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PERIODICITY OF MATRICES AND ORBITS IN FUZZY ALGEBRA

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ABSTRACT. Periodicity of vector orbits in the fuzzy algebra is studied. Relations between the period of a matrix $\mathbb{A}$ and the periods of the orbits of all vectors with respect to $\mathbb{A}$ are described.

1. Introduction

Fuzzy matrix operations are useful for expressing basic properties of fuzzy relations. The convergence and periodicity in special classes of matrices were studied by M. G. Thomason [10], and subsequently by many other authors. Li Jian-Xin [8], [9] considered the periodicity of fuzzy matrices in the general case and gave an upper estimate for the period of a matrix.

The convergence of the power sequence of a square matrix in fuzzy algebra was studied, by means of digraphs, by K. Cechlárová [3]. Other connections between eigenvectors of a fuzzy matrix and its associated digraphs were described in [4].

It was proved in [6] that the period $\operatorname{per}(\mathbb{A})$ of a matrix $\mathbb{A}$ is equal to the least common multiple of the periods of all non-trivial strongly connected components in all threshold digraphs of $\mathbb{A}$ and to the least common multiple of the orbit periods $\operatorname{per}(\mathbb{A}, x)$ for all vectors $x \in \mathcal{B}(n)$. An algorithm was suggested which enables to compute the period in $O(n^3)$ time.

In this paper the conditions are considered under which, for a given matrix $\mathbb{A}$, there is a vector x such that $\operatorname{per}(\mathbb{A}) = \operatorname{per}(\mathbb{A}, x)$.

2. Notions and notation

In this section we define the notions mentioned informally in the introduction. For simpler notation of index sets we shall use the convention by which any

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natural number n is considered as the set of all smaller natural numbers, i.e., $n = \{0, 1, \dots, n-1\}$. By $\mathbb{N}$, $\mathbb{N}^+$ we denote the set of all non-negative integers and the set of all positive integers, respectively. The greatest common divisor and the least common multiple of a set $S \subseteq \mathbb{N}$ are denoted by the abbreviations $\gcd S$ and $\operatorname{lcm} S$, respectively.

If $\mathcal{G} = (V, E)$ is a digraph (directed graph), then by *strongly connected component* of $\mathcal{G}$ we mean a subdigraph $\mathcal{K} = (K, E \cap K^2)$ generated by a non-empty subset $K \subseteq V$ such that any two vertices $x, y \in K$ are contained in a common cycle, and K is a maximal subset with this property. $\mathcal{K}$ is called *trivial*, if K contains only one vertex x, and there is no loop from x to x in $\mathcal{K}$. If $\mathcal{G}_1 = (V_1, E_1)$, $\mathcal{G}_2 = (V_2, E_2)$ are digraphs, we say that $\mathcal{G}_1$ is a *subdigraph* of $\mathcal{G}_2$ (in notation: $\mathcal{G}_1 \subseteq \mathcal{G}_2$), if $V_1 \subseteq V_2$ and $E_1 \subseteq E_2$. The intersection of digraphs $\mathcal{G}_1$, $\mathcal{G}_2$ is a digraph $\mathcal{G}_1 \cap \mathcal{G}_2 = (V_1 \cap V_2, E_1 \cap E_2)$; we say that $\mathcal{G}_1 \cap G_2 = \emptyset$ if $V_1 \cap V_2 = \emptyset$ (and $E_1 \cap E_2 = \emptyset$).

DEFINITION 2.1. The *fuzzy algebra* $\mathcal{B}$ is a triple $(\mathcal{B}, \oplus, \otimes)$, where $\mathcal{B}$ is a linearly ordered set and $\oplus, \otimes$ are the binary operations of maximum and minimum, respectively, on $\mathcal{B}$. For any natural $n > 0$, $\mathcal{B}(n)$ denotes the set of all n-dimensional column vectors over $\mathcal{B}$, and $\mathcal{B}(n, n)$ denotes the set of all square matrices of order n over $\mathcal{B}$. The matrix operations over $\mathcal{B}$ are defined formally in the same manner (with respect to $\oplus$, $\otimes$) as the matrix operations over any field.

DEFINITION 2.2. Let $\mathbf{A} \in \mathcal{B}(n, n)$, $h \in \mathcal{B}$.

(i) The *threshold digraph* $\mathcal{G}(\mathbf{A}, h)$ is the digraph $\mathcal{G} = (n, E)$, with the vertex set $n = \{0, 1, \dots, n-1\}$ and with the arc set $E = \{(i, j);\ i, j \in n,\ a_{ij} \geq h\}$.

(ii) For any natural r and for any $i, j \in n$, we denote by $W_{\mathcal{G}}^{(r)}(i, j)$ the set of all walks in $\mathcal{G}$, of length r, beginning in i and ending in j.

