TOPOLOGIES COMPATIBLE WITH ORDER.

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Dedicated to Professor J. Jakubík on the occasion of his 70th birthday

ABSTRACT. In paper [2] there were studied two kinds of compatibility of a
topology in an ordered set \((P; \leq)\) with the order relation \(\leq\). The present paper
deals with some other kind of such compatibility.

By an ordered set \((= o.\ set)\) we mean a partially ordered set. Two elements
\(a, b\) of an o. set \((P; \leq)\) are said to be incomparable, in notation, \(a \parallel b\), whenever
\(a \not\leq b\) and \(b \not\leq a\).

DEFINITION. (see [2]). Let \((P; \leq)\) be an o. set. A \(T_1\)-topology \(T\) in \(P\) is said
to be strongly compatible with the order relation \(\leq\), whenever for any \(a, b \in P\)
with \(a < b\) there exist neighborhoods \(U\) and \(V\) of \(a\) and \(b\), respectively, such
that \(x \in U, y \in V\) imply \(x < y\) or \(x \parallel y\).

Now we can prove

THEOREM 1. Let \((P; \leq)\) be an o. set. Then a \(T_1\)-topology \(T\) in \(P\) is strongly
compatible with the order relation \(\leq\) if and only if for any two convergent nets
\(\{x_\alpha : \alpha \in D\}, \{y_\alpha : \alpha \in D\}\) of \(P\) with \(x_\alpha \leq y_\alpha\) for every \(\alpha \in D\),

\[\lim x_\alpha \leq \lim y_\alpha\] or \[\lim x_\alpha \parallel \lim y_\alpha\]

holds true.

Proof. Let the topology \(T\) in \(P\) be strongly compatible with \(\leq\). Let
\(\{x_\alpha : \alpha \in D\}, \{y_\alpha : \alpha \in D\}\) be convergent nets in \(P\) such that \(x_\alpha \leq y_\alpha\) for
all \(\alpha \in D\). Denote by \(a = \lim x_\alpha\) and \(b = \lim y_\alpha\). Assume to the contrary that
\(a > b\). Then there exist neighborhoods \(U\) and \(V\) of \(a\) and \(b\), respectively, such
that \(u \in U\) and \(v \in V\) imply \(u > v\) or \(u \parallel v\). By assumption, there exists
\(\beta \in D\) such that \(x_\alpha \in U\) and \(y_\alpha \in V\) for all \(\alpha \geq \beta\). Since \(x_\alpha \leq y_\alpha\), we have a
contradiction. Thus \(a \leq b\) or \(a \parallel b\).

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Conversely, let \( \lim x_\alpha \leq \lim y_\alpha \) or \( \lim x_\alpha \parallel \lim y_\alpha \) for any two convergent nets \( \{x_\alpha : \alpha \in D\} \), \( \{y_\alpha : \alpha \in D\} \) with \( x_\alpha \leq y_\alpha \) for all \( \alpha \in D \) in \( P \). Assume to the contrary that \( T \) is not strongly compatible with \( \leq \). Then there exists \( a < b \) in \( P \) such that for any neighborhoods \( U \) and \( V \) of \( a \) and \( b \), respectively, there exist \( x_U \in U \) and \( y_V \in V \) such that \( x_U \geq y_V \). It is easy to construct a directed set \( D \) and nets \( \{x_\alpha : \alpha \in D\} \), \( \{y_\alpha : \alpha \in D\} \) such that \( x_U \in \{x_\alpha : \alpha \in D\} \), \( y_V \in \{y_\alpha : x \in D\} \), \( \lim x_\alpha = a \) and \( \lim y_\alpha = b \), a contradiction. \( \square \)

The last result leads us to the following

**Definition.** Let \( (P; \leq) \) be an o. set. A \( T_1 \)-topology \( T \) in \( P \) is said to be **extremely compatible with the order relation \( \leq \)**, whenever the following condition is fulfilled:

If \( \{x_\alpha : \alpha \in D\} \) and \( \{y_\alpha : \alpha \in D\} \) are convergent nets in \( P \) such that \( x_\alpha \leq y_\alpha \) for any \( \alpha \in D \), then \( \lim x_\alpha \leq \lim y_\alpha \).

The following example shows that there exist an o. set and an extremely compatible topology with the given order.

**Example.** (See [1, Chapt. X]). Consider a complete lattice \( (L; \leq) \) endowed with their order topology. Recall that for a net \( \{x_\alpha : \alpha \in D\} \) in \( L \), \( \lim x_\alpha = a \) in the order topology if and only if

\[
\liminf \{x_\alpha\} = \limsup \{x_\alpha\} = a,
\]

where \( \liminf \{x_\alpha\} = \sup_\beta \{\inf_\alpha \} \), \( \limsup \{x_\alpha\} = \inf_\beta \{\sup_\alpha \} \). Now, it is routine to verify that the order topology in \( L \) is extremely compatible with \( \leq \).

**Theorem 2.** Let \( (P; \leq) \) be an o. set and let a \( T_1 \)-topology \( T \) in \( P \) be given such that

(i) If \( a \parallel b \) in \( P \), then there exist neighborhoods \( U \) and \( V \) of \( a \) and \( b \), respectively, such that \( x \in U \) and \( y \in V \) imply \( x \parallel y \);

(ii) If \( a < b \) in \( P \), then there exist neighborhoods \( U \) and \( V \) of \( a \) and \( b \), respectively, such that \( x \in U \) and \( y \in V \) imply \( x < y \).

Then \( T \) is extremely compatible with the order relation \( \leq \).

**Proof.** Consider two convergent nets \( \{x_\alpha : \alpha \in D\} \) and \( \{y_\alpha : \alpha \in D\} \) in \( P \). Assume that \( x_\alpha \leq y_\alpha \) for every \( \alpha \in D \). Denote by \( a = \lim x_\alpha \) and \( b = \lim y_\alpha \). According to Theorem 1 we have \( a \leq b \) or \( a \parallel b \). Suppose that \( a \parallel b \). Then by (i) there exist neighborhoods \( U \) and \( V \) of \( a \) and \( b \), respectively, such that \( x \in U \) and \( y \in V \) imply \( x \parallel y \). Since \( a = \lim x_\alpha \) and \( b = \lim y_\alpha \), there exists \( \beta \in D \) such that \( x_\alpha \in U \) and \( y_\alpha \in V \) for all \( \alpha \geq \beta \). As \( x_\alpha \leq y_\alpha \), by assumption, we have come to a contradiction. Thus \( a \leq b \) and the proof is complete. \( \square \)
REFERENCES


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