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# VALUES OF GROUP WORDS IN TOTALLY ORDERED GROUPS

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*Dedicated to Professor J. Jakubík on the occasion of his 70th birthday*

ABSTRACT. It is proved that for each nilpotent totally ordered group $G$ of nilpotent class $\leq 5$ and each group word $w(x_1, \dots, x_n)$ in variables $x_1, \dots, x_n$, the inequality $w(\bar{x}_1, \dots, \bar{x}_n) > e$ for some $\bar{x}_1, \dots, \bar{x}_n \in G$ implies that there are $x_1', \dots, x_n' \in G$ such that $w(x_1', \dots, x_n') < e$ in $G$.

In the Black Swamp Problem Book [1] problem 49 is the following: "Let $G$ be a totally ordered group (o-group) and let $w(x, y, \dots)$ be a word in free group with countable set of free generators. Suppose that for some substitution $\overline{x}, \overline{y}, \dots \in G$ we have $w(\bar{x}, \bar{y}, \dots) > e$ in $G$. Does it imply that for some substitution $x', y', \dots \in G$ we also have $w(x', y', \dots) < e$?"

In [2] examples of the nilpotent o-group $G$ of nilpotent class 6 and the group word $w(x, y)$ in two variables $x$ and $y$ were constructed, such that all values of $w(x, y)$ in o-group $G$ have the same sign. The following question arises: is there a similar counterexample to problem 49 of nilpotent o-group of nilpotent class $\leq 5$? The present paper gives the negative answer to this question. It is shown that for each nilpotent o-group $G$ of nilpotent class $\leq 5$ and each group word $w(x_1, \dots, x_n)$ in variables $x_1, \dots, x_n$ from the inequality $w(\bar{x}_1, \dots, \bar{x}_n) > e$ for some $\bar{x}_1, \dots, \bar{x}_n \in G$ there are $x_1', \dots, x_n' \in G$ such that $w(x_1', \dots, x_n') < e$ in o-group $G$.

All basic facts and definitions on ordered groups and groups can be found in [3], [4] and [5], respectively.

It is known that each group word $w(x_1, \dots, x_n)$ in the variety of nilpotent groups of nilpotent class $\leq 5$ can be represented as a product of basic commutators of weight $\leq 5$ in the variables $x_1, \dots, x_n$ ([5], Theorem 11.2.4). As usual,

$$[x_1, x_2] = x_1^{-1} x_2^{-1} x_1 x_2, \qquad [x_1, \dots, x_n] = \big[[x_1, \dots, x_{n-1}], x_n\big],$$

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and the set of integers is denoted by $\mathbb{Z}$.

**PROPOSITION 1.** *Let $G$ be a nilpotent o-group of nilpotent class $\leq 5$ and $w(x_1, x_2)$ be a group word in two variables $x_1$, $x_2$. If $w(\bar{x}_1, \bar{x}_2) > e$ in o-group $G$ for some $\bar{x}_1$, $\bar{x}_2 \in G$, then there are $x_1'$, $x_2' \in G$ such that $w(x_1', x_2') < e$.*

Proof. By the above arguments we may assume that

$$\begin{aligned} w(x_1,x_2) = {} & x_1^{k_1} x_2^{k_2} [x_1,x_2]^{p_1} [x_1,x_2,x_1]^{\alpha_{11}} [x_1,x_2,x_2]^{\alpha_{21}} [x_1,x_2,x_1,x_1]^{\beta_{11}} \times \\ & \times [x_1,x_2,x_2 x_1]^{\beta_{21}} [x_1,x_2,x_2,x_2]^{\beta_{31}} [x_1,x_2,x_1,x_1,x_1]^{\gamma_{11}} [x_1,x_2,x_2,x_1,x_1]^{\gamma_{21}} \times \\ & \times [x_1,x_2,x_2,x_2,x_1]^{\gamma_{31}} [x_1,x_2,x_2,x_2,x_2]^{\gamma_{41}} \times \\ & \times \big[[x_1,x_2,x_2],[x_1,x_2]\big]^{\gamma_{51}} \big[[x_1,x_2,x_1],[x_1,x_2]\big]^{\gamma_{61}} \end{aligned}$$

