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PRE–SOLID VARIETIES OF COMMUTATIVE SEMIGROUPS

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Dedicated to Professor J. Jakubík on the occasion of his 70th birthday

ABSTRACT. We determine all pre-solid varieties of commutative semigroups. Each of these varieties is contained in the pre-hyperequational class defined by the associative and the commutative law and every subvariety of this variety is pre-solid.

1. Introduction

An identity $t \approx t'$ of terms of any type τ is called a *pre-hyperidentity* for a universal algebra $\mathcal{A} = \left(A;\ (f_i^A)_{i\in I}\right)$ if $t \approx t'$ holds identically for every choice of n-ary term operations different from projections to represent n-ary operation symbols occurring in t and in t'. The concept of a pre-hyperidentity weakens that of a hyperidentity ([10]) where substitution of arbitrary n-ary term operations (including projections) for n-ary operation symbols must give identities.

An algebra or a variety for which every identity is a pre-hyperidentity is called *pre-solid.* Solid varieties are defined by the property that every identity is a hyperidentity ([9]). Since the commutative law cannot be a hyperidentity in any nontrivial variety, there is no solid nontrivial variety of commutative semigroups. The variety of all zero semigroups (defined by the identity $xy \approx zt$) is an example for a nontrivial pre-solid variety of commutative semigroups.

In this paper we determine the set of all pre-solid varieties of commutative semigroups. In [3] it was shown that a variety V is pre-solid if and only if it is a *pre-hyperequational class*, i.e., the class of all algebras of a given type τ satisfying all equations from a given set Σ as pre-hyperidentities. Using this result we will determine the greatest pre-solid variety V_{PC} of commutative semigroups. The subvariety lattice of V_{PC} agrees with the set of all pre-solid varieties of

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commutative semigroups. For each of these pre-solid varieties an identity basis is given.

In section 2 we set up the basic notation to be used throughout, and recall some preliminary results.

2. Basic concepts

We begin with a more precise definition of the concept of a pre-hyperidentity. Let us fix a type $\tau = (n_i)_{i \in I}$, $n_i > 0$ for all $i \in I$, and operation symbols $(f_i)_{i \in I}$, where f_i is n_i-ary. Let $W_\tau(X)$ be the set of all terms of type τ over some fixed alphabet X, and let $\mathrm{Alg}(\tau)$ be the class of all algebras of type τ.

A mapping

$$\sigma_p : \{f_i \mid i \in I\} \to W_\tau(X) \setminus X$$

which assigns to every n_i-ary operation symbol f_i an n_i-ary term different from a variable will be called a *pre-hypersubstitution of type* τ. (Note that we consider the first n_i variables $x_0, \dots, x_{n_i-1}$ of the standard alphabet $X = \{x_0, \dots x_{n_i-1}, \dots\}$ as n_i-ary terms). Recall that a mapping $\sigma : \{f_i \mid i \in I\} \to W_\tau(X)$ which assigns to every n_i-ary operation symbol f_i an arbitrary n_i-ary term is called a *hypersubstitution of type* τ.

Applying a pre-hypersubstitution σ_p (or a hypersubstitution σ) to a term $t \in W_\tau(X)$ we get a term $\hat{\sigma}_p[t]$ ($\hat{\sigma}[t]$) which can be defined inductively by:

(i) $\hat{\sigma}_p[x] := x$ for any variable x in the alphabet X, and

(ii) $\hat{\sigma}_p\big[f_i(t_1, \cdots, t_{n_i})\big] := \sigma_p(f_i)^{\mathcal{W}_\tau(X)}\big(\hat{\sigma}_p[t_1], \cdots, \hat{\sigma}_p[t_{n_i}]\big)$.

It is clear that $\sigma_p(f_i)^{\mathcal{W}_\tau(X)}$ on the right hand side of (ii) is the operation induced by the term $\sigma_p(f_i)$ on the term algebra $\mathcal{W}_\tau(X)$.

If $t \approx t'$ is an equation, then we denote by $\Xi_p[t \approx t']$ the set

$$\big\{\hat{\sigma_p}[t] \approx \hat{\sigma}_p[t'] \mid \sigma_p \text{ is a pre-hypersubstitution}\big\}.$$

If Σ is a set of equations, we use $\Xi_p[\Sigma]$ for the union of the sets $\Xi_p[t \approx t']$, for $t \approx t'$ in Σ. In the same way we define $\Xi[\Sigma]$.

