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# THE CONNECTION BETWEEN INTERPOLATION IN VAGUE ENVIRONMENTS AND FUZZY CONTROL

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ABSTRACT. This paper emphasizes the concept of indistinguishability or similarity as a possible underlying semantics for fuzzy sets. The correspondence between equality or similarity relations and fuzzy sets is elucidated. As an application we examine interpolation in vague environments characterized by equality relations. As a result we rediscover the max-min rule as the appropriate inference mechanism. The consequences of this result for fuzzy control are discussed.

## 1. Introduction

Fuzzy sets are often used to model linguistic terms like *small, medium, big,...* , especially in fuzzy control applications. The membership grade of a certain element is intuitively interpreted as the degree to which this element fits the concept represented by the linguistic term associated with the fuzzy set.

In this paper we propose the notion of vague environment that is based on the concept of similarity or indistinguishability. As it is shown in Section 2, similarity or equality relations can be defined by the very simple idea of scaling factors or functions that represent the indistinguishability. Section 3 motivates how fuzzy sets are induced by crisp elements in vague environments. In Section 4 we see that a vague environment can also be derived from a given fuzzy partition so that we obtain a duality between vague environments and fuzzy sets. Section 5 is devoted to interpolation in vague environments. The main result yields that the max-min rule commonly applied in fuzzy control can be justified in the view of interpolation in vague environments.

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## 2. Indistinguishability and similarity

Equality, similarity, and indistinguishability are concepts that describe a relation between objects. In classical logic the following three axioms are imposed on a relation $\approx$ if it is intended to model one of these concepts.

(i) $x \approx x$ (reflexivity),

(ii) $x \approx y \leftrightarrow y \approx x$ (symmetry),

(iii) $x \approx y \wedge y \approx z \rightarrow x \approx z$ (transitivity).

These are the well known axioms for an equivalence relation. Unfortunately, there are phenomena in the real world that can be classified as a kind of similarity or indistinguishability, but do not obey these restrictions. Consider for example the indistinguishability induced by a measuring instrument that has a precision of 0.6. It is neither possible to distinguish the value 0.9 nor 1.9 from 1.4 with this instrument. But the values 0.9 and 1.9 can be distinguished. The relation $R$, containing the pairs of indistinguishable values, i.e.,

$$R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid |x - y| \leq 0.6\}$$

is reflexive and symmetric, but not transitive, so that it is impossible to build appropriate equivalence classes. Taking the transitive hull of $R$ leads to the indiscrete relation, where any two values are identified.

Giving up the concept of having only two truth values, it is possible to overcome these problems. Equality, similarity, and indistinguishability are therefore considered as a matter of degree. For reasons of simplicity let us restrict to the unit interval as the set of truth values. In order to formulate adequate requirements for such a fuzzified concept of equivalence, we have to assign truth functions to the logical connectives $\rightarrow$, $\wedge$, and $\leftrightarrow$. The truth value of a formula $\varphi$ is denoted by $[\![\varphi]\!]$. We assume that $\wedge$ is associated with a lower semi-continuous t-norm $T$, i.e., we have

$$[\![\varphi \wedge \psi]\!] = T([\![\varphi]\!], [\![\psi]\!]) .$$

For valuating the implication we choose residuation, which leads to

$$[\![\varphi \rightarrow \psi]\!] = \vec{T}([\![\varphi]\!], [\![\psi]\!]) = \sup\{\alpha \mid T([\![\varphi]\!], \alpha) \leq [\![\psi]\!]\} .$$

For the biimplication we define accordingly

$$\begin{aligned}[\![\varphi \leftrightarrow \psi]\!] = \overleftrightarrow{T}([\![\varphi]\!], [\![\psi]\!]) &= [\![(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)]\!] = \\ &= \vec{T}(\max\{[\![\varphi]\!], [\![\psi]\!]\}, \min\{[\![\varphi]\!], [\![\psi]\!]\}) .\end{aligned}$$

Using these notions we can now introduce the concept of an equality relation. The idea is to allow any number between 0 and 1 as the truth value for the expression $x \approx y$. Thus an equality relation corresponds to a binary predicate such that the truth valuation of each of the three axioms for equivalence relations yields the value 1. An equality relation $\approx$ is characterized by the truth function

$$E_{\approx}\colon X \times X \to [0,1]\,, \qquad (x,y) \mapsto [\![x \approx y]\!]\,,$$

where $X$ is the underlying domain. It is easy to check that $\approx$ satisfies the axioms for an equivalence relation if and only if

(E1) $E_{\approx}(x,x) = 1$,

(E2) $E_{\approx}(x,y) = E_{\approx}(y,x)$,

(E3) $T\big(E_{\approx}(x,y),\, E_{\approx}(y,z)\big) \le E_{\approx}(x,z)$

hold. This motivates the following definition.