for some integers $k_1$, $k_2$, $p_1$, $\alpha_{11}$, $\alpha_{21}$, $\beta_{11}$, $\beta_{21}$, $\beta_{31}$, $\gamma_{i1}$ $(1 \leq i \leq 6)$. By our assumption $w(\bar{x}_1, \bar{x}_2) > e$ in o-group $G$. Suppose that $w(x_1, x_2) \geq e$ for all $x_1$, $x_2 \in G$. Then $w(x_1^{-1}, x_2^{-1}) \geq e$ in $G$ for all $x_1$, $x_2$ too. Let $w_2 = w(x_1, x_2) \times w(x_1^{-1}, x_2^{-1})$. Direct verification shows that the group word $w_2(x_1, x_2)$ can be represented in the form

$$\begin{aligned} w_2(x_1,x_2) = {} & [x_1,x_2]^{p_2} [x_1,x_2,x_1]^{\alpha_{12}} [x_1,x_2,x_2]^{\alpha_{22}} [x_1,x_2,x_1,x_1]^{\beta_{12}} \times \\ & \times [x_1,x_2,x_2 x_1]^{\beta_{22}} [x_1,x_2,x_2,x_2]^{\beta_{32}} [x_1,x_2,x_1,x_1,x_1]^{\gamma_{12}} [x_1,x_2,x_2,x_1,x_1]^{\gamma_{22}} \times \\ & \times [x_1,x_2,x_2,x_2,x_1]^{\gamma_{32}} [x_1,x_2,x_2,x_2,x_2]^{\gamma_{42}} \times \\ & \times [[x_1,x_2,x_2],[x_1,x_2]]^{\gamma_{52}} [[x_1,x_2,x_1],[x_1,x_2]]^{\gamma_{62}} \end{aligned}$$

for some integers $p_2$, $\alpha_{12}$, $\alpha_{22}$, $\beta_{12}$, $\beta_{22}$, $\beta_{32}$, $\gamma_{i2}$ $(1 \leq i \leq 6)$.

Then $w_2(\bar{x}_1, \bar{x}_2) > e$ and $w_2(x_1, x_2) \geq e$ in o-group $G$ for all $x_1$, $x_2 \in G$. Let now $w_3(x_1, x_2) = w_2(x_1, x_2) \cdot w_2(x_1^{-1}, x_2)$. It is clear that

$$w_3(x_1, x_2) \geq e$$

in o-group $G$ for all $x_1$, $x_2 \in G$ and $w_3(\bar{x}_1, \bar{x}_2) > e$. Again, direct verification shows that

$$\begin{aligned} w_3(x_1,x_2) = {} & [x_1,x_2,x_1]^{\alpha_{13}} [x_1,x_2,x_2]^{\alpha_{23}} [x_1,x_2,x_1,x_1]^{\beta_{13}} \times \\ & \times [x_1,x_2,x_2 x_1]^{\beta_{23}} [x_1,x_2,x_2,x_2]^{\beta_{33}} [x_1,x_2,x_1,x_1,x_1]^{\gamma_{13}} [x_1,x_2,x_2,x_1,x_1]^{\gamma_{23}} \times \\ & \times [x_1,x_2,x_2,x_2,x_1]^{\gamma_{33}} [x_1,x_2,x_2,x_2,x_2]^{\gamma_{43}} \times \\ & \times [[x_1,x_2,x_2],[x_1,x_2]]^{\gamma_{53}} [[x_1,x_2,x_1],[x_1,x_2]]^{\gamma_{63}} \end{aligned}$$

for some integers $\alpha_{13}$, $\alpha_{23}$, $\beta_{13}$, $\beta_{23}$, $\beta_{33}$, $\gamma_{i3}$ $(1 \leq i \leq 6)$.