(iii) For any natural r and for $I, J \subseteq n$, we denote

$$W_{\mathcal{G}}^{(r)}(I, J) := \bigcup \{ W_{\mathcal{G}}^{(r)}(i, j) ;\ i \in I,\ j \in J \},$$

$$W_{\mathcal{G}}(I, J) := \bigcup \{ W_{\mathcal{G}}^{(r)}(I, J) ;\ r \in \mathbb{N} \}.$$

Several authors studied the graph properties using power sequences of associated matrices (R. A. Cuninghame-Green [5], M. Gondran and M. Minoux [7], U. Zimmermann [12]). Conversely, powers of a matrix over $(\mathcal{B}, \oplus, \otimes)$ can be characterized by walks in the corresponding threshold graphs. The following formulation is due to K. Cechlárová [3].

Lemma 2.1. *Let* $\mathbf{A} \in \mathcal{B}(n,n)$, $\mathbf{x} \in \mathcal{B}(n)$, $h \in \mathcal{B}$, $r \in \mathbb{N}^+$, $i,j \in n$. *Then*

(i) $(\mathbf{A}^r)_{ij} \geq h \iff W^{(r)}_{\mathcal{G}(\mathbf{A},h)}(i,j) \neq \emptyset$,

(ii) $(\mathbf{A}^r x)_i \geq h \iff (\exists j \in n)\big[x_j \geq h,\ W^{(r)}_{\mathcal{G}(\mathbf{A},h)}(i,j) \neq \emptyset\big]$.

Proof. By induction on r. □

3. Matrix periods

The necessary definitions and facts from [6] concerning matrix periods in fuzzy algebra are presented in this section.

Definition 3.1. Let $\mathbf{A} \in \mathcal{B}(n,n)$, $h \in \mathcal{B}$. We define

$$\operatorname{Per}(\mathbf{A}) := \big\{p \in \mathbb{N}^+;\ (\exists R)(\forall r > R)\ \mathbf{A}^r = \mathbf{A}^{r+p}\big\},$$
$$\operatorname{per}(\mathbf{A}) := \min \operatorname{Per}(\mathbf{A}).$$

The number $\operatorname{per}(\mathbf{A})$ is called the *period* of the matrix $\mathbf{A}$.

Remark 3.1. By the linearity of $\mathcal{B}$, any element of any power of the matrix $\mathbf{A}$ is equal to some element of $\mathbf{A}$. Therefore, the sequence of powers of $\mathbf{A}$ contains only finitely many different matrices. As a consequence, the set $\operatorname{Per}(\mathbf{A})$ is always non-empty and the period of $\mathbf{A}$ is well-defined.

Definition 3.2. Let $\mathbf{A} \in \mathcal{B}(n,n)$, $\mathbf{x} \in \mathcal{B}(n)$. We define by recursion

$$\mathbf{x}^{(0)} := \mathbf{x}, \qquad \text{if } r = 0,$$
$$\mathbf{x}^{(r)} := \mathbf{A}\mathbf{x}^{r-1}, \qquad \text{if } r > 0.$$

The sequence $(\mathbf{x}^{(r)}; r \in \mathbb{N})$ is called the *orbit* of $\mathbf{x}$ (with respect to $\mathbf{A}$).

Definition 3.3. Let $\mathbf{A} \in \mathcal{B}(n,n)$, $\mathbf{x} \in \mathcal{B}(n)$. We define

$$\operatorname{Per}(\mathbf{A},\mathbf{x}) := \big\{p \in \mathbb{N}^+;\ (\exists R)(\forall r > R)\ \mathbf{x}^{(r)} = \mathbf{x}^{(r+p)}\big\},$$
$$\operatorname{per}(\mathbf{A},\mathbf{x}) := \min \operatorname{Per}(\mathbf{A},\mathbf{x}).$$

The number $\operatorname{per}(\mathbf{A},\mathbf{x})$ is called the *orbit period* of $\mathbf{x}$ (with respect to $\mathbf{A}$).

The following lemma is based on the properties of divisibility of natural numbers. The denotation $(\mathbb{N}, |)$ is used for the set of all natural numbers partially ordered by the relation of divisibility.

Lemma 3.1. *Let* $\mathbf{A} \in \mathcal{B}(n,n)$, $\mathbf{x} \in \mathcal{B}(n)$. *Then*

(i) $\operatorname{Per}(\mathbf{A})$ *is the principal filter in* $(\mathbb{N}, |)$ *generated by* $\operatorname{per}(\mathbf{A})$,

(ii) $\operatorname{Per}(\mathbf{A}, \mathbf{x})$ *is the principal filter in* $(\mathbb{N}, |)$ *generated by* $\operatorname{per}(\mathbf{A}, \mathbf{x})$,

(iii) $\operatorname{Per}(\mathbf{A}) = \bigcap_{\mathbf{x} \in \mathcal{B}(n)} \operatorname{Per}(\mathbf{A}, \mathbf{x})$,

(iv) $\operatorname{per}(\mathbf{A}) = \operatorname{lcm}_{\mathbf{x} \in \mathcal{B}(n)} \operatorname{Per}(\mathbf{A}, \mathbf{x})$.