Let $\mathcal{A} = (A;\ (f_i^A)_{i \in I})$ be an algebra in $\mathrm{Alg}(\tau)$, let $K \subseteq \mathrm{Alg}(\tau)$, and let σ_p be a pre-hypersubstitution. Then we make the following definitions:

$$\sigma_p[\mathcal{A}] := \big(A;\ (\sigma_p(f_i)^A)_{i \in I}\big),$$
$$\Xi_p[\mathcal{A}] := \big\{\sigma_p[\mathcal{A}] \mid \sigma_p \text{ is a pre-hypersubstitution of type } \tau\big\},$$
$$\Xi_p[K] := \bigcup_{\mathcal{A} \in K} \Xi_p[\mathcal{A}].$$

Using hyperidentities one defines $\Xi[K]$. For the operators Ξ and Ξ_p we have the following properties:

PROPOSITION 2.1([3]). *Let K be a class of algebras of type τ and let Σ be a class of identities of type τ. Then*

(i) $\Sigma \subseteq \Xi_p[\Sigma]$,

(i) $\Sigma' \subseteq \Sigma \Rightarrow \Xi_p[\Sigma'] \subseteq \Xi_p[\Sigma]$,

(iii) $\Xi_p[\Xi_p[\Sigma]] = \Xi_p[\Sigma]$,

(iv) $K \subseteq \Xi_p[K]$,

(v) $K' \subseteq K \Rightarrow \Xi_p[K'] \subseteq \Xi_p[K]$,

(vi) $\Xi_p[\Sigma] \subseteq \Xi[\Sigma]$,

(vii) $\Xi_p[\Xi_p[K]] = \Xi_p[K]$.

(ii) $\Xi_p[K] \subseteq \Xi[K]$.

Now we can give a more precise definition of a hyperidentity and of a pre-hyperidentity:

DEFINITION 2.2. Let t, t' be terms of a type τ. Then the identity $t \approx t'$ is called a *hyperidentity* (a *pre-hyperidentity*) in a variety V of type τ if $\hat{\sigma}[t] \approx \hat{\sigma}[t']$ $\big(\hat{\sigma}_p[t] \approx \hat{\sigma}_p[t']\big)$ is an identity in V for every hypersubstitution (pre-hypersubstitution).

Clearly, every hyperidentity of type τ is a pre-hyperidentity of this type. In general, the converse is false.

For a class K of algebras of type τ and for a set Σ of identities of this type we fix the following notations:

$\operatorname{Id} K$ — the class of all identities of K,

$H \operatorname{Id} K$ — the class of all hyperidentities of K,

$H_p \operatorname{Id} K$ — the class of all pre-hyperidentities of K,

$\operatorname{Mod} \Sigma = \{\mathcal{A} \in \operatorname{Alg}(\tau) \mid \mathcal{A} \text{ satisfies } \Sigma\}$ — the variety defined by Σ,

$H \operatorname{Mod} \Sigma = \{\mathcal{A} \in \operatorname{Alg}(\tau) \mid \mathcal{A} \text{ hypersatisfies } \Sigma\}$ — the hyperequational class defined by Σ,

$H_p \operatorname{Mod} \Sigma = \{\mathcal{A} \in \operatorname{Alg}(\tau) \mid \mathcal{A} \text{ pre-hypersatisfies } \Sigma\}$ — the pre-hyperequational class defined by Σ.

For hyperequational classes and pre-hyperequational classes we have the inclusion $H \operatorname{Mod} \Sigma \subseteq H_p \operatorname{Mod} \Sigma$.

By definition, every hyperidentity or every pre-hyperidentity is an identity. Very naturally there arises a problem of finding algebras or varieties for which every identity is either a hyperidentity or a pre-hyperidentity. We define:

DEFINITION 2.3. Let V be a variety of type τ. Then V is called *solid* if $\Xi[V] = V$. The variety V is called *pre-solid* if $\Xi_p[V] = V$.