Let now $w_4(x_1, x_2) = w_3(x_1, x_2) \cdot w_3(x_1^{-1}, x_2^{-1})$.

Clearly, $w_4(x_1, x_2) \geq e$ in o-group $G$ for all $x_1,\ x_2 \in G$, $w_4(\bar{x}_1, \bar{x}_2) > e$ and

$$\begin{aligned}&w_4(x_1,x_2) = [x_1,x_2,x_1,x_1]^{\beta_{14}}[x_1,x_2,x_2x_1]^{\beta_{24}}\times\\ &\times [x_1,x_2,x_2,x_2]^{\beta_{34}}[x_1,x_2,x_1,x_1,x_1]^{\gamma_{14}}\times\\ &\times [x_1,x_2,x_2,x_1,x_1]^{\gamma_{24}}[x_1,x_2,x_2,x_2,x_1]^{\gamma_{34}}[x_1,x_2,x_2,x_2,x_2]^{\gamma_{44}}\times\\ &\times \big[[x_1,x_2,x_2],[x_1,x_2]\big]^{\gamma_{54}}\big[[x_1,x_2,x_1],[x_1,x_2]\big]^{\gamma_{64}}\end{aligned}$$

for some integers $\beta_{14}$, $\beta_{24}$, $\beta_{34}$, $\gamma_{i4}$ $(1 \leq i \leq 6)$.

Similarly, let $w_5(x_1, x_2) = w_4(x_1, x_2) \cdot w_4(x_1^{-1}, x_2)$. Then $w_5(x_1, x_2) \geq e$ in o-group $G$ for all $x_1,\ x_2 \in G$, $w_5(\bar{x}_1, \bar{x}_2) > e$ and

$$\begin{aligned}&w_5(x_1,x_2) = [x_1,x_2,x_2x_1]^{\beta_{25}}[x_1,x_2,x_1,x_1,x_1]^{\gamma_{15}}[x_1,x_2,x_2,x_1,x_1]^{\gamma_{25}}\times\\ &\times [x_1,x_2,x_2,x_2,x_1]^{\gamma_{35}}[x_1,x_2,x_2,x_2,x_2]^{\gamma_{45}}\times\\ &\times \big[[x_1,x_2,x_2],[x_1,x_2]\big]^{\gamma_{55}}\big[[x_1,x_2,x_1],[x_1,x_2]\big]^{\gamma_{65}}\end{aligned}$$

for some integers $\beta_{25}$, $\gamma_{i5}$ $(1 \leq i \leq 6)$. Let $w_6(x_1, x_2) = w_5(x_1, x_2) \cdot w_5(x_2, x_1)$. Then $w_6(x_1, x_2) \geq e$ in o-group $G$ for all $x_1,\ x_2 \in G$ and $w_6(\bar{x}_1, \bar{x}_2) > e$. By the use of Hall's commutator identity [5] we have

$$\begin{aligned}&w_6(x_1,x_2) = [x_1,x_2,x_1,x_1,x_1]^{\gamma_{16}}[x_1,x_2,x_2,x_1,x_1]^{\gamma_{26}}\times\\ &\times [x_1,x_2,x_2,x_2,x_1]^{\gamma_{36}}[x_1,x_2,x_2,x_2,x_2]^{\gamma_{46}}\times\\ &\times \big[[x_1,x_2,x_2],[x_1,x_2]\big]^{\gamma_{56}}\big[[x_1,x_2,x_1],[x_1,x_2]\big]^{\gamma_{66}}\end{aligned}$$

for some integers $\gamma_{i6}$ $(1 \leq i \leq 6)$.