Definition 3.4. Let $\mathbf{A} \in \mathcal{B}(n,n)$, $\mathbf{x} \in \mathcal{B}(n)$, $i \in n$. We define

$$\begin{aligned}
&\operatorname{Per}(\mathbf{A}, \mathbf{x}, i) := \{p \in \mathbb{N}^+;\ (\exists R)(\forall r > R)\ x_i^{(r)} = x_i^{(r+p)}\}, \\
&\operatorname{per}(\mathbf{A}, \mathbf{x}, i) := \min \operatorname{Per}(\mathbf{A}, \mathbf{x}, i).
\end{aligned}$$

Lemma 3.2. *Let* $\mathbf{A} \in \mathcal{B}(n,n)$, $\mathbf{x} \in \mathcal{B}(n)$. *Then*

(i) $\operatorname{Per}(\mathbf{A}, \mathbf{x}, i)$ *is the principal filter in* $(\mathbb{N}, |)$ *generated by* $\operatorname{per}(\mathbf{A}, \mathbf{x}, i)$,

(ii) $\operatorname{Per}(\mathbf{A}, \mathbf{x}) = \bigcap_{i \in n} \operatorname{Per}(\mathbf{A}, \mathbf{x}, i)$,

(iii) $\operatorname{per}(\mathbf{A}, \mathbf{x}) = \operatorname{lcm}_{i \in n} \operatorname{Per}(\mathbf{A}, \mathbf{x}, i)$.

Definition 3.5. Let $\mathbf{A} \in \mathcal{B}(n,n)$, $h \in \mathcal{B}$, let $\mathcal{G}(\mathbf{A}, h)$ be a threshold digraph of $\mathbf{A}$. By $\mathrm{SCC}^* \mathcal{G}(\mathbf{A}, h)$ we denote the set of all non-trivial strongly connected components of $\mathcal{G}(\mathbf{A}, h)$. For any $\mathrm{SCC}^* \mathcal{G}(\mathbf{A}, h)$, we define

(i) $\operatorname{per}(\mathcal{K}) := \gcd\{|C|;\ C \text{ is a cycle in } \mathcal{K}\}$,

(ii) $\bar{\mathbf{A}} = \{a_{ij};\ i, j \in n\}$,

(iii) $\mathrm{SCC}^*(\mathbf{A}) = \bigcup\{\mathrm{SCC}^* \mathcal{G}(\mathbf{A}, h);\ h \in \bar{\mathbf{A}}\}$,

(iv) $\mathrm{SCC}^{\min}(\mathbf{A}) = \{\mathcal{K} \in \mathrm{SCC}^*(\mathbf{A});\ \mathcal{K} \text{ is minimal in } (\mathrm{SCC}^*(\mathbf{A}), \subseteq)\}$.

Lema 3.3. *Let* $\mathbf{A} \in \mathcal{B}(n,n)$. *Then*

(i) $(\forall \mathcal{K},\ \mathcal{K}' \in \mathrm{SCC}^*(\mathbf{A}))\,[\mathcal{K} \subseteq \mathcal{K}' \implies \operatorname{per}(\mathcal{K})')|\operatorname{per}(\mathcal{K})]$,

(ii) $(\forall h,\ h' \in \bar{\mathbf{A}})(\forall\, \mathrm{SCC}^* \mathcal{G}(\mathbf{A}, h),\ \mathcal{K}' \in \mathrm{SCC}^* \mathcal{G}(\mathbf{A},\ h'))$
$[(h \geq h' \wedge \mathcal{K} \cap \mathcal{K}' \neq \emptyset) \implies \mathcal{K} \subseteq \mathcal{K}']$,

(iii) $(\forall \mathcal{K}, \mathcal{K}' \in \mathrm{SCC}^{\min}(\mathbf{A}))\,[\mathcal{K} \neq \mathcal{K}' \implies \mathcal{K} \cap \mathcal{K}' = \emptyset]$,

(iv) $|\mathrm{SCC}^{\min}(\mathbf{A})| \leq n$.

In [6] the period of a fuzzy matrix was characterized by the orbit periods and by the periods of the non-trivial strongly connected components in the threshold graphs of the matrix. The following theorem was proved.

THEOREM 3.4. *Let $\mathbf{A} \in \mathcal{B}(n,n)$, $d \in \mathbb{N}$. Then the following statements are equivalent*

(i) $\operatorname{per}(\mathbf{A}) \mid d$,

(ii) $\big(\forall \mathbf{x} \in \mathcal{B}(n)\big) \operatorname{per}(\mathbf{A}, \mathbf{x}) \mid d$,

(iii) $(\forall h \in \mathcal{B}) \big(\forall \operatorname{SCC}^* \mathcal{G}(\mathbf{A}, h)\big) \operatorname{per}(\mathcal{K}) \mid d$.

The first two statements in the next theorem present a more compact formulation of Theorem 3.4. The third statement is a consequence of Lemma 3.3 (i).

THEOREM 3.5. *Let $\mathbf{A} \in \mathcal{B}(n,n)$. Then*

(i) $\operatorname{per}(\mathbf{A}) = \operatorname{lcm}\big\{\operatorname{per}(\mathbf{A}, \mathbf{x});\ \mathbf{x} \in \mathcal{B}(n)\big\}$,

(ii) $\operatorname{per}(\mathbf{A}) = \operatorname{lcm}\big\{\operatorname{per}(\mathcal{K});\ \mathcal{K} \in \operatorname{SCC}^*(\mathbf{A})\big\}$,

(iii) $\operatorname{per}(\mathbf{A}) = \operatorname{lcm}\big\{\operatorname{per}(\mathcal{K});\ \mathcal{K} \in \operatorname{SCC}^{\min}(\mathbf{A})\big\}$.

4. Orbit periods

From Theorem 3.5 a natural question arises: Is the value of the matrix period $\operatorname{per}(\mathbf{A}) = d$ necessarily achieved by some orbit period, i.e., is there a vector $\mathbf{x} \in \mathcal{B}(n)$ such that $\operatorname{per}(\mathbf{A}, \mathbf{x}) = d$?

If the matrix period is a prime power, then the answer to this question is positive.

THEOREM 4.1. *Let $\mathbf{A} \in \mathcal{B}(n,n)$, let $d = p^\alpha$ be a prime power. Then the following statements are equivalent*

(i) $d \mid \operatorname{per}(\mathbf{A})$,

(ii) $\big(\exists \mathbf{x} \in \mathcal{B}(n)\big)\, d \mid \operatorname{per}(\mathbf{A}, \mathbf{x})$,

(iii) $\big(\exists \mathcal{K} \in \operatorname{SCC}^*(\mathbf{A})\big)\, d \mid \operatorname{per}(\mathcal{K})$.

P r o o f. In view of the well-known properties of prime numbers, a prime power p^α divides the least common multiple of some given natural numbers if and only if p^α divides at least one of the numbers. By this fact, the equivalence of the statements (i) and (ii) follows from Theorem 3.5 (i). Similarly, the equivalence of (i) and (iii) is a consequence of Theorem 3.5 (ii). □

THEOREM 4.2. *Let $\mathbf{A} \in \mathcal{B}(n,n)$ and let $d = p^\alpha$ be a prime power. If $\operatorname{per}(\mathbf{A}) \mid d$, then the following statements are equivalent*

(i) $\operatorname{per}(\mathbf{A}) = d$,

(ii) $\big(\exists \mathbf{x} \in \mathcal{B}(n)\big) \operatorname{per}(\mathbf{A}, \mathbf{x}) = d$,

(iii) $\big(\exists \mathcal{K} \in \operatorname{SCC}^*(\mathbf{A})\big) \operatorname{per}(\mathcal{K}) = d$.

Proof. By Theorem 4.1. $\square$

In the general case, when the considered matrix period is a product of prime powers, the situation is somewhat more complicated. Theorem 4.1 can be then expressed in the following general form.

THEOREM 4.3. *Let $\mathbb{A} \in \mathcal{B}(n,n)$, let $d_j = p_j^{\alpha_j}$, for $j \in s$, be powers of different primes, let $d = \prod_{j \in s} d_j$. Then the following statements are equivalent*

(i) $d \mid \mathrm{per}(\mathbb{A})$,

(ii) $(\forall j \in s)\big(\exists \boldsymbol{x}_j \in \mathcal{B}(n)\big)\, d_j \mid \mathrm{per}(\mathbb{A}, \boldsymbol{x}_j)$,

(iii) $(\forall j \in s)\big(\exists \mathcal{K}_j \in \mathrm{SCC}^*(\mathbb{A})\big)\, d_j \mid \mathrm{per}(\mathcal{K}_j)$.

Proof. The theorem can be proved by using the same arguments as in the proof of Theorem 4.1, for any $p_j^{\alpha_j}$, $j \in s$. $\square$

If we change the order of the quantifiers in Theorem 4.3 (ii), (iii), in order to get one common $\boldsymbol{x} \in \mathcal{B}(n)$, or one common $\mathcal{K} \in \mathrm{SCC}^*(\mathbb{A})$, then we obtain a weaker form of the theorem, with implications instead of equivalences.

