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GROUP HYPERREALS: ITERATED SEQUENTIAL COMPLETIONS OF RATIONALS

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Dedicated to Professor J. Jakubík on the occasion of his 70th birthday

ABSTRACT. We construct sequential convergence group completions of the rational numbers the underlying groups of which differ from the real numbers. First, we equip the group of real numbers with an incomplete compatible sequential convergence which preserves the usual metric convergence of rational sequences. Second, every completion of the resulting group has the desired properties. We show that there are exactly $\exp\exp\omega$ such nonhomeomorphic completions; the sequential order of each completion is at least 2 and there are $\exp\exp\omega$ completions the sequential order of which is ω_1.

1. Introduction

Sequential convergence groups, or $\mathcal{L}$-groups, were introduced by O. Schreier in [SCH] as groups equipped with a compatible convergence of sequences (cf. [NOG]). As shown by J. Novák in [NOV], the completion theory for $\mathcal{L}$-groups differs from the completion theory for topological groups. The group of rational numbers equipped with the usual convergence is a surprisingly rich example of the difference (cf. [FZS], [FKP], [FCB]). Observe that the rational numbers can be equipped with various nondiscrete compatible convergences having interesting properties. P. Simon proved recently that there exists a convergence finer than the usual one such that the resulting group is complete, and another, incomplete one, such that the underlying group of its completion is a proper subgroup of the real numbers ([SIM]). F. Zanolin proved that each coarse convergence (no convergent sequence can be added) compatible with the group structure of rational numbers is complete ([FZB]).

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By $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{N}$ we denote the set of real, rational and natural numbers, respectively. The set of all monotone mappings of $\mathbb{N}$ into $\mathbb{N}$ is denoted by MON. Let X be a set. A *sequential convergence*, or an *$\mathcal{L}$-convergence*, $\mathscr{L}$ on X is a subset of $X^{\mathbb{N}} \times X$; $(S,x) \in \mathscr{L}$ means that the sequence S converges to x. We always assume the axioms on constant sequences and subsequences. The pair $(X,\mathscr{L})$ is an $\mathcal{L}$-space; an $\mathcal{L}_0$-space if limits are unique and an $\mathcal{L}_0^*$-space if the Urysohn axiom is also satisfied (if S is a sequence and x is a point such that every subsequence of S contains another subsequence converging to x, then S itself converges to x). Recall that the *Urysohn modification* $\mathscr{L}^*$ of $\mathscr{L}$ is defined as follows: S is $\mathscr{L}^*$-convergent to x whenever each subsequence of S contains another subsequence $\mathscr{L}$-converging to x. If $\mathscr{L}$ has unique limits, then $\mathscr{L}^*$ has unique limits as well. If X is an algebra with a convergence, then it is an $\mathcal{L}$-algebra if the convergence is compatible, i.e., if the algebraic operations are sequentially continuous (cf. [SCH]). The Urysohn modification preserves compatibility. In an $\mathcal{L}$-group (ring, field) the convergence is homogeneous and the set of all neutral sequences determines the convergence.

Let X be a set equipped with an $\mathcal{L}$-convergence. For each subset A of X define $\mathrm{cl}A$ to be the set of all limits of convergent sequences ranging in A. Put $0-\mathrm{cl}A = A$ and, inductively, for each ordinal number $\alpha \geq 1$ put $\alpha - \mathrm{cl}A = \bigcup_{\beta<\alpha} \mathrm{cl}(\beta - \mathrm{cl}A)$. Each $\alpha - \mathrm{cl}$ is a closure operator on X which is not necessarily idempotent. If $X = \mathrm{cl}A$, then A is said to be *closure dense* and if $X = \omega_1 - \mathrm{cl}A$, then A is said to be *topologically dense.* The sequential order of X is the least ordinal number $\sigma \geq 1$ such that $\mathrm{cl}(\sigma - \mathrm{cl}A) = \sigma - \mathrm{cl}A$ for each subset A of X. Clearly $1 \leq \sigma \leq \omega_1$. The sequential order of every metrizable topological group is 1 and there are many interesting topological groups the sequential order of which is ω_1. However, it is an open problem whether the sequential order of a topological group or of an $\mathcal{L}_0^*$-group can be anything between 1 and ω_1 (Problem 4 in [NYI] and Problem 1.5 in [FKP]).

