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RECENT RESULTS ON NULL–ADDITIVE SET FUNCTIONS

ENDRE PAP

ABSTRACT. Lebesgue decomposition type theorem is proved for null-σ-additive set functions. Daniell–Greco–Stone type representation theorem of monotone and autocontinuous from above functional by Choquet integral with respect to special null-additive set function is presented.

1. Introduction

A wide class of non-additive set functions known as null-additive set functions includes many important fuzzy measures (as submeasures [7], capacities, belief function, possibility measure, decomposable measures [13], [14], [29], k-triangular set functions [15], [16], etc.). These set functions were earlier investigated in mathematics by I. Dobrakov [6] and L. Drewnowski [7]. The relation with fuzzy measures and the name were given by Z. Wang [27]. Many interesting properties for these set functions were proved in [24], [17], [18], [19], [21]. The books of Wang and Klir [28] and Pap [22] contain a lot of properties and applications (for example in statistics, decision theory, game theory, functional analysis, potential theory, nonlinear differential equations) of null-additive set functions.

We shall first give some general definitions and properties of null-additive set functions.

V. Ficker's and P. Capek's algebraic approach by ideals enables us to obtain a Lebesgue decomposition theorem for a wide class of non-additive set functions called null-additive set functions ([4]). In special cases decomposition theorems for $\oplus$-decomposable measures and k-triangular set functions are obtained.

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We shall present a Daniell–Greco–Stone type representation theorem of a monotone and autocontinuous from above functional by Choquet integral with respect to special null-additive set function.

2. Null-additive set functions

Let $\mathcal{L}$ be a ring of subsets of the given set X such that $\emptyset \in \mathcal{L}$.

Definition 1. A set function m, $m\colon \mathcal{L} \to [0,\infty]$ with $m(\emptyset) = 0$ is called *null-additive*, if we have

$$m(A \cup B) = m(A)\,,$$

whenever $A, B \in \mathcal{L}$, $A \cap B = \emptyset$ and $m(B) = 0$.

For properties of null-additive set functions see E. Pap [17], [18], [22], H. Suzuki [24] and Z. Wang [27], Z. Wang and G. Klir [28].

Let $\mathcal{R}$ be a σ-ring of subsets of the given set X. We have introduced in the paper [4] the following generalization of the notion of the countable additive set function.

Definition 2. A set function $m\colon \mathcal{R} \to [0,\infty]$ is *null-σ-additive* if for any sequence (B_i) of pairwise disjoint sets from $\mathcal{R}$ such that $A \cap B_i = \emptyset$ and $m(B_i) = 0$ we have

$$m\Big(A \cup \bigcup_{i=1}^{\infty} B_i\Big) = m(A)\,.$$

We have proved in [4] the following two propositions.

Proposition 1. *m is null-σ-additive iff m is null-additive and $m(B_i) = 0$ $(i \in \mathbb{N})$ implies $m(\bigcup_{i=1}^{\infty} B_i) = 0$ for a sequence (B_i) of pairwise disjoint sets from $\mathcal{R}$.*

Proposition 2. *If m is null-additive and continuous from below, i.e., $A_1 \subset A_2 \subset \ldots$ imply*

$$\lim_{n\to\infty} m(A_n) = m\Big(\bigcup_{n=1}^{\infty} A_n\Big)\,,$$

then it is null-σ-additive.

Example 1. Let $\mathcal{S}$ be a *t-conorm*, i.e., a binary operation on $[0,1]$ such that it is associative commutative and non-decreasing with a neutral element 0. A set function $m\colon \mathcal{R} \to [0,1]$ is called *$\mathcal{S}$-decomposable measure* if

$$m(\emptyset) = 0$$

and

$$m(A \cup B) = m(A) \mathcal{S} m(B),$$

whenever $A, B \in \mathcal{R}$ and $A \cap B = \emptyset$.

The set function m is a monotone null-additive set function.

EXAMPLE 2. ($\oplus$-decomposable measure). The operation $\oplus$ (*pseudo-addition*) is a function $\oplus \colon [0,\infty] \times [0,\infty] \to [0,\infty]$ which is commutative, nondecreasing, associative continuous and has a zero element 0.

A set function $m \colon \mathcal{R} \to [0,\infty]$ is a *$\oplus$-decomposable measure* if

$$m(\emptyset) = 0,$$
$$m(A \cup B) = m(A) \oplus m(B),$$

whenever $A, B \in \mathcal{R}$ and $A \cap B = \emptyset$. It is obvious that m is a null-additive set function. There are further generalizations for operations defined on $[a,b] \subset [-\infty,+\infty]$.

EXAMPLE 3. (k-triangular set function). A set function $m \colon \mathcal{R} \to [0,\infty)$ is said to be *k-triangular* for $k \geq 1$ if $m(\emptyset) = 0$ and

$$m(A) - km(B) \leq m(A \cup B) \leq m(A) + km(B),$$

whenever $A, B \in \mathcal{R}$, $A \cap B = \emptyset$. It is obvious that k-triangular set function is always null-additive, although it may not be monotone. Special 1-triangular set functions are submeasures.

