of type $\diamond(1, n)$.

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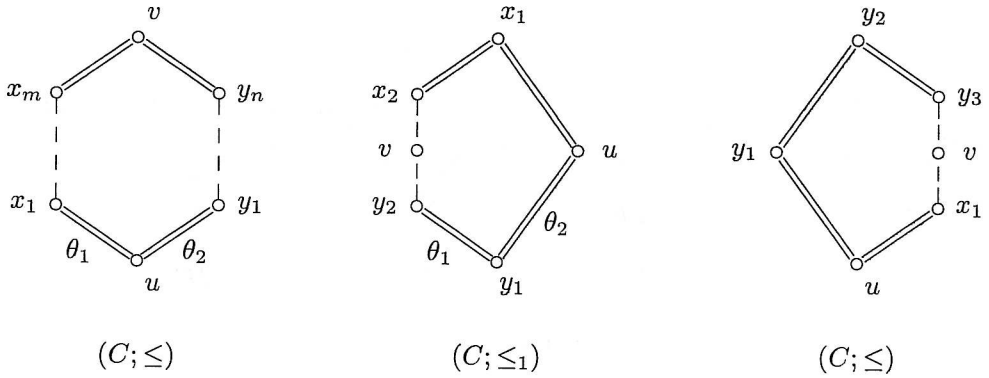


FIGURE 4.

To prove Condition (C), let $A = \{u \prec x_1 \prec \dots \prec x_m \prec v \succ y_n \succ \dots \succ y_1 \succ u\}$ be a proper cell of S of type $\vee(m, n)$ ($m > 1$ and $n > 1$). Then $x_1 \vee y_1 = v$. Suppose that $x_1 \theta_1 u \theta_2 y_1$. Then $x_i \theta_2 v \theta_1 y_j$ for all $1 \leq i < m$ and $1 \leq j \leq n$, which together with the transitivity of θ_1 and θ_2 implies that $x_i \theta_2 x_k$ and $y_j \theta_1 y_\ell$ for all $1 \leq i, k \leq m$ and $1 \leq j, \ell \leq n$. So $C = \{y_1 \prec_1 y_2 \prec_1 \dots \prec_1 y_n \prec_1 v \prec_1 x_m \prec_1 \dots \prec_1 x_1 \succ_1 u \succ_1 y_1\}$ is a cell of S_1 of type $\diamond(1, n + m - 1)$ (see Figure 4). By Condition (B), since $y_2 \theta_1 y_1 \theta_2 u$, we have that D^∂ is a cell of S of type $\diamond(1, n + m - 1)$ which yields $y_2 = y_3 = \dots = y_n = v$; i.e., $n = 1$, a contradiction.

We will get a similar contradiction if $x_1 \theta_2 u \theta_1 y_1$. Therefore, either $x_1 \theta_1 u \theta_1 y_1$ or $x_1 \theta_2 u \theta_2 y_1$. Hence A is preserved or reversed. \square

In the following lemmata, we shall assume that S and S_1 are semilattices satisfying Conditions (A), (B) and (C).

LEMMA 4. *Let $a, b, c \in S$ with $a \succ c \prec b$. Then*

- (i) $c \prec_1 a$ implies $b \leq_1 a \vee b$, and
- (ii) $a \prec_1 c$ implies $a \vee b \leq_1 b$.

Proof. We only prove (i) as (ii) follows by duality. Assume $c \prec_1 a$. Since the case $a = b$ is trivial, we assume $a \neq b$.