Let X be an abelian group equipped with an $\mathcal{L}$-group convergence $\mathscr{L}$. A sequence $S \in X^{\mathbb{N}}$ is $\mathscr{L}$-Cauchy if for each $s,t \in MON$ the sequence $S\circ s - S\circ t$ converges to zero. If each $\mathscr{L}$-Cauchy sequence converges, then X is *$\mathcal{L}$-complete.* In an $\mathcal{L}$-ring or an $\mathcal{L}$-field the Cauchy sequences are always defined with respect to the underlying group structure. By an *$\mathcal{L}$-completion* we understand an $\mathcal{L}$-complete group, ring or field, respectively, in which the original one is embedded as a subalgebra and a topologically dense $\mathcal{L}$-subspace; if only Cauchy sequences ranging in the original group (ring, field) are assumed to converge, then we speak of a precompletion. We usually assume the uniqueness of limits and the Urysohn axiom.

2. Existence

In this section we construct an incomplete $\mathcal{L}_0^*$-group precompletion of the rational numbers and describe some of its properties. In what follows, $\mathscr{M}$ denotes the usual convergence on the real line and $\mathscr{M}_{\mathbb{Q}}$ denotes its restriction to $\mathbb{Q}$.

Construction.

Let $\{1,\pi\}\cup B$ be a Hamel basis of $\mathbb{R}$ considered as a linear space over the scalar field $\mathbb{Q}$. Define $\mathscr{L}_{\mathbb{R}} \subset \mathbb{R}^{\mathbb{N}} \times \mathbb{R}$ to be the set of all pairs (S,x) such that $\langle S(n)\rangle$ is of the form

$$S(n) = S_1(n) + S_2(n)\pi + q_1 b_1 + \cdots + q_k b_k\,, \quad n \in \mathbb{N},$$

where $k\in\mathbb{N}$, $b_i \in B$, $q_i \in \mathbb{Q}$, $i = 1,\dots,k$, S_1 and S_2 are Cauchy sequences of rational numbers, and

$$x = x_1 + x_2\pi + q_1 b_1 + \cdots + q_k b_k\,,$$

where $(S_1,x_1),(S_2,x_2) \in \mathscr{M}$ and $x_2 \in \mathbb{Q}$. Trivially, $\mathscr{L}_{\mathbb{R}}$ is an $\mathcal{L}$-group convergence on $\mathbb{R}$. Let $\mathscr{L}_{\mathbb{R}}^*$ be the Urysohn modification of $\mathscr{L}_{\mathbb{R}}$.

CLAIM. $(\mathbb{R},\mathscr{L}_{\mathbb{R}}^*)$ *is an incomplete* $\mathcal{L}_0^*$*-group precompletion of* $(\mathbb{Q},\mathscr{M}_{\mathbb{Q}})$.

Our claim follows immediately from the next lemma.

LEMMA 2.1.

(i) $\mathscr{L}_{\mathbb{R}} \subset \mathscr{M}$.

(ii) *Let* $(S,x)\in\mathscr{M}$ *and* $S\in\mathbb{Q}^{\mathbb{N}}$. *Then* $(S,x)\in\mathscr{L}_{\mathbb{R}}$.

(iii) $\mathscr{L}_{\mathbb{R}} \neq \mathscr{L}_{\mathbb{R}}^*$.

(iv) *Let* $(S,x)\in\mathscr{M}$, $S\in\mathbb{Q}^{\mathbb{N}}$ *and* $x\in\mathbb{R}\setminus\mathbb{Q}$. *Then* $\langle S(n)\pi\rangle$ *is an* $\mathscr{L}_{\mathbb{R}}^*$*-Cauchy sequence no subsequence of which* $\mathscr{L}_{\mathbb{R}}^*$*-converges.*

Proof. (i) and (ii) are obvious. Observe that, since $\mathscr{M}$ has unique limits, $\mathscr{L}_{\mathbb{R}} \subset \mathscr{M}$ implies that $\mathscr{L}_{\mathbb{R}}$ has unique limits, too.