Finally, let $w_7(x_1, x_2) = w_6(x_1, x_2) \cdot w_6(x_1^{-1}, x_2^{-1})$. By the above arguments $w_7(\bar{x}_1, \bar{x}_2) > e$ and $w_7(x_1, x_2) \geq e$ in o-group $G$ for all $x_1,\ x_2 \in G$. Direct verification shows that $w_7(x_1, x_2) = e$ for all $x_1,\ x_2 \in G$, a contradiction. □

**Proposition 2.** *Let $G$ be a nilpotent o-group of nilpotent class $\leq 5$ and $w(x_1, x_2, x_3)$ be a group word in three variables $x_1, x_2, x_3$. If*

$$w(\bar{x}_1, \bar{x}_2, \bar{x}_3) > e$$

*in o-group $G$ for some $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3 \in G$ then there are $x_1'$, $x_2'$, $x_3' \in G$ such that $w(x_1', x_2', x_3') < e$.*

Proof. Assume that $w(x_1, x_2, x_3) \geq e$ for all $x_1,\ x_2,\ x_3 \in G$. The group word $w(x_1, x_2, x_3)$ can be represented as a product of basic commutators in variables $x_1$, $x_2$, $x_3$. By the use of Hall's commutator identities [5] we can represent the group word $w(x_1, x_2, x_3)$ as product $w_1(x_1, x_2) \cdot w_2(x_1, x_2, x_3)$, where $w_1(x_1, x_2,)$ is a product of basic commutators in variables $x_1$, $x_2$ and $w_2(x_1, x_2, x_3)$ is a product of basic commutators in variables $x_1$, $x_2$, $x_3$, containing variable $x_3$.

If $w_1(\bar{x}_1, \bar{x}_2) > e$ for some $\bar{x}_1$, $\bar{x}_2$, then by Proposition 1 there are $x_1'$, $x_2' \in G$ such that $w(x_1', x_2') < e$. Then $w(x_1', x_2', e) = w_1(x_1', x_2') < e$. Therefore, we may assume that $w(x_1, x_2) = e$ for all $x_1$, $x_2 \in G$ and $w(x_1, x_2, x_3) = w_2(x_1, x_2, x_3)$ for all $x_1$, $x_2$, $x_3 \in G$. By our assumption $w(\bar{x}_1, \bar{x}_2, \bar{x}_3) = w_2(\bar{x}_1, \bar{x}_2, \bar{x}_3) > e$. Similarly

$$w_2(x_1, x_2, x_3) = w_3(x_1, x_3) \cdot w_4(x_1, x_2, x_3)$$

where $w_3(x_1, x_3)$ is a product of basic commutators in variables $x_1$, $x_3$ and $w_4(x_1, x_2, x_3)$ is a product of basic commutators containing variables $x_2$, $x_3$.

By the above arguments we may suppose that $w_3(x_1, x_3) = e$ for all $x_1$, $x_3 \in G$. Therefore, $w(x_1, x_2, x_3) = w_4(x_1, x_2, x_3)$ for all $x_1$, $x_2$, $x_3 \in G$ and $w_4(x_1, x_2, x_3)$ is a product of basic commutators containing variables $x_2$, $x_3$. Let now $w_4(x_1, x_2, x_3) = w_5(x_1, x_2, x_3) \cdot w_6(x_1, x_2, x_3)$ where $w_5(x_1, x_2, x_3)$ is a product of basic commutators in variables $x_2$, $x_3$ and $w_6(x_1, x_2, x_3)$ is a product of basic commutators containing variables $x_1$, $x_2$, $x_3$. By the above arguments we may assume that $w(x_1, x_2, x_3) = w_6(x_1, x_2, x_3)$ for all $x_1$, $x_2$, $x_3 \in G$ and $w_6(\bar{x}_1, \bar{x}_2, \bar{x}_3) > e$. Then the group word

$$w_7(x_1, x_2, x_3) = w_6(x_1, x_2, x_3) \cdot w_6(x_1^{-1}, x_2^{-1}, x_3^{-1})$$

has the following properties:

1) $w_7(\bar{x}_1, \bar{x}_2, \bar{x}_3) > e$;

2) $w_7(x_1, x_2, x_3) \geq e$ for all $x_1, x_2, x_3 \in G$;

3) $w_7(x_1, x_2, x_3)$ is a product of basic commutators of the weight $\geq 4$

and each basic commutator contains variables $x_1$, $x_2$, $x_3$.