THEOREM 4.4. *Let $\mathbb{A} \in \mathcal{B}(n,n)$, let $d \in \mathbb{N}$. Then the statements*

(i) $d \mid \mathrm{per}(\mathbb{A})$,

(ii) $\big(\exists \boldsymbol{x} \in \mathcal{B}(n)\big)\, d \mid \mathrm{per}(\mathbb{A}, \boldsymbol{x})$,

(iii) $\big(\exists \mathcal{K} \in \mathrm{SCC}^*(\mathbb{A})\big)\, d \mid \mathrm{per}(\mathcal{K})$,

fulfil the implications (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i).

Proof. The theorem follows from Theorem 3.5. $\square$

The following theorem presents another formulation, in terms of equality.

THEOREM 4.5. *Let $\mathbb{A} \in \mathcal{B}(n,n)$, let $d \in \mathbb{N}$. If $\mathrm{per}(\mathbb{A}) \mid d$, then the statements*

(i) $\mathrm{per}(\mathbb{A}) = d$,

(ii) $\big(\exists \boldsymbol{x} \in \mathcal{B}(n)\big)\ \mathrm{per}(\mathbb{A}, \boldsymbol{x}) = d$,

(iii) $\big(\exists \mathcal{K} \in \mathrm{SCC}^*(\mathbb{A})\big)\ \mathrm{per}(\mathcal{K}) = d$,

fulfil the implications (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i).

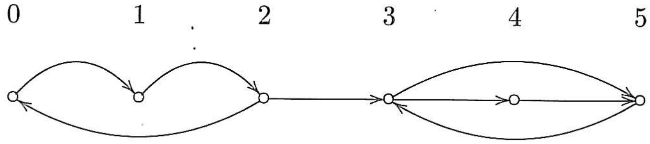
Proof. By Theorem 4.4. $\square$

Remark 4.1. In general, the converse implications to those of Theorem 4.4 and Theorem 4.5 do not hold true. This can be shown by the following two examples.

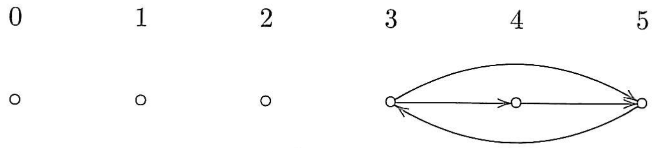
EXAMPLE 1. Let be $\mathcal{B} = \langle 0, 4\rangle$, $n = 6$, $d = 6$ and

$$\mathbb{A} = \begin{pmatrix} 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & & \\ 1 & 0 & 0 & 1 & & \\ & & & 0 & 2 & 3 \\ & & & 0 & 0 & 2 \\ & & & 3 & 0 & 0 \end{pmatrix}.$$

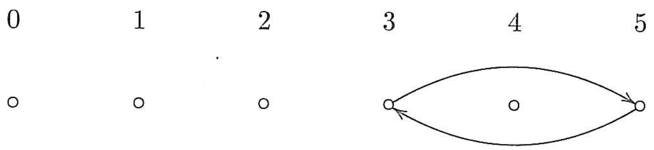
The threshold graphs $\mathcal{G}(\mathbb{A}, h)$ are of the following five types:
for $h = 0$ we have the complete digraph $\mathcal{G}_0$ with 6 vertices $\{0, 1, \ldots, 5\}$,
for $0 < h \leq 1$ we have the digraph $\mathcal{G}_1$:



for $1 < h \leq 2$ we have the digraph $\mathcal{G}_2$:



for $2 < h \leq 3$ we have the digraph $\mathcal{G}_3$:



and for $h > 3$ we have the discrete digraph $\mathcal{G}_4$ with 6 vertices.

We can see that the only non-trivial strongly connected components of matrix $\mathbb{A}$ are the subgraphs $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$, generated by subsets

$K_1 = \{0, 1, 2\}$ in $\mathcal{G}_1$,
$K_2 = \{3, 4, 5\}$ in $\mathcal{G}_1$ and in $\mathcal{G}_2$,
$K_3 = \{3, 5\}$ in $\mathcal{G}_3$.

The component periods are $\mathrm{per}(\mathcal{K}_1) = 3$, $\mathrm{per}(\mathcal{K}_2) = 1$, $\mathrm{per}(\mathcal{K}_3) = 2$.

By Theorem 3.5 (ii), we have $\mathrm{per}(\mathbb{A}) = \mathrm{lcm}(3, 1, 2) = 6 = d$, therefore, the statements Theorem 4.4 (i) and Theorem 4.5 (i) are satisfied.

On the other hand, we can verify that there is no $\mathbf{x} \in \mathcal{B}(n)$ for which $\operatorname{per}(\mathbb{A}, \mathbf{x}) = 6$. Let us choose a fixed vector $\mathbf{x} = (x_0, x_1, x_2, x_3, x_4, x_5) \in \mathcal{B}(n)$.

If any of the elements x_3, x_4, x_5 is ≥ 1, then the orbit elements $x_0^{(r)}$, $x_1^{(r)}$, $x_2^{(r)}$ stabilize on the value 1. The remaining elements $x_3^{(r)}$, $x_4^{(r)}$, $x_5^{(r)}$ obviously cannot have their periods other than 1 or 2. By Lemma 3.2 (iii), the orbit period $\operatorname{per}(\mathbb{A}, \mathbf{x}) \neq 6$.

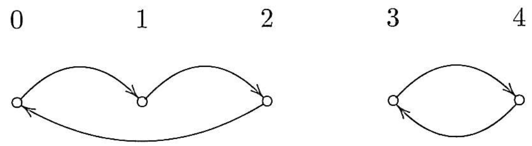
If all the elements x_3, x_4, x_5 are < 1, then the orbit elements $x_0^{(r)}$, $x_1^{(r)}$, $x_2^{(r)}$ can only have periods 1 or 3 and all orbit elements $x_3^{(r)}$, $x_4^{(r)}$, $x_5^{(r)}$ stabilize on the value $h = \max(x_3, x_4, x_5)$. Again, by Lemma 3.2 (iii), the orbit period $\operatorname{per}(\mathbb{A}, \mathbf{x}) \neq 6$.

We have shown that in Theorem 4.4 and Theorem 4.5, the implication (i) $\Longrightarrow$ (ii) is not satisfied.

EXAMPLE 2. Let be $\mathcal{B} = \langle 0, 2 \rangle$, $n = 5$, $d = 6$ and

$$\mathbb{A} = \begin{pmatrix} 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \\ 1 & 0 & 0 & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{pmatrix}.$$

For $0 < h \leq 1$ we have the threshold digraph



The non-trivial strongly connected components of matrix $\mathbb{A}$ are the subgraphs $\mathcal{K}_1$, $\mathcal{K}_2$ generated by subsets $K_1 = \{0, 1, 2\}$ and $K_2 = \{3, 4\}$, respectively. The component periods are $\operatorname{per}(\mathcal{K}_1) = 3$, $\operatorname{per}(\mathcal{K}_2) = 2$.

If we define $\mathbf{x} = (x_0, x_1, x_2, x_3, x_4) = (1, 0, 0, 1, 0)$, then clearly the orbit elements $x_0^{(r)}$, $x_1^{(r)}$, $x_2^{(r)}$ have their periods equal to 3 and $x_3^{(r)}$, $x_4^{(r)}$ have the periods equal to 2. Thus, $\operatorname{per}(\mathbb{A}) = \operatorname{per}(\mathbb{A}, \mathbf{x}) = 6 = d$, i.e., the statements Theorem 4.4 (ii) and Theorem 4.5 (ii) are satisfied.

On the other hand, there is no strongly connected component with a period equal to 6; therefore, the statements (iii) of the mentioned theorems are not true.

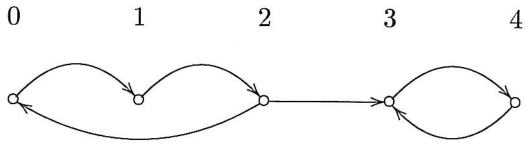
We have shown that in Theorem 4.4 and Theorem 4.5, the implication (ii) $\Longrightarrow$ (iii) is not satisfied.

R e m a r k 4.2. In our previous considerations we have used the fact that in Example 2 the components $\mathcal{K}_1$, $\mathcal{K}_2$ are not reachable from each other. However, the non-reachability of components is not necessary. With a little effort we can construct an example in which the situation is different.

EXAMPLE 3. Let be $\mathcal{B} = \langle 0, 2\rangle$, $n = 5$, $d = 6$ and

$$\mathbf{A} = \begin{pmatrix} 0 & 2 & 0 & & \\ 0 & 0 & 2 & & \\ 2 & 0 & 0 & 2 & \\ & & & 0 & 2 \\ & & & 2 & 0 \end{pmatrix}.$$

For $0 < h \leq 2$ we have the threshold digraph



Similarly as in the previous example, the only non-trivial strongly connected components of matrix $\mathbf{A}$ are the subgraphs $\mathcal{K}_1$, $\mathcal{K}_2$ generated by subsets $K_1 = \{0, 1, 2\}$ and $K_2 = \{3, 4\}$, respectively. Again the component periods are $\mathrm{per}(\mathcal{K}_1) = 3$, $\mathrm{per}(\mathcal{K}_2) = 2$.