**Definition 2.1.** An *equality relation* on a set $X$ is a mapping $E\colon X \times X \to [0,1]$ fulfilling the axioms (E1), (E2), and (E3).

Equality relations are also called *similarity relations* [16] or *indistinguishability* operators [15]. Generalizations of this definition of equality relations can be found in [4, 5].

Example 2.2. Let $T$ be the Łukasiewicz t-norm, i.e., $T(\alpha,\beta) = \max\{\alpha+\beta-1, 0\}$. A (pseudo-)metric $\delta$ on $X$ induces an equality relation on $X$ by

$$E_{\delta}(x,y) = 1 - \min\big\{\delta(x,y), 1\big\}\,.$$

On the other hand, an equality relation determines a (pseudo-)metric by

$$\delta_E(x,y) = 1 - E(x,y)\,.$$

For $X = \mathbb{R}$ and $\delta(x,y) = |x-y|$ we obtain the canonical equality relation

$$E_{|\cdot|}(x,y) = 1 - \min\big\{|x-y|, 1\big\}\,,$$

that solves the problem for our measurement instrument with precision 0.6. We have $E(1.4, 0.9) = E(1.4, 1.9) = 0.5$, but $E(0.9, 1.9) = 0$.

Example 2.2 shows that (pseudo-)metrics bounded by 1 and equality relations with respect to the Łukasiewicz t-norm are dual concepts.

Example 2.3. Let $X = [0,1]^{X_0} = \big\{\mu\colon X_0 \to [0,1]\big\}$ be the set of fuzzy sets on $X_0$. The formula

$$\mu \equiv \nu \Leftrightarrow (\forall x \in X_0)(x \in \mu \leftrightarrow x \in \nu)$$

induces an equality relation on $X_0$ by

$$[\![\mu \equiv \nu]\!] = \big[\!\big[(\forall x \in X_0)(x \in \mu \leftrightarrow x \in \nu)\big]\!\big] = \inf_{x \in X_0} \big\{\overset{\leftrightarrow}{T}(\mu(x),\, \nu(x))\big\}\,.$$

This equality relation was used in [2] to define approximate solutions of fuzzy relational equations.

Choosing the Łukasiewicz t-norm, we obtain

$$[\![\mu \equiv \nu]\!] = 1 - \sup_{x \in X_0} \{|\mu(x) - \nu(x)|\}.$$

The minimum yields

$$[\![\mu \equiv \nu]\!] = \inf \{ \min\{\mu(x), \nu(x)\} \mid \mu(x) \neq \nu(x)\}.$$

## 3. From equality relations to fuzzy sets

The notion of an equality relation was introduced in the previous section. Accepting such an idea of fuzzy equality, we have to make sure that other concepts behave well with respect to a given equality relation.

As an example consider the concept of membership. For any set $M$

$$x \in M \wedge x = y \rightarrow y \in M$$

is obviously satisfied.

**Definition 3.1.** Let $E$ be an equality relation on $X$ with respect to the t-norm $T$. A fuzzy set $\mu \colon X \rightarrow [0,1]$ is *extensional* if

$$T(\mu(x), E(x,y)) \leq \mu(y)$$

holds for all $x, y \in X$.

Note that a fuzzy set is extensional if and only if

$$[\![x \in \mu \wedge x \approx y \rightarrow y \in \mu]\!] = 1.$$

**Definition 3.2.** Let $E$ be an equality relation on $X$ with respect to the t-norm $T$ and let $\mu \colon X \rightarrow [0,1]$ be a fuzzy set on $X$. The *extensional hull* of $\mu$ is the fuzzy set

$$\hat{\mu} \colon X \rightarrow [0,1], \qquad x \mapsto \sup_{y \in X} \{T(\mu(y), E(x,y))\}.$$

It is easy to prove that $\widehat{\mu}$ is the smallest extensional fuzzy set containing $\mu$. As a special case we can compute extensional hulls of ordinary sets. With a set $M \subseteq X$ we associate its characteristic function $\chi_M$ taking the value 1 for $x \in M$ and 0 otherwise. We abbreviate the extensional hull of $M$ or $\chi_M$, respectively, by

$$\mu_M(x) = \hat{\chi}_M(x) = \sup_{y \in M} \{E(x,y)\}.$$

If $M$ contains only one element, say $M = \{x_0\}$, we write

$$\mu_{x_0}(x) = \mu_{\{x_0\}}(x) = E(x_0, x).$$

Note that in this way crisp elements and crisp sets induce fuzzy sets as representations of crisp data in a *vague environment* that is characterized by an equality relation.