In [3] for pre-solid varieties we obtained the following characterization:

THEOREM 2.4. *Let $K \subseteq \mathrm{Alg}(\tau)$ be a variety. Then the following conditions are equivalent:*

(i) *K is a pre-hyperequational class,*

(ii) *K is pre-solid,*

(iii) $\mathrm{Id}\, K \subseteq H_p \mathrm{Id}\, K$, *i.e., every identity of K is a pre-hyperidentity,*

(iv) $\Xi_p[\mathrm{Id}\, K] = \mathrm{Id}\, K$, *i.e.,* $\mathrm{Id}\, K$ *is closed under pre-hypersubstitutions.*

Let $\mathcal{L}(\tau)$ be the lattice of all varieties of type τ, let $S(\tau)$ be the set of all solid varieties of type τ and let $S_p(\tau)$ be the set of all pre-solid varieties of this type. In [8] it was shown that for any finite type the least nontrivial solid variety is the variety RA_τ of all rectangular algebras of this type (that is the variety generated by all projection algebras of type τ.) If $\tau = (2)$, then RA_τ agrees with the variety RB of rectangular bands.

Then we have the following results:

PROPOSITION 2.5. ([9], [3]):

(i) *The set $S(\tau)$ forms a complete sublattice of $\mathcal{L}(\tau)$ with RA_τ as unique atom.*

(ii) *$S_p(\tau)$ forms a complete sublattice of $\mathcal{L}(\tau)$ containing $S(\tau)$ as a sublattice.*

(iii) *Let V be a variety of type $\tau = (2)$ such that $RB \subseteq V$. Then V is solid if and only if V is pre-solid.*

For semigroups we described in [3] the greatest solid variety. This variety is the hyperequational class defined by the associative law:

$$H\,\mathrm{Mod}\,\{F(F(x,y),z) \approx F(x,F(y,z))\}\,.$$

An equational basis of this solid variety is given by the following set of identities ([7]):

$$\{x(yz) \approx (xy)z,\ x^2 \approx x^4,\ xyxzxyx \approx xyzyx,$$
$$xy^2z^2 \approx xyz^2yz^2,\ x^2y^2z \approx x^2yx^2yz\}$$

In [3] we proved the following proposition:

PROPOSITION 2.6. *The variety V_{HS} is pre-solid and for any pre-solid variety V of semigroups, $V \subseteq V_{HS}$.*

3. The class of all pre-solid varieties of commutative semigroups

The commutative law fails to be a hyperidentity in any nontrivial variety since by substitution of one of the binary projections for the binary operation symbol we get the identity $x \approx y$. We will determine all pre-solid varieties of commutative semigroups. At first we will determine the greatest of them. By Theorem 2.4 this variety is the pre-hyperequational class defined by the associative and the commutative law. We will give an equational basis for this variety.

THEOREM 3.1. *Every pre-solid variety of commutative semigroups is included in*

$$\begin{aligned} &H_p \operatorname{Mod}\{F(F(x,y),z) \approx F(x,F(y,z)),\ F(x,y) \approx F(y,x)\} \\ &= \operatorname{Mod}\{(xy)z \approx x(yz),\ xy \approx yx,\ xy^2 \approx x^2y,\ x^2 \approx y^2\} =: V_{PC}\,. \end{aligned}$$

Proof. If V is a pre-solid variety of commutative semigroups, then the commutative law and the associative law must be pre-hyperidentities in V. We have to show the proposition concerning the equational basis. Substituting the binary term $t_1(x,y) = x^2$ in the hyperidentity $F(x,y) \approx F(y,x)$, we get $x^2 \approx y^2$. With $t_2(x,y) = xy$ we obtain $xy \approx yx$ and $t_3(x,y) = xy^2$ leads to $xy^2 \approx x^2y$. The associative hyperidentity gives the associative identity. This shows that $V \subseteq V_{PC}$.

Now assume that V is a subvariety of V_{PC}. By Proposition 2.6 $V \subseteq V_{HS}$ and the associative law is a pre-hyperidentity satisfied in V. We show that the commutative law is a pre-hyperidentity in V_{PC}, too. Firstly, from the identities $x^2 \approx y^2$ and $x^2y \approx xy^2$ we get the identity $x^2y \approx x^2y^2$ and then $x^2y \approx x^2$. Let now $t(x,y)$ be a binary term different from a variable. Substituting $t(x,y)$ in the pre-hypercommutative law and using the identities $xy \approx yx, x^2y \approx xy^2$ and $x^2y \approx x^2$ we obtain

a) $t(x,y) \approx x^2 \approx y^2 \approx t(y,x)$, or

b) $t(x,y) \approx xy \approx yx \approx t(y,x)$.