If we define $\boldsymbol{x} = (x_0, x_1, x_2, x_3, x_4) = (2, 0, 0, 1, 0)$, then clearly the orbit elements $x_3^{(r)}, x_4^{(r)}$ have the periods equal to 2. On the other hand, in the sequence of orbit elements $x_0^{(r)}$, the value 2 repeats with the periodicity equal to 3, whereas the value 1 occurs at any other position (for sufficiently large r). This is a consequence of Lemma 2.1 and of the fact that $\gcd\big(\mathrm{per}(\mathcal{K}_1),\ \mathrm{per}(\mathcal{K}_2)\big) = 1$. Therefore, we have $\mathrm{per}(\mathbf{A}, \boldsymbol{x}, 0) = 3$. Thus, $\mathrm{per}(\mathbf{A}) = \mathrm{per}(\mathbf{A}, \boldsymbol{x}) = \mathrm{lcm}(2, 3) = 6 = d$.

Also in this example there is no strongly connected component with a period equal to 6. Therefore we can conclude from this example that in Theorem 4.4 and Theorem 4.5, the implication (ii) $\Longrightarrow$ (iii) is not satisfied.

Remark 4.3. If we compare Example 1 and Example 3, we can see that the existence of a vector $\boldsymbol{x} \in \mathcal{B}(n)$ with the property $\mathrm{per}(\mathbf{A}) = \mathrm{per}(\mathbf{A}, \boldsymbol{x})$ depends on the level $h \in \mathcal{B}$ on which the strongly connected components are reachable from each other. The arguments used in Example 3 form the base of the following theorem.

THEOREM 4.6. *Let $\mathbf{A} \in \mathcal{B}(n, n)$, let $d_j = p_j^{\alpha_j}$, for $j \in s$, be powers of different primes, let $d = \prod\limits_{j \in s} d_j$. Then the statements*

(i) $\big(\exists \boldsymbol{x} \in \mathcal{B}(n)\big)\, d \,|\, \mathrm{per}(\mathbf{A}, \boldsymbol{x})$,

$$\text{(ii)}\quad (\forall j \in s)(\exists h_j \in \mathcal{B})\big(\exists \mathcal{K}_j \in \mathrm{SCC}^* \mathcal{G}(\mathbf{A}, h_j)\big)$$
$$\big[d_j \,|\, \mathrm{per}(\mathcal{K}_j) \wedge (\forall j \in s)[W_{\mathcal{G}(\mathbf{A},h_j)}(\mathcal{K}_j, \mathcal{K}_m) \neq \emptyset \implies h_j > h_m]\big]$$

fulfil the implication (ii) $\implies$ (i).

P r o o f. Let us suppose that (ii) holds true. For any $j \in s$ we choose an element $i_j \in K_j$ in such a way that $(\forall j,\ m \in s)[\mathcal{K}_j = \mathcal{K}_m \Leftrightarrow i_j = i_m]$. This is always possible because of Lemma 3.3.

We define $\boldsymbol{x} \in \mathcal{B}(n)$ as follows:

$$\begin{aligned} x_i &:= h_j\,, && \text{if } i = i_j \text{ for some } j \in s\,, \\ x_i &:= 0\,, && \text{otherwise}. \end{aligned}$$

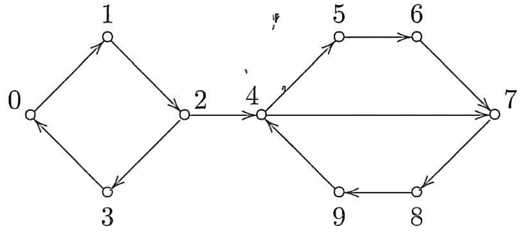
Then for any $j \in s$, no vertex i_m with $h_m > h_j$ can be reached from the vertex i_j in $\mathcal{G}(\mathbf{A}, h_j)$. Therefore, the value $x_{i_j} = h_j$ will be repeated in the orbit elements $x_{i_j}^{(r)}$, for sufficiently large r, with periodicity $\mathrm{per}(\mathcal{K}_j)$ which is a multiple of d_j. This implies that $d_j \,|\, \mathrm{per}(\mathbf{A}, \boldsymbol{x}, i_j)$. By Lemma 3.2 we get $\operatorname*{lcm}_{j\in s} d_j \,|\, \operatorname*{lcm}_{j \in s} \mathrm{per}(\mathbf{A}, \boldsymbol{x}, i_j)$, i.e., $d \,|\, \mathrm{per}(\mathbf{A}, \boldsymbol{x})$. □

R e m a r k 4.4. It will be shown by the last example that neither the statement (ii) in Theorem 4.6 is necessary for the existence of $\boldsymbol{x} \in \mathcal{B}(n)$ such that $\mathrm{per}(\mathbf{A}) = \mathrm{per}(\mathbf{A}, \boldsymbol{x})$. It turns out that in deciding the question it is important to consider also the lengths of the walks connecting the components of threshold digraphs.