EXAMPLE 3.3. Let us recall Example 2.2, where we considered $X = \mathbb{R}$ and the equality relation $E_{|\cdot|}(x,y) = 1 - \min\{|x-y|, 1\}$ with respect to the Łukasiewicz t-norm. We observe that the extensional hull of the value $x_0 \in \mathbb{R}$ corresponds to a triangular membership function whereas we obtain a trapezoidal membership function as the extensional hull of the interval $[a,b]$ (see Figures 1 and 2).

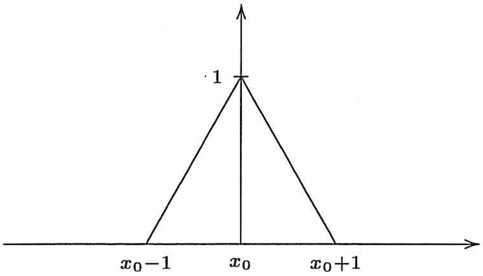


FIGURE 1. The extensional hull of $x_0$.

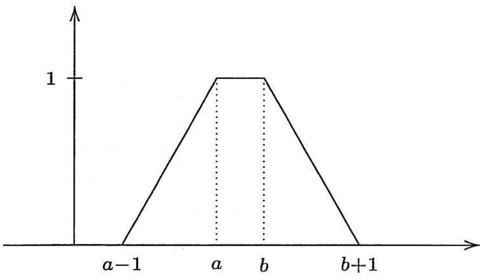


FIGURE 2. The extensional hull of the interval $[a,b]$.

For this equality relation these fuzzy sets can also be interpreted as representing the corresponding crisp sets with respect to different tolerance bounds. The $\alpha$-cut of $\mu_{x_0}$ contains exactly the values whose distance to $x_0$ is not greater than $1-\alpha$. For $\mu_{[a,b]}$ the $\alpha$-cut contains all elements whose distance is not greater than $1-\alpha$ to at least one of the elements of $[a,b]$, or more precisely, all elements $x$ with $\inf_{m \in M} \{|x-m|\} \le 1-\alpha$. A detailed discussion of this interpretation of fuzzy sets can be found in [8].

Example 3.3 shows how triangular and trapezoidal membership functions with slope 1 can be seen as representations of a single value or an interval in the vague environment that is characterized by the equality relation induced by the standard metric on $\mathbb{R}$.

Choosing other equality relations it is possible to obtain other shapes of membership functions for fuzzy sets representing crisp values or intervals in the corresponding vague environment. Using the metric $\delta_c(x,y) = |c\cdot x - c\cdot y|$ for the equality relation, triangular and trapezoidal membership functions with slope $c$ are obtained. The scaling factor $c$ describes how strong the distinguishability is. One can generalize this concept of a scaling factor to scaling functions $c\colon X \to [0,\infty[$ that assign an individual scaling factor $c(x)$ to each value $x \in \mathbb{R}$. $c(x)$ determines how strong the distinguishability is in the neighbourhood of $x$. The corresponding metric induced by such a scaling function is

$$\delta_c(x,y) = \left| \int_x^y c(s)\,ds \right| .$$

Using such transformation functions, arbitrary fuzzy sets $\mu$ on $\mathbb{R}$ satisfying the conditions

(C1) there exists $x_0 \in \mathbb{R}$ such that $\mu(x_0) = 1$,
(C2) $\mu$ is a non-decreasing function on $]-\infty, x_0]$,
(C3) $\mu$ is a non-increasing function on $[x_0, \infty[$,
(C4) $\mu$ is continuous,
(C5) $\mu$ is almost everywhere differentiable,

can be obtained as representations of crisp values in the corresponding vague environment [8]. Jacas and Recasens [7] considered arbitrary transformation functions, i.e., any monotonous transformation function $t\colon \mathbb{R} \to \mathbb{R}$ inducing the metric $\delta^{(t)}(x,y) = |t(x) - t(y)|$. They proved that any fuzzy set fulfilling $(C1)$–$(C3)$ can be seen as a representation of a crisp value in the vague environment characterized by an equality relation that is the infimum of equality relations induced by monotonous transformation functions (with respect to the Łukasiewicz t-norm). Without restrictions for the equality relation arbitrary fuzzy sets can be generated.