Therefore the commutative law is a pre-hyperidentity and

$$V \subseteq H_p \operatorname{Mod}\{F(F(x,y),z) \approx F(x,F(y,z)), F(x,y) \approx F(y,x)\}.$$

□

To give more examples of commutative pre-solid semigroup varieties we fix the following notations:

$p_n : x_0x_1 \dots x_n \approx y_0y_1 \dots y_n$,

$I_n = \{(xy)z \approx x(yz)\,,\ xy \approx yx\,,\ xy^2 \approx x^2y,\ x^2 \approx y^2,\ p_n\}$,

$P_n = \mathrm{Mod}(I_n)$ for every natural number n.

Further we set $P = \{p_n \mid n \in \mathbb{N}\}$.

Then we have:

LEMMA 3.2. *For every $n \in \mathbb{N}$ the variety P_n is pre-solid.*

Proof. For every natural number $n \in \mathbb{N}$ the variety P_n is included in the pre-solid variety V_{PC}. Therefore, the commutative and the associative law are pre-hyperidenties in P_n for every n.

Now let us check the equations

$$F\big(F(\dots F(x_0,x_1),\dots,x_{i-1}),x_i\big) \approx F\big(F(\dots F(F(y_0,y_1),y_2)\,,\dots,y_{i-1}),y_i\big) \quad (1)$$

for every $i \in \mathbb{N}$. Substituting a binary term $t(x,y)$ different from a variable in (1) and using the identity $x_0 \dots x_i \approx y_0 \dots y_i$ we obtain that there exist terms $u_0,\dots,u_i,v_0,\dots,v_i$ such that

$$\begin{aligned} &t\big(t(\dots(t(x_0,x_1),\dots),x_{i-1}),x_i\big) = u_0u_1 \dots u_i \approx v_0v_1 \dots v_i = \\ &= t\big(t(\dots(t(y_0,y_1),\dots),y_{i-1}),y_i\big)\,. \end{aligned}$$

It is easy to see that $F\big(x,F(y,y)\big) \approx F\big(F(x,x),y\big)$ and $F(x,x) \approx F(y,y)$ are pre-hyperidentities. That means $P_n \subseteq H_p\,\mathrm{Mod}\{F\big(F(x,F(y,z)\big) \approx F\big(F(x,y),z)\big)$, $F(x,y) \approx F(y,x)$, (1), $F(x,x) \approx F(y,y)$, $F\big(x,F(y,y)\big) \approx F\big(F(x,x),y\big)\}$. Clearly, $H_p\,\mathrm{Mod}\{F\big(F(x,F(y,z)\big) \approx (F(x,y),z)\big)$, $F(x,y) \approx F(y,x)$, (1), $F(x,x) \approx F(y,y)$, $F\big(x,F(y,y)\big) \approx F\big(F(x,x),y\big)\} \subseteq P_n$. □

Each of the varieties P_n, $n \in \mathbb{N}$, is included in V_{PC}. Now we will show that $\{P_n \mid n \in \mathbb{N}\} \cup \{V_{PC}\}$ are all commutative pre-solid varieties of semigroups. Indeed, we can show that V_{PC} has no subvarieties different from P_n, $n \in \mathbb{N}$, and V_{PC}.

THEOREM 3.3. *The set of all commutative pre-solid semigroup varieties forms a lattice, namely, the subvariety lattice of V_{PC}.*

Proof. In V_{PC} the identity $x^3 \approx x^2$ can be derived. Therefore, every commutative pre-solid variety of semigroups is included in the variety $\mathrm{Mod}\{x(yz) \approx (xy)z,\ xy \approx yx,\ x^3 \approx x^2\}$. All commutative subvarieties of this variety are given in [5]. Using this result we see that $\{P_n \mid n \in \mathbb{N}\} \cup \{V_{PC}\}$ are all commutative pre-solid varieties of semigroups and that this is the subvariety lattice of V_{PC}. □

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