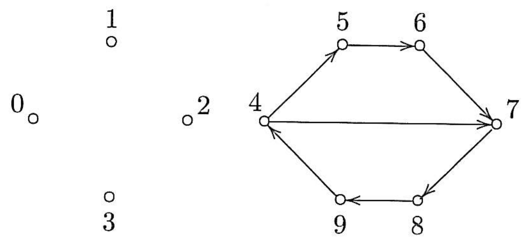
EXAMPLE 4. Let be $\mathcal{B} = \langle 0, 4\rangle$, $n = 10$, $d = 12$ and

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & & & & & & \\ 0 & 0 & 1 & 0 & & & & & & \\ 0 & 0 & 0 & 1 & & & & & & \\ 1 & 0 & 0 & 0 & 1 & & & & & \\ & & & & 0 & 3 & 0 & 2 & 0 & 0 \\ & & & & 0 & 0 & 3 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 3 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 & 3 & 0 \\ & & & & 0 & 0 & 0 & 0 & 0 & 3 \\ & & & & 3 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

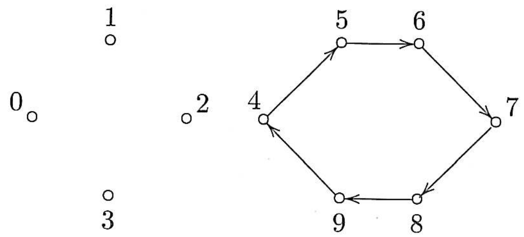
for $0 < h \leq 1$ we have the digraph $\mathcal{G}_1$:



for $1 < h \leq 2$ we have the digraph $\mathcal{G}_2$:



for $2 < h \leq 3$ we have the digraph $\mathcal{G}_3$:



Similarly as in the first example, the only non-trivial strongly connected components of matrix $\mathbf{A}$ are the subgraphs $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$, generated by subsets

$K_1 = \{0,1,2,3\}$ in $\mathcal{G}_1$,

$K_2 = \{4,5,6,7,8,9\}$ in $\mathcal{G}_1$ and in $\mathcal{G}_2$,

$K_3 = \{4,5,6,7,8,9\}$ in $\mathcal{G}_3$.

The component periods are $\operatorname{per}(\mathcal{K}_1) = 4$, $\operatorname{per}(\mathcal{K}_2) = 2$, $\operatorname{per}(\mathcal{K}_3) = 6$. By Theorem 3.5 (ii), we have $\operatorname{per}(\mathbf{A}) = \operatorname{lcm}(4,2,6) = 12 = d$.

If we define $\mathbf{x} = (1,0,0,0,3,0,0,0,0,0)$, then the orbit elements $x_4^{(r)}, \dots, x_9^{(r)}$ evidently have their periods equal to 6. Namely, the value 3 repeats with periodicity equal to 6 (the value 2 also occurs on infinitely many places).

On the other hand, in the sequence of orbit elements $x_0^{(r)}$, the value 1 repeats on even places with the periodicity equal to 4, whereas on the odd places the value 1 occurs with periodicity equal to 2 (for sufficiently large r). This is a consequence of the fact that $\gcd\big(\operatorname{per}(\mathcal{K}_1), \operatorname{per}(K_2)\big) = 2$ and that the length of any walk from the vertex 0 to the vertex 4 is odd. On the whole we have $\operatorname{per}(\mathbf{A}, \mathbf{x}, 0) = 4$. Thus, $\operatorname{per}(\mathbf{A}) = \operatorname{per}(\mathbf{A}, \mathbf{x}) = \operatorname{lcm}(4,6) = 12 = d$.

It can be easily seen that the statement Theorem 4.6 (ii) is not fulfilled in Example 4. Therefore, in Theorem 4.6, the converse implication (i) $\Longrightarrow$ (ii) does not hold true.

R e m a r k 4.5. The following three problems remain open:

(i) Find a necessary and sufficient condition for the existence of a vector $\mathbf{x} \in \mathcal{B}(n)$ with the property $\operatorname{per}(\mathbb{A}, \mathbf{x}) = \operatorname{per}(\mathbb{A})$, if matrix $\mathbb{A} \in \mathcal{B}(n, n)$ is given.

(ii) Find a polynomial algorithm for deciding the previous problem.

(iii) Find a polynomial algorithm for computing the orbit period $\operatorname{per}(\mathbb{A}, \mathbf{x})$, if $\mathbb{A} \in \mathcal{B}(n, n)$ and $\mathbf{x} \in \mathcal{B}(n)$ are given.

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