Finally, let us remark that a set of crisp values in a vague environment induces a family of fuzzy set which can be seen as a fuzzy partition.

## 4. From fuzzy partitions to equality relations

In the previous section we have seen that a fuzzy set can be induced by a crisp value in vague environment that is characterized by an equality relation. A canonical question that turns up is whether the fuzzy sets of a fuzzy partition can be seen as representations of crisp values in an appropriate vague environment. The following theorem [6, 13] answers this question for the most general case.

**THEOREM 4.1.** *Let $T$ be a lower semi-continuous t-norm. Let $(\mu_i)_{i\in I}$ be a non-empty family of fuzzy sets on $X$ and let $(x_i)_{i\in I}$ be a family of elements of*

*$X$ such that $\mu_i(x_i) = 1$ holds for all $i \in I$. The following two statements are equivalent.*

(i) *There exists an equality relation (with respect to $T$) on $X$ such that $\mu_i = \mu_{x_i}$ holds for all $i \in I$.*

(ii) *For all $i, j \in I$ the inequality*

$$\sup_{x\in X} \{T(\mu_i(x), \mu_j(x))\} \le \inf_{y\in X} \{\overset{\leftrightarrow}{T}(\mu_i(y), \mu_j(y))\} \tag{1}$$

*is satisfied.* □

If condition (ii) of Theorem 1 is fulfilled,

$$E(x,y) = \inf_{i\in I} \{\overset{\leftrightarrow}{T}(\mu_i(x), \mu_i(y))\}$$

is an appropriate equality relation and $E$ is even the greatest (coarsest) equality relation for which $\mu_i = \mu_{x_i}$ holds.

(1) can be rewritten as

$$[\![(\exists x)(x \in \mu_i \wedge x \in \mu_j) \to (\forall y)(y \in \mu_i \leftrightarrow y \in \mu_j)]\!] = 1\,,$$

or, more conveniently,

$$[\![\neg(\mu_i \cap \mu_j \equiv \emptyset) \to \mu_i \equiv \mu_j]\!] = 1\,,$$

where the intersection of $\mu_i$ and $\mu_j$ is computed with respect to the t-norm $T$. This is a very canonical condition since it is exactly what is required from a family of ordinary sets to be a partition. Although this is a very appealing condition, it might be difficult to check it. The following theorem states that the disjointness of the fuzzy sets is a sufficient condition.

**THEOREM 4.2.** *Let $T$ be a lower semi-continuous t-norm. Let $(\mu_i)_{i\in I}$ be a non-empty family of fuzzy sets on $X$ and let $(x_i)_{i\in I}$ be a family of elements of $X$ such that $\mu_i(x_i) = 1$ holds for all $i \in I$. If*

$$T(\mu_i(x), \mu_j(x)) = 0 \tag{2}$$

*holds for all $x \in X$ and all $i \ne j$, then condition* (ii) *and therefore also condition* (i) *of theorem 4.1 is satisfied.* □

For the Łukasiewicz t-norm (2) is equivalent to

$$\mu_i(x) + \mu_j(x) \le 1\,,$$

a condition that is satisfied for many fuzzy partitions used in fuzzy control.

Theorems 4.1 and 4.2 did not put any restriction on the equality relation. When the equality relation (with respect to the Łukasiewicz t-norm) is required to be induced by a scaling function, additional assumptions for the fuzzy partition are necessary [8].