Let

$$w_8(x_1, x_2, x_3) = w_7(x_1, x_2, x_3) \cdot w_7(x_1^{-1}, x_2, x_3).$$

Then $w_8(\bar{x}_1, \bar{x}_2, \bar{x}_3) > e$, $w_8(x_1, x_2, x_3) \geq e$ for all $x_1, x_2, x_3 \in G$ and $w_8(x_1, x_2, x_3)$ is a product of basic commutators and each basic commutator of the weight 4 has two occurrences of the variable $x_1$. Now consider the group word

$$w_9(x_1, x_2, x_3) = w_8(x_1, x_2, x_3) \cdot w_8(x_1, x_2^{-1}, x_3).$$

It is clear that $w_9(\bar{x}_1, \bar{x}_2, \bar{x}_3) > e$, $w_9(x_1, x_2, x_3) \geq e$ for all $x_1$, $x_2$, $x_3 \in G$ and $w_9(x_1, x_2, x_3)$ is a product of basic commutators of the weight 5 containing variables $x_1$, $x_2$, $x_3$.

Therefore $w_{10}(x_1, x_2, x_3) = w_9(x_1, x_2, x_3) \cdot w_9(x_1^{-1}, x_2^{-1}, x_3^{-1}) \geq e$ for all $x_1$, $x_2$, $x_3 \in G$ and $w_{10}(\bar{x}_1, \bar{x}_2, \bar{x}_3) > e$. A contradiction to the equality $w_{10}(x_1, x_2, x_3) = e$ for all $x_1$, $x_2$, $x_3 \in G$. □

**Proposition 3.** *Let $G$ be a nilpotent o-group of nilpotent class $\leq 5$ and $w(x_1, x_2, x_3, x_4)$ is a group word in four variables $x_1$, $x_2$, $x_3$, $x_4$.*

*If* $w(\bar{x}_1,\bar{x}_2,\bar{x}_3,\bar{x}_4) > e$ *in o-group* $G$ *for some* $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$, $\bar{x}_4$ *then there are* $x'_1$, $x'_2$, $x'_3$, $x'_4$ *such that* $w(x'_1,x'_2,x'_3,x'_4) < e$.

Proof. Let us assume that $w(x_1,x_2,x_3,x_4) \geq e$ for all $x_1$, $x_2$, $x_3$, $x_4 \in G$ and

$$w(x_1,x_2,x_3,x_4) = w_1(x_1,x_2,x_3) \cdot w_2(x_1,x_2,x_3,x_4)$$

where $w_1(x_1,x_2,x_3)$ is a product of basic commutators in variables $x_1$, $x_2$, $x_3$, $x_4$ and $w_2(x_1,x_2,x_3,x_4)$ is a product of basic commutators in variables $x_1$, $x_2$, $x_3$, $x_4$ containing variable $x_4$. Arguments similar to the proof of Proposition 2 show that

$$w(x_1,x_2,x_3,x_4) = w_8(x_1,x_2,x_3,x_4)$$

for all $x_1$, $x_2$, $x_3$, $x_4 \in G$, where $w_8(x_1,x_2,x_3,x_4)$ is a product of basic commutators containing variables $x_1$, $x_2$, $x_3$, $x_4$. Let now

$$w_9(x_1,x_2,x_3,x_4) = w_8(x_1,x_2,x_3,x_4) \cdot w_8(x_1^{-1},x_2,x_3,x_4).$$

Then the word $w_9(x_1,x_2,x_3,x_4)$ is a product of basic commutators of the weight 5 containing variables $x_1$, $x_2$, $x_3$, $x_4$. It is evident, that

$$w_9(x_1,x_2,x_3,x_4) \geq e$$

for all $x_1$, $x_2$, $x_3$, $x_4 \in G$ and $w_9(\bar{x}_1,\bar{x}_2,\bar{x}_3,\bar{x}_4) > e$. Let

$$w_{10}(x_1,x_2,x_3,x_4) = w_9(x_1,x_2,x_3,x_4) \cdot w_9(x_1^{-1},x_2^{-1},x_3^{-1},x_4^{-1}).$$

Clearly, $w_{10}(x_1,x_2,x_3,x_4) = e$ for all $x_1$, $x_2$, $x_3$, $x_4 \in G$. A contradiction to the inequality $w_{10}(\bar{x}_1,\bar{x}_2,\bar{x}_3,\bar{x}_4) > e$. □

**THEOREM 1.** *Let* $G$ *be an arbitrary nilpotent o-group of nilpotent class* $\leq 5$ *and* $w(x_1,\dots,x_n)$ *be a group word in variables* $x_1,\dots,x_n$. *If*

$$w(\bar{x}_1,\dots,\bar{x}_n) > e$$

*in o-group* $G$ *for some* $\bar{x}_1,\dots,\bar{x}_n \in G$, *then there are* $x'_1,\dots,x'_n \in G$ *such that* $w(x'_1,\dots,x'_n) < e$.

Proof. By Propositions 1–3 we may assume that $n \geq 5$ and $w(x_1,\dots,x_n)$ is a product of basic commutators of weight $n \leq 5$ in variables $x_1,\dots,x_n$. The group word $w(x_1,\dots,x_n)$ can be represented in the form

$$w(x_1,\dots,x_n) = w_1(x_1,\dots,x_{n-1}) \cdot w_2(x_1,\dots,x_n),$$

where $w_1(x_1,\dots,x_{n-1})$ is a product of basic commutators in variables $x_1,\dots,x_{n-1}$ and $w_2(x_1,\dots,x_n)$ is a product of basic commutators containing variable $x_n$. If for some $\bar{x}_1,\dots,\bar{x}_{n-1} \in G$ is valid $w_1(\bar{x}_1,\dots,\bar{x}_{n-1}) > e$ in $G$, then by inductive arguments

$$w_1(x'_1,\dots,x'_{n-1}) < e, \qquad w(x'_1,\dots,x'_{n-1},x'_n) = w_1(x'_1,\dots,x'_{n-1},e) < e$$

in $G$ for some $x'_1, \dots, x'_{n-1}$. So we may assume that

$$w(x_1, \dots, x_n) = w_2(x_1, \dots, x_n)$$

for all $x_1, \dots, x_n \in G$. By these arguments we may assume that

$$w(x_1, \dots, x_n) = \hat{w}(x_1, \dots, x_n)$$

for all $x_1, \dots, x_n \in G$, where $\hat{w}(x_1, \dots, x_n)$ is a product of basic commutators of the weight 5 containing variables $x_{n-4}, \dots, x_n$ and $\hat{w}(\bar{x}_1, \dots, \bar{x}_n) > e$. Then

$$w^*(x_1, \dots, x_n) = \hat{w}(x_1, \dots, x_n)\, \hat{w}(x_1^{-1}, \dots, x_n^{-1}) = e\,,$$

a contradiction to inequalities

$$\hat{w}(\bar{x}_1, \dots, \bar{x}_n) > e, \qquad \hat{w}(\bar{x}_1^{-1}, \dots, \bar{x}_n^{-1}) \geq e\,.$$

□

REFERENCES

[1] *The Black Swamp Problem Book*, kept by Ch. Holland at Bowling Green State University.

[2] MEDVEDEV, N. Ya.: *On covers in the lattice of quasivarieties of l-groups*, Ordered Algebraic Structures (The 1991 Conrad conference), Kluwer Academic Publishers, Dordrecht, 1993.

[3] KOKORIN, A. I.—KOPYTOV, V. M.: *Fully Orderable Groups*, John Wiley and Sons, Halstead Press, New York – Toronto, 1974.

[4] ANDERSON, A.—FEIL, T.: *Lattice-Ordered Groups. An Introduction*, D. Reidel, Boston, 1988.

[5] HALL, M.: *The Theory of Groups*, MacMillan, New York, 1959.

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