If $c \prec a \prec a \vee b \succ b \succ c$, then using (B) we obtain immediately $b \prec_1 a \vee b$. If $c \prec a \not\prec a \vee b \not\prec b \succ c$, then condition (C) applies. We may assume that $c \prec a \prec a \vee b \geq b \succ c$ (we can prove analogously if $c \prec b \prec a \vee b \geq a \succ c$). Then $C = \{c \prec a \prec a \vee b \succ y_n \succ \dots \succ y_1 \succ b \succ c\}$ for some $y_1, \dots, y_n \in S$ is a cell of S of type $\diamond(1, n)$ with $[c, a]$ preserved (reversed). Hence, by Condition (B), either C or $D = \{b \prec_1 y_1 \prec_1 \dots \prec_1 y_n \prec_1 a \vee b \prec_1 a \succ_1 c \succ_1 b\}$ (resp. C^∂ or D^∂) is a cell of S_1 of type $\diamond(1, n)$ (see Figure 5). In either case we have $b \leq_1 a \vee b$ (resp. $a \vee b \geq_1 b$). \square

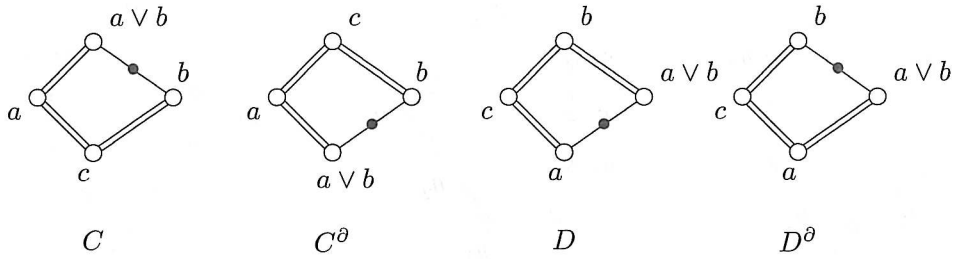


FIGURE 5.

LEMMA 5. Let $a, b, c \in S$ with $a \prec b$. Then

- (i) $a \prec_1 b$ implies $a \vee c \leq_1 b \vee c$, and
- (ii) $b \prec_1 a$ implies $b \vee c \leq_1 a \vee c$.

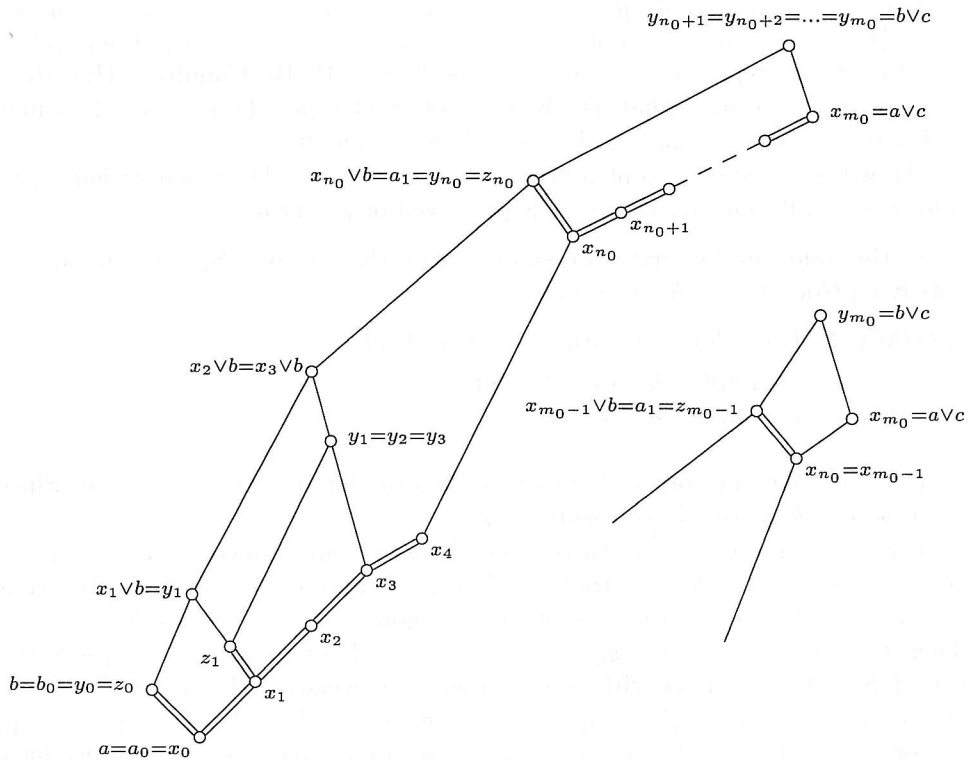


FIGURE 6.

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Proof. Let $a = x_0 \prec x_1 \prec \dots \prec x_{m_0} = a \vee c$, and define $y_i = x_i \vee z_{i-1}$ where $y_0 = b = z_0$ and in general z_{i-1} is chosen so that $x_{i-1} \prec z_{i-1} \leq y_{i-1}$ (if there is an i ($0 < i < m_0$) such that $z_i = x_{i+1}$, we will continue the process by considering $z_i = z_{i-1}$ and $y_{i+1} = y_i$ (see Figure 6)). Then, by Lemma 4, the interval $[x_{i+1}, y_{i+1}]$ is preserved (reversed) if the interval $[x_i, y_i]$ is preserved (reversed) since $x_i \prec z_i \leq y_i$. Hence by induction $[x_{m_0}, y_{m_0}]$ is preserved (reversed) since $[a, b]$ is preserved (reversed). Since $x_i \leq y_i \leq x_i \vee b$ for all $0 \leq i \leq m_0$, we have $a \vee c = x_{m_0} \leq y_{m_0} \leq x_{m_0} \vee b = a \vee c \vee b = b \vee c$.

Let $a_0 = a$, $b_0 = b$, $x_i^{(o)} = x_i$, $y_i^{(o)} = y_i$, $z_i^{(o)} = z_i$ for all $i = 0, 1, \dots, m_0$. Note that $z_0^{(o)} = b_0$. Let $a_1 = z_{n_0}^{(o)}$ where n_0 is the least number such that $0 \leq n_0 < m_0$ and $x_{n_0+1} \vee z_{n_0} = y_{n_0+1} = y_{n_0+2} = \dots = y_{m_0}$ (see Figure 6 and 7).

Case 1: If $a_1 = z_{n_0}^{(o)} = x_{n_0} \vee b (= y_{n_0})$, then $y_{m_0} = x_{m_0} \vee z_{n_0} = x_{m_0} \vee (x_{n_0} \vee b) = x_{m_0} \vee b = (a \vee c) \vee b = b \vee c$; hence, by using (B) or (C), the preservation of $[x_{n_0}, y_{n_0}]$ implies the preservation of $[x_{m_0}, y_{m_0}] = [a \vee c, b \vee c]$ (see Figure 6).

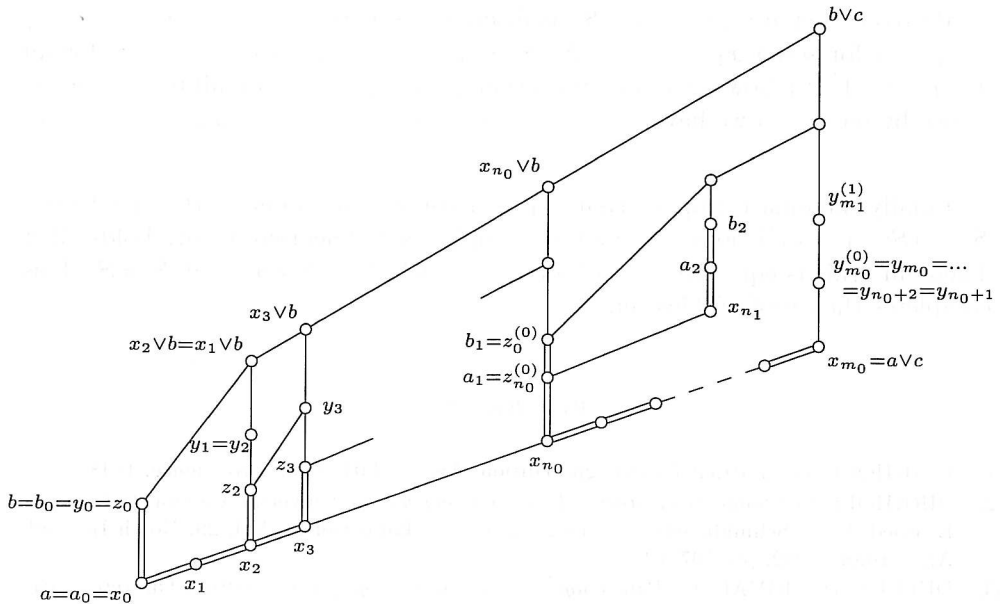


FIGURE 7.