**THEOREM 4.3.** *Let $(\mu_i)_{i\in I}$ be an at most countable family of fuzzy sets on $\mathbb{R}$ and let $(x_0^{(i)})_{i\in I}$ be a family of real numbers such that $\mu_i(x_0^{(i)}) = 1$ holds and the conditions $(C1)$–$(C5)$ are satisfied for all $i \in I$. There exists a scaling function $c\colon \mathbb{R} \to [0,\infty[$ such that $\mu_i$ coincides with the fuzzy set $\mu_{x_0^{(i)}}$ (for each $i \in I$), which represents the value $x_0^{(i)}$ in the vague environment induced by $c$, if and only if*

$$\min\{\mu_i(x),\, \mu_j(x)\} > 0 \Rightarrow |\mu_i'(x)| = |\mu_j'(x)| \tag{3}$$

*holds almost everywhere for all $i$, $j \in I$.*

For the proof choose

$$c\colon \mathbb{R} \to [0,\infty[, \qquad x \mapsto \begin{cases} |\mu_i'(x)|, & \text{if there exists } i \in I \text{ with } \mu_i(x) > 0, \\ 0, & \text{otherwise}, \end{cases}$$

as the scaling function. □

## 5. Interpolation in vague environments

The previous sections have explained the connections between fuzzy sets and equality relations. In this section we apply the notions developed in the previous sections to interpolation and rediscover the max-min rule.

Let us consider $n$ "input" domains $X_1, \ldots, X_n$ and one "output" domain $Y$. We assume that each of these domains is endowed with an equality relation $E_1, \ldots, E_n$, $F$, respectively. On the product space we define the equality relation

$$E\big((x_1,\ldots,x_n,y),\,(\tilde{x}_1,\ldots,\tilde{x}_n,\tilde{y})\big) = \min\big\{E_1(x_1,\tilde{x}_1),\ldots,E_n(x_n,\tilde{x}_n),\,F(y,\tilde{y})\big\}.$$

$E$ is the greatest equality relation on $X_1 \times \cdots \times X_n \times Y$ satisfying

$$E\big((x_1,\ldots,x_n,y),\,(\tilde{x}_1,\ldots,\tilde{x}_n,\tilde{y})\big) \le E_i(x_i,\tilde{x}_i)$$

and

$$E\big((x_1,\ldots,x_n,y),\,(\tilde{x}_1,\ldots,\tilde{x}_n,\tilde{y})\big) \le F(y,\tilde{y}).$$

Consider a mapping $\varphi\colon X_1 \times \cdots \times X_n \to Y$. Taking the equality relation $E$ into account, we can represent the graph of $\varphi$ in the product space $X_1 \times \cdots \times X_n \times Y$ by its extensional hull

$$\mu_\varphi\colon X_1 \times \cdots \times X_n \times Y \to [0,1],$$

$$(x_1,\ldots,x_n,y) \mapsto \sup_{\varphi(\tilde{x}_1,\ldots,\tilde{x}_n)=\tilde{y}} \big\{E\big((x_1,\ldots,x_n,y),\,(\tilde{x}_1,\ldots,\tilde{x}_n,\tilde{y})\big)\big\}.$$

For a given input $(x_1,\ldots,x_n) \in X_1 \times \cdots \times X_n$ we can compute the corresponding output taking the vague environment into account by projecting the intersection of the cylindrical extension of $(x_1, \dots, x_n)$ and the extensional hull of the graph of $\varphi$ to $Y$, i.e.,

$$\mu^{\text{output}}_{x_1,\dots,x_n} : Y \to [0,1]\,, \qquad y \mapsto \mu_\varphi(y)\,.$$

Note that we obtain the same result, when we first compute the extensional hull of $(x_1, \dots, x_n)$ and project its intersection with $\mu_\varphi$ to $Y$.

Now let us assume that we do not know the mapping $\varphi$ everywhere, but only for the inputs $(x_1^{(k)}, \dots, x_n^{(k)})$ for $k = 1, \dots, r$, say $\varphi(x_1^{(k)}, \dots, x_n^{(k)}) = y^{(k)}$, i.e., we are given a partial mapping $\varphi_0$ that coincides with $\varphi$ where $\varphi_0$ is defined. Without taking the vague environments into account, we can only make a pure guess how to extend $\varphi_0$ to obtain $\varphi$. But the vague environments enable us to gain information about $\varphi$ in the neighbourhood of the points for which $\varphi_0$ is defined. We can exploit the extensional hull of the graph of $\varphi_0$ to obtain an output fuzzy set for any input in the same way as we did it for $\varphi$. This leads to the output fuzzy set

$$\begin{aligned}\mu^{\text{output}}_{x_1,\dots,x_n}(y) &= \max_{k\in\{1,\dots,r\}} \big\{E\big((x_1^{(k)},\dots,x_n^{(k)},y^{(k)}),\ (x_1,\dots,x_n,y)\big)\big\} \\ &= \max_{k\in\{1,\dots,r\}} \big\{\min\{E_1(x_1^{(k)},x_1),\dots,E_n(x_n^{(k)},x_n),\ F(y^{(k)},y)\}\big\} \\ &= \max_{k\in\{1,\dots,r\}} \big\{\min\{\mu_{x_1^{(k)}}(x_1),\dots,\mu_{x_n^{(k)}}(x_n),\ \mu_{y^{(k)}}(y)\}\big\}\,.\end{aligned} \tag{4}$$