DEFINITION 3. A set function m is called *autocontinuous from above* (resp. *from below*) if for every $\varepsilon > 0$ and every $A \in \mathcal{R}$, there exists $\delta = \delta(A,\varepsilon) > 0$ such that

$$m(A) - \varepsilon \leq m(A \cup B) \leq m(A) + \varepsilon$$
$$(\text{resp. } m(A) - \varepsilon \leq m(A \setminus B) \leq m(A) + \varepsilon),$$

whenever $B \in \mathcal{R}$, $A \cap B = \emptyset$ (resp. $B \subset A$) and $m(B) < \delta$ holds.

3. Lebesgue decomposition

DEFINITION 4. Let $\mu \colon \mathcal{R} \to [0,+\infty)$ be a set function and $\mu(\emptyset) = 0$. We shall say that μ is *order continuous* if for each sequence (E_n) such that $E_n \searrow \emptyset$ we have

$$\lim \mu\left(E_n \bigcap Y\right) = 0$$

uniformly with respect to $Y \in \mathcal{R}$.

DEFINITION 5. Let m and v be two finite set functions defined on $\mathcal{R}$. The set function m is called *singular with respect to* v if there exists a set A from $\mathcal{R}$ such that

$$m(E\backslash A) = v(E \cap A) = 0 \quad (E \in \mathcal{R}).$$

THEOREM 1. *Let m be a null-additive set function, and μ a finite null-σ-additive set function. If μ is exhaustive, then there exist two null-additive set functions m_c and m_s, such that*

$$m_c(E) = m(E\backslash A) \quad \text{and} \quad m_s(E) = m(E \cap A)$$

for a set $A \in \mathcal{N}(\mu)$, where

$$\mathcal{N}(\mu) = \{A \in \mathcal{R} \colon B \subset A \Rightarrow \mu(B) = 0, \ \forall B \in \mathcal{R}\}$$

and m_c is absolutely continuous with respect to μ and m_s is singular with respect to μ.

In the proof of Theorem 1 we shall also need the following.

THEOREM 2. *If $m \colon \mathcal{R} \to [0,\infty]$ is null-additive set function then the set*

$$\mathcal{N}(m) = \{A \in \mathcal{R} \colon \quad B \subset A \Rightarrow m(B) = 0, \quad \forall B \in \mathcal{R}\}$$

is an ideal in $\mathcal{R}$.

Proof. By the definition, for $B \in \mathcal{R}$ and $A \in \mathcal{N}(m)$ such that $B \subset A$ we have $m(B) = 0$, i.e., $B \in \mathcal{N}(m)$.

For A_1, $A_2 \in \mathcal{N}(m)$ and for arbitrary but fixed subset B of $A_1 \cup A_2$ which belongs to $\mathcal{R}$ we have $B\backslash A_1$ and $B \cap A_1 \in \mathcal{N}(m)$, and so

$$m(B\backslash A_1) = m(B \cap A_1) = 0.$$

Hence, since $B\backslash A_1$ and $B \cap A_1$ are disjoint sets and m is null-additive,

$$m(B) = m((B\backslash A_1) \cup (B \cap A_1)) = 0,$$

i.e., $A_1 \cup A_2 \in \mathcal{N}(m)$. □

COROLLARY 1. *If m is null-σ-additive then the set*

$$\mathcal{N}(m) = \{A \in \mathcal{R} \colon \quad B \subset A \Rightarrow m(B) = 0, \quad \forall B \in \mathcal{R}\}$$

is a σ-ideal of $\mathcal{R}$.

Proof. Consequence of Theorem 2 and Proposition 1. □

In the proof of Theorem 1 we shall need the following consequence of Theorem 3.3 from [26].

LEMMA 1. *Let m be a group valued function on $\mathcal{R}$. If M is a σ-complete ideal of $\mathcal{R}$ such that $M \backslash \mathcal{N}(m)$ satisfies C.C.C., then there exists $A \in M$ such that $m^A(E) := m(E \cap A)$ is M–continuous and $m_A(E) = m(E \backslash A)$ is M–singular.*

Note that C.C.C. means the *countable chain condition.* Further, m is called *M–continuous* if $M \subset \mathcal{N}(m)$ and m is called *M–singular* if $\mathcal{N}(m)$ is M–singular, i.e., if there exists

$$A \in M_* = \{E \in \mathcal{R}\colon\ E \cap B \in M \quad \text{for all} \quad B \in \mathcal{R}\}$$

such that $\{B \in \mathcal{R}\colon B \backslash A \in M\} = \mathcal{R}$.

We shall also need the following lemma from [4].

LEMMA 2. *If μ is a finite exhaustive set function and if $\{E_\gamma\colon \gamma \in C\}$ is a disjoint family of subsets of X then the index set*

$$C_+ = \{\gamma \in C\colon \mu(E_\gamma) > 0\}$$

is at most countable, i.e., μ satisfies C.C.C.

P r o o f o f T h e o r e m 1. Apply Lemma 1 on $M = \mathcal{N}(\mu)$. Namely, by Corollary 1, $\mathcal{N}(\mu)$ is a σ-ideal of $\mathcal{R}$ and by Lemma 2 μ satisfies C.C.C. □

In [4] the following theorem has been proved.

THEOREM 3. *Let the suppositions be the same as in the previous theorem. Then m_c and m_s are independent of the choice of set $A \in \mathcal{R}$.*

Furthermore, if $m = m_c \oplus m_s$ holds, where $\oplus$ is a pseudo-addition, then this is the unique decomposition of m by means of the sum of two null-additive functions where one is absolutely continuous with respect to μ and the other is singular with respect to μ.

4. Integral and functionals

In this section we shall present a representation theorem of a functional with Chouqet integral.

Let $\mathcal{F}$ be a family of functions $f\colon \mathcal{X} \to [0,\infty]$ with the properties:

$$af,\ f \wedge a,\ f - f \wedge a \in \mathcal{F} \quad (f \in \mathcal{F},\ a \in [0,\infty)) \quad (\textit{Stone condition}),$$

$$f \wedge g,\ f \vee g \in \mathcal{F} \quad (f, g \in \mathcal{F}) \quad (\textit{Lattice condition}).$$

The family of all upper level sets of the function f is denoted by $\mathcal{U}_f$, i.e.,

$$\mathcal{U}_f := \{\{x\colon f(x) > t\}\colon t \in [0,\infty]\} \cup \{\{x\colon f(x) \ge t\}\colon t \in [0,\infty]\}.$$

A class of functions $\mathcal{F}_o \subset \mathcal{F}$ is *comonotonic* (*common monotonic*) if

$$\bigcup_{f \in \mathcal{F}_o} \mathcal{U}_f$$

is a chain. A class of functions $\mathcal{F}_o$ is comonotonic iff each pair of functions from $\mathcal{F}_o$ is comonotonic. The equivalent condition for a pair of functions f and g to be comonotonic is that there is no pair $x_1,\ x_2 \in X$ such that $f(x_1) < f(x_2)$ and $g(x_1) > g(x_2)$.

Let $m\colon \mathcal{P}(X) \to [0,\infty]$ be a monotone set function with the property $m(\emptyset)=0$ and let $f\colon X \to [0,\infty]$ be a function, then $m(\{x\colon f(x) > t\})$ is a decreasing function on $[0,\infty]$. For a monotone set function $m\colon \mathcal{L} \to [0,\infty]$ and an upper m-measurable function $f\colon X \to [0,\infty]$ the *Choquet integral* is defined by

$$(\mathcal{C}) \int f\, dm := \int_0^\infty m(\{x\colon f(x) > t\})\, dt\,.$$

Let M be a functional $M\colon \mathcal{F} \to [0,\infty]$. We list the following properties of M for every $f, g \in \mathcal{F}$ and every $a \in [0,\infty)$:

(F1) $f \le g \;\Rightarrow\; M(f) \le M(g)$ (*monotonicity*);

(F2) $M(f+g) = M(f)+M(g)$ for comonotonic f and g such that $f+g \in \mathcal{F}$ (*comonotonic additivity*);

(F3) $\lim_{a\downarrow 0} M(f - f\wedge a) = M(f)$ (*lower marginal continuity*);

(F4) $\lim_{a\to\infty} M(f\wedge a) = M(f)$ (*upper marginal continuity*);

(F5) $(\forall\varepsilon > 0)(\exists\delta > 0)(\forall g \in \mathcal{F})(\forall f \in \mathcal{F})$
$\big(M(g) < \delta \;\Rightarrow\; M(f\vee g) < M(f) + \varepsilon\big)$ (*autocontinuity from above*);

(F6) $M(f\vee g) = M(f)$ whenever $M(g) = 0$ (*null-additivity*);

(F7) $M(f\vee g) + M(f\wedge g) \le M(f) + M(g)$ (*submodularity*);

(F8) $M(f\vee g) + M(f\wedge g) \ge M(f) + M(g)$ (*supermodularity*).

Definition 6. For a given class of functions $\mathcal{F}$, $f\colon X \to [0,\infty]$, a monotone set function $m\colon \mathcal{P}(X) \to [0,\infty]$ and a functional $M\colon \mathcal{F} \to [0,\infty]$ we say that *m represents M* if we have

$$M(f) = (\mathcal{C}) \int f\, dm \quad (f \in \mathcal{F})\,.$$

In [21] we have proved the following representation theorem.

THEOREM 4. *If M is a functional $M\colon \mathcal{F} \to [0,\infty]$ with the properties $(F1)$–$(F5)$, then there exists a monotone autocontinuous from above set function $m\colon \mathcal{P}(X) \to [0,\infty]$ which represents M.*

Many interesting results on representation theorems of non-additive functionals by integrals can be found in the papers of B. Anger [1], R. C. Bassanezi and G. H. Greco [2], G. H. Greco [8], in the book of D. Dennenberg [5] and in the paper of D. Schmeidler [23]. Further results on comonotonicity and Choquet–like integrals can be find in the papers of J. Šipoš [25] and R. Mesiar [12].

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Institute of Mathematics
University of Novi Sad
Trg Dositeja Obradovića 4
YU–21000 Novi Sad
YUGOSLAVIA