(iii) For each $k\in\mathbb{N}$, let $\langle a_{kn}\rangle$ be a one-to-one sequence of rational numbers which $\mathscr{M}$-converges to $1/k$ such that for all $n\in\mathbb{N}$ we have $1 < a_{1n}$ and $1/k < a_{kn} < 1/(k-1)$ for $k = 2,\dots$. Arrange the set $\{a_{kn}; k,n\in\mathbb{N}\}$ into a one-to-one sequence S_2. For each $n\in\mathbb{N}$, define π_n to be π truncated to n decimal places (i.e., $\pi_1 = 3,1$, $\pi_2 = 3,14,\dots$), $S_1(n) = -S_2(n)\pi_n$ and $S(n) = S_1(n) + S_2(n)\pi$. Then the sequence S does not $\mathscr{L}_{\mathbb{R}}$-converge, but $\mathscr{L}_{\mathbb{R}}^*$-converges to 0. Indeed, for each $s\in MON$ there exists $t\in MON$ such that $S_2\circ s\circ t$ is either a subsequence of the sequence $\langle a_{kn}\rangle$ for some $k\in\mathbb{N}$, or $S_2\circ s\circ t$ is $\mathscr{M}$-converging to 0. Since in both cases $S\circ s\circ t$ is $\mathscr{L}_{\mathbb{R}}$-converging to 0, the sequence S is $\mathscr{L}_{\mathbb{R}}^*$-converging to 0, too.

(iv) Recall that $\{1,\pi\}\cup B$ is a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$. It follows directly from the definition of $\mathscr{L}_{\mathbb{R}}$ that the sequence $\langle S(n)\pi\rangle$ has the desired properties. □

As shown by J. Novák in [NOV], every incomplete abelian $\mathcal{L}_0^*$-group has an $\mathcal{L}_0^*$-group completion but it can have many nonequivalent completions; even $(\mathbb{Q},\mathscr{M}_{\mathbb{Q}})$ has $\exp\exp\omega$ completions finer than $(\mathbb{R},\mathscr{M})$. Clearly, each $\mathcal{L}_0^*$-group completion of $(\mathbb{R},\mathscr{L}_{\mathbb{R}}^*)$ is also an $\mathcal{L}_0^*$-group completion of $(\mathbb{Q},\mathscr{M}_{\mathbb{Q}})$. As already stated in [FKP], $\mathbb{R}$ can be equipped in many different ways with an $\mathcal{L}_0^*$-group convergence so that it becomes an incomplete $\mathcal{L}_0^*$-group precompletion of $(\mathbb{Q},\mathscr{M}_{\mathbb{Q}})$. It might be interesting to investigate the class of all completions of $(\mathbb{Q},\mathscr{M}_{\mathbb{Q}})$. In the present paper we describe some of the $\mathcal{L}_0^*$-group completions of $(\mathbb{R},\mathscr{L}_{\mathbb{R}}^*)$.

Let S be an $\mathscr{L}_{\mathbb{R}}^*$-Cauchy sequence. Then S is either $\mathscr{L}_{\mathbb{R}}^*$-convergent or no subsequence of S is $\mathscr{L}_{\mathbb{R}}^*$-convergent. Recall that the equivalence class [S] containing S consists of all $\mathscr{L}_{\mathbb{R}}^*$-Cauchy sequences T such that $S-T$ is $\mathscr{L}_{\mathbb{R}}^*$-converging to 0. Further, S and $S\circ s$ are equivalent for all $s\in MON$ and the equivalence relation is a congruence (cf. [NOV]).