We observe that (4) is exactly the output fuzzy set obtained from the max-min rule for the linguistic rules.

*If* $\xi_1$ *is approximately* $x_1^{(k)}$ *and … and* $\xi_n$ *is approximately* $x_n^{(k)}$, *then* $\eta$ *is approximately* $y^{(k)}$, $\quad (k = 1, \dots, r)$,

where the linguistic term *approximately* $z$ is associated with the fuzzy set $\mu_z$ representing the value $z$ in the corresponding vague environment. In this way we have rediscovered the max-min rule in the context of interpolation in vague environments.

Fuzzy control is a typical field where this rule is applied. Thus we can translate our approach of interpolation in vague environments to a standard fuzzy controller and can make use of standard hard- and software tools. However, the basic parameters in our approach differ from those of common fuzzy control. We need a characterization of vague environments in the form of equality relations, which can for instance be defined by appropriate scaling functions, and a specification of a partial (control) mapping. These concepts are intuitively appealing and have a clear interpretation.

We have seen that we can always translate a fuzzy controller developed on the basis of interpolation in vague environments to a standard fuzzy controller. Applying the results of section 4, we can in most cases also reformulate a fuzzy controller using the max-min rule as interpolation in vague environments. From the fuzzy partitions we can derive the corresponding equality relations and we can associate with each fuzzy set the crisp value which it represents in the vague environment. These values together with the rule base determine the partial control mapping. Of course, the fuzzy partitions must satisfy the preliminaries for one of the theorems in section 4. Respecting these constraints seems to be reasonable and we can provide a clear interpretation by the translations to interpolation in vague environments.

Our approach may also elucidate certain heuristic requirements for fuzzy control. The remark after Theorem 4.2 explains why neighbouring membership functions of a fuzzy partition meet at hight 0.5. We can also explain, why it is reasonable to use fuzzy partitions in which the support of each fuzzy set is chosen in such a way that it exactly covers the range between the points where its neighbouring fuzzy sets reach their maximum. For triangular membership functions this always guarantees the existence of a corresponding scaling function. This strategy also implies that the fuzzy sets are more dense in ranges where they have smaller supports. Since smaller supports induce greater scaling factors, i.e., higher distinguishability, this leads to having more points for interpolation where the distinguishability is high.

## 6. Conclusions

This paper was devoted to vague environments characterized by equality relations. We have seen that crisp elements and sets in vague environments induce fuzzy sets. On the other hand it is possible to derive under very general constraints an equality relation from a fuzzy partition so that the fuzzy sets represent crisp values in the vague environment. In this sense, fuzzy sets and equality relations are dual concepts. We do not claim that fuzzy sets should always be interpreted on the basis of equality relations, since there are other semantics for fuzzy sets using probabilistic notions [3] or seeing them as possibility distributions [1]. However, especially in fuzzy control the concept of indistinguishability is very appealing. Within our approach we do not only have a correspondence between fuzzy sets and equality relation, but we can also motivate concepts like the max-min rule.

Our approach to fuzzy control as interpolation in vague environments does enforce certain reasonable constraints on the choice of the fuzzy partitions and the rule base of a fuzzy controller. Taking these constraints into account, when designing or tuning a fuzzy controller, can simplify these tasks. We have only discussed some aspects of these constraints. More detailed investigations in this direction, especially with respect to defuzzification can be found in [11, 12, 13].

[8] examines the connection between fuzzy control and equality relations induced by scaling functions. An application of our model to idle speed control of a Volkswagen GTI is described in [10].

At first sight, fuzzy control does not fit into a formal logical setting in spite of the name fuzzy logic control, since interpreting the rules as logical implications would lead to different computations [9]. On the basis of equality relations it is possible to embed fuzzy control in an appropriate first order fuzzy theory [14].

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