Case 2: If $a_1 = x_{n_0}^{(o)} < x_{n_0} \vee b$ (note that $x_{n_0}^{(o)} \leq y_{n_0} \leq x_{n_0} \vee b$), we choose $b_1 (= z_0^{(1)})$ with $a_1 \prec b_1 \leq x_{n_0}^{(o)} \vee b$ (see Figure 7). Note that since $[a_0, b_0]$ is preserved (reversed) so is $[a_1, b_1]$. Now repeat the construction with a_0 replaced

by a_1 and b_0 replaced by b_1 . This then produces the elements $z_{n_1}^{(1)}$ and $y_{m_1}^{(1)}$ which are needed to begin the next step of the induction.

(In general, we define $a_{i+1} = z_{n_i-1}^{(i)}$ where n_i is the least number such that $0 \leq n_i < m_i$ and $x_{n_i+1} \vee z_{n_i} = y_{n_i+1} = y_{n_i+2} = \dots = y_{m_i}$ and choose $b_{i+1} = z_0^{(i+1)}$ with $a_{i+1} = z_{n_i}^{(i)} \prec z_0^{(i+1)} = b_{i+1} \leq x_{n_i}^{(i)} \vee b_i$ (for the case $a_{i+1} = z_{n_i}^{(i)} = x_{n_i}^{(i+1)} \vee b_i = y_{n_i}^{(i)}$, (as Case 1) the preservation of $[a_i, b_i]$ implies the preservation of $[y_{m_i}^{(i)}, y_{m_{i+1}}^{(i+1)}]$) and, finally, repeat the process with the covering chain:

$$a_{i+1} = z_{n_i-1}^{(i)} = x_0^{(i+1)} \prec x_1^{(i+1)} \prec \dots \prec x_{m_{i+1}}^{(i+1)} = y_{m_i}^{(i)}.$$

Since S is discrete, there exists N such that $y_{m_N}^{(N)} = b \vee c$ and hence we have a chain:

$$a \vee c \leq y_{m_0}^{(0)} \leq y_{m_1}^{(1)} \leq \dots \leq y_{m_N}^{(N)} = b \vee c,$$

and each step in this chain is preserved (reversed) since each interval $[a_i, b_i]$ is preserved (reversed). □

LEMMA 6. *Let $a, b, c \in S$ with $a \leq_1 b$. Then $a \vee c \leq_1 b \vee c$.*

Proof. Let $a \leq_1 b$. Since S_1 is discrete, we have $a = x_0 \prec_1 x_1 \prec_1 \dots \prec_1 x_{n+1} = b$ for some $x_1, \dots, x_n \in S$. So either $x_i \prec x_{i+1}$ or $x_{i+1} \prec x_i$ for all $0 \leq i < n+1$. It follows from Lemma 5 that $x_i \vee c \leq_1 x_{i+1} \vee c$ for all $0 \leq i < n+1$. Hence by induction we have $a \vee c = x_0 \vee c \leq_1 x_1 \vee c \leq_1 \dots \leq_1 x_{n+1} \vee c = b \vee c$. □

Finally, Lemma 6 implies that \leq_1 is a compatible order of the semilattice $S = (S; \leq)$. Conditions (B) and (C) imply that Theorem 1 (iii) holds. But Theorem 1(iii) is equivalent to Theorem 2 (i) which is $S \# S_1$ and $S_1 \# S$. This completes the proof of Theorem 3.

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