Let $(G,\mathscr{L}_G^*)$ be an $\mathcal{L}_0^*$-group completion of $(\mathbb{R},\mathscr{L}_{\mathbb{R}}^*)$. Then the set of all $\mathscr{L}_G^*$-limits of sequences ranging in $\mathbb{R}$ can be identified in a natural way with the group of all equivalence classes of $\mathscr{L}_{\mathbb{R}}^*$-Cauchy sequences; denote it F. As a rule, we identify $r\in\mathbb{R}$ and the equivalence class $[\langle r\rangle]$ containing the constant sequence $\langle r\rangle$.

LEMMA 2.2. *Let S be an $\mathscr{L}_{\mathbb{R}}^*$-Cauchy sequence. Then:*

(i) *There are $n_0,k\in\mathbb{N}$, $b_i\in B$ and $q_i\in\mathbb{Q}$, $i=1,\dots,k$, and $S_1,S_2\in\mathbb{Q}^{\mathbb{N}}$ such that for all $n\in\mathbb{N}$, $n>n_0$, we have $S(n)=S_1(n)+S_2(n)\pi+q_1b_1+\dots+q_kb_k$.*

(ii) *For each $s\in MON$ there exists $t\in MON$ such that $S_1\circ s\circ t$ and $S_2\circ s\circ t$ are $\mathscr{M}_{\mathbb{Q}}$-Cauchy sequences.*

Proof. Each $S(n)$ can be expressed as a $\mathbb{Q}$-linear combination of finitely many elements of $\{1,\pi\}\cup B$. Since S is $\mathscr{L}_{\mathbb{R}}^*$-Cauchy, there is a finite subset $\{b_1,\dots,b_k\}$ of B such that each $S(n)$, $n\in\mathbb{N}$, is a $\mathbb{Q}$-linear combination of $\{1,\pi\}\cup B$. This proves (i).

(ii) Clearly, the sequences S_1 and S_2 are bounded. The rest is trivial. □

Consider the set $\mathbb{R}\otimes\pi$ of all symbols $r\otimes\pi$, $r\in\mathbb{R}$. For $r\in\mathbb{R}$ and $q\in\mathbb{Q}$ define $q(r\otimes\pi)=(qr)\otimes\pi$ and for $r_1,r_2\in\mathbb{R}$ define $(r_1\otimes\pi)+(r_2\otimes\pi)=(r_1+r_2)\otimes\pi$. Then $\mathbb{R}\otimes\pi$ is a linear space over the scalar field $\mathbb{Q}$ and it is isomorphic to $\mathbb{R}$ over $\mathbb{Q}$. Denote $\mathbb{R}_{\{1\}\cup B}$ the linear subspace of $\mathbb{R}$ over $\mathbb{Q}$ generated by $\{1\}\cup B$. Then the set X of all sums of the form $p+r\otimes\pi+q_1b_1+\dots+q_kb_k$, where $p\in\mathbb{Q}$, $r\in\mathbb{R}$, $k\in\mathbb{N}$, $q_i\in\mathbb{Q}$, $i=1,\dots,k$, can be considered as a linear space over $\mathbb{Q}$

and a direct sum of its linear subspaces $\mathbb{R}\otimes\pi$ and $\mathbb{R}_{\{1\}\cup B}$ (i.e., each $x\in X$ can be written in a unique way as x_1+x_2 with $x_1\in\mathbb{R}\otimes\pi$ and $x_2\in\mathbb{R}_{\{1\}\cup B}$). Clearly, the subgroup of X consisting of all $p+r\otimes\pi+q_1b_1+\cdots+q_rb_r$, $r\in\mathbb{Q}$, is isomorphic to $\mathbb{R}$. Identifying $q\otimes\pi$ and $q\pi$ for $q\in\mathbb{Q}$ we can consider $\mathbb{R}$ as a subgroup of X.

According to Lemma 2.2, each $\mathscr{L}^*_{\mathbb{R}}$-Cauchy sequence S is equivalent to a sum of three $\mathscr{L}^*_{\mathbb{R}}$-Cauchy sequences S_1,S_2,S_3 such that S_1 is an $\mathscr{L}^*_{\mathbb{R}}$-convergent sequence of rational numbers, S_3 is a constant sequence, and S_2 is of the form $\langle T(n)\pi\rangle$, where T is an $\mathscr{M}$-convergent sequence of rational numbers. Hence S_2 either $\mathscr{L}^*_{\mathbb{R}}$-converges to some $q\pi$, $q\in\mathbb{Q}$, or no subsequence of S_2 is $\mathscr{L}^*_{\mathbb{R}}$-convergent and S_2 is $\mathscr{M}$-convergent to some $r\pi$, $r\in\mathbb{R}\setminus\mathbb{Q}$. This yields a canonical mapping h of the group F of all equivalence classes $[S]$ of $\mathscr{L}^*_{\mathbb{R}}$-Cauchy sequences onto the group X. In particular, if $\langle T(n)\pi\rangle$ is $\mathscr{L}^*_{\mathbb{R}}$-convergent to $q\pi$, $q\in\mathbb{Q}$, then $h\big[\langle T(n)\pi\rangle\big]=q\otimes\pi$.

PROPOSITION 2.3. *The mapping h is an isomorphism of F onto X pointwise fixed on $\mathbb{R}$.*

P r o o f. A straightforward calculation shows that h has the desired properties. □

In what follows we identify F and X and consider $\mathbb{R}$ as a subgroup of F.

R e m a r k 2.4. Let $(G,\mathscr{L}^*_G)$ be an $\mathcal{L}^*_0$-group completion of $(\mathbb{R},\mathscr{L}^*_{\mathbb{R}})$. Since $\mathbb{Q}\subsetneqq\mathbb{R}\subsetneqq G$, each point of $\mathbb{R}$ is an $\mathscr{L}^*_G$-limit of a sequence ranging in $\mathbb{Q}$ and no sequence ranging in $\mathbb{Q}$ can $\mathscr{L}^*_G$-converge to a point in $G\setminus\mathbb{R}$, the sequential order of $(G,\mathscr{L}^*_G)$ is at least 2. In the next section we show that there are $\exp\exp\omega$ $\mathcal{L}^*_0$-group completions of $(\mathbb{R},\mathscr{L}^*_{\mathbb{R}})$, hence of $(\mathbb{Q},\mathscr{M}_Q)$, the sequential order of which is ω_1.

3. Completions

To construct an $\mathcal{L}^*_0$-group completion of $(\mathbb{R},\mathscr{L}^*_{\mathbb{R}})$, we have to equip the set F of all equivalence classes of $\mathscr{L}^*_{\mathbb{R}}$-Cauchy sequences with a suitable $\mathcal{L}^*_0$-group convergence. Since F is a direct sum of $\mathbb{R}\otimes\pi$ and $\mathbb{R}_{\{1\}\cup B}$, we can start with an $\mathcal{L}_0$-group convergence on $\mathbb{R}\otimes\pi$ and extend it in a canonical way to F. Since $\mathbb{R}$ and $\mathbb{R}\otimes\pi$ are isomorphic, we can utilize all $\mathcal{L}^*_0$-group completions of $(\mathbb{Q},\mathscr{M}_{\mathbb{Q}})$ described in [FCB], [FKP] and [FZS]. We start with the categorical completion $(\mathbb{R},\mathscr{L}_1^*)$ of $(\mathbb{Q},\mathscr{M}_{\mathbb{Q}})$ constructed by J. N o v á k in [NOV].

EXAMPLE 3.1. Recall that $\mathscr{L}_1$ consists of all pairs of the form $\big(\langle q_n-r+x\rangle,x\big)$, where $\langle q_n\rangle$ is a sequence of rational numbers $\mathscr{M}$-convergent to $r\in\mathbb{R}$ and $x\in\mathbb{R}$. Then the set of all pairs $\big(\langle S(n)\otimes\pi\rangle,x\otimes\pi\big)$, $(S,x)\in\mathscr{L}_1$, is an

$\mathcal{L}_0$-group convergence on $\mathbb{R}\otimes\pi$. Define $\mathscr{L}_F \subