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TOPOLOGIES COMPATIBLE WITH ORDER

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Dedicated to Professor J. Jakubík on the occasion of his 70th birthday

ABSTRACT. In paper [2] there were studied two kinds of compatibility of a topology in an ordered set $(P;\leq)$ with the order relation $\leq$. The present paper deals with some other kind of such compatibility.

By an ordered set (= o. set) we mean a partially ordered set. Two elements a, b of an o. set $(P;\leq)$ are said to be *incomparable*, in notation, $a \parallel b$, whenever $a \nleq b$ and $b \nleq a$.

DEFINITION. (see [2]). Let $(P;\leq)$ be an o. set. A T_1-topology T in P is said to be *strongly compatible with the order relation* $\leq$, whenever for any $a, b \in P$ with $a < b$ there exist neighborhoods U and V of a and b, respectively, such that $x \in U$, $y \in V$ imply $x < y$ or $x \parallel y$.

Now we can prove

THEOREM 1. *Let $(P;\leq)$ be an o. set. Then a T_1-topology T in P is strongly compatible with the order relation $\leq$ if and only if for any two convergent nets $\{x_\alpha\colon \alpha \in D\}$, $\{y_\alpha\colon \alpha \in D\}$ of P with $x_\alpha \leq y_\alpha$ for every $\alpha \in D$,*

$$\lim x_\alpha \leq \lim y_\alpha \qquad \text{or} \qquad \lim x_\alpha \parallel \lim y_\alpha$$

holds true.

Proof. Let the topology T in P be strongly compatible with $\leq$. Let $\{x_\alpha\colon \alpha \in D\}$, $\{y_\alpha\colon \alpha \in D\}$ be convergent nets in P such that $x_\alpha \leq y_\alpha$ for all $\alpha \in D$. Denote by $a = \lim x_\alpha$ and $b = \lim y_\alpha$. Assume to the contrary that $a > b$. Then there exist neighborhoods U and V of a and b, respectively, such that $u \in U$ and $v \in V$ imply $u > v$ or $u \parallel v$. By assumption, there exists $\beta \in D$ such that $x_\alpha \in U$ and $y_\alpha \in V$ for all $\alpha \geq \beta$. Since $x_\alpha \leq y_\alpha$, we have a contradiction. Thus $a \leq b$ or $a \parallel b$.

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Conversely, let $\lim x_\alpha \le \lim y_\alpha$ or $\lim x_\alpha \parallel \lim y_\alpha$ for any two convergent nets $\{x_\alpha : \alpha \in D\}$, $\{y_\alpha : \alpha \in D\}$ with $x_\alpha \le y_\alpha$ for all $\alpha \in D$ in P. Assume to the contrary that T is not strongly compatible with $\le$. Then there exists $a < b$ in P such that for any neighborhoods U and V of a and b, respectively, there exist $x_U \in U$ and $y_V \in V$ such that $x_U \ge y_V$. It is easy to construct a directed set D and nets $\{x_\alpha : \alpha \in D\}$, $\{y_\alpha : \alpha \in D\}$ such that $x_U \in \{x_\alpha : \alpha \in D\}$, $y_V \in \{y_\alpha : x \in D\}$, $\lim x_\alpha = a$ and $\lim y_\alpha = b$, a contradiction. □

The last result leads us to the following

DEFINITION. Let $(P; \le)$ be an o. set. A T_1-topology T in P is said to be *extremely compatible with the order relation* $\le$, whenever the following condition is fulfilled:

> If $\{x_\alpha : \alpha \in D\}$ and $\{y_\alpha : \alpha \in D\}$ are convergent nets in P such that $x_\alpha \le y_\alpha$ for any $\alpha \in D$, then $\lim x_\alpha \le \lim y_\alpha$.

The following example shows that there exist an o. set and an extremely compatible topology with the given order.

EXAMPLE. (See [1, Chapt. X]). Consider a complete lattice $(L; \le)$ endowed with their order topology. Recall that for a net $\{x_\alpha : \alpha \in D\}$ in L, $\lim x_\alpha = a$ in the order topology if and only if

$$\liminf\{x_\alpha\} = \limsup\{x_\alpha\} = a\,,$$

where $\liminf\{x_\alpha\} = \sup\limits_\beta\{\inf\limits_{\alpha \ge \beta} x_\alpha\}$, $\limsup\{x_\alpha\} = \inf\limits_\beta\{\sup\limits_{\alpha \ge \beta} x_\alpha\}$. Now, it is routine to verify that the order topology in L is extremely compatible with $\le$.

THEOREM 2. *Let $(P; \le)$ be an o. set and let a T_1-topology T in P be given such that*

(i) *If $a \parallel b$ in P, then there exist neighborhoods U and V of a and b, respectively, such that $x \in U$ and $y \in V$ imply $x \parallel y$;*
(ii) *If $a < b$ in P, then there exist neighborhoods U and V of a and b, respectively, such that $x \in U$ and $y \in V$ imply $x < y$.*

Then T is extremely compatible with the order relation $\le$.

Proof. Consider two convergent nets $\{x_\alpha : \alpha \in D\}$ and $\{y_\alpha : \alpha \in D\}$ in P. Assume that $x_\alpha \le y_\alpha$ for every $\alpha \in D$. Denote by $a = \lim x_\alpha$ and $b = \lim y_\alpha$. According to Theorem 1 we have $a \le b$ or $a \parallel b$. Suppose that $a \parallel b$. Then by (i) there exist neighborhoods U and V of a and b, respectively, such that $x \in U$ and $y \in V$ imply $x \parallel y$. Since $a = \lim x_\alpha$ and $b = \lim y_\alpha$, there exists $\beta \in D$ such that $x_\alpha \in U$ and $y_\alpha \in V$ for all $\alpha \ge \beta$. As $x_\alpha \le y_\alpha$, by assumption, we have come to a contradiction. Thus $a \le b$ and the proof is complete. □

REFERENCES

[1] BIRKHOFF, G.: *Lattice Theory, 3rd ed.*, American Mathematical Society, Providence, R. I., 1967.

[2] SEKANINA, A.—SEKANINA, M.: *Topologies compatible with ordering*, Arch. Math. (Brno) **2** (1966), 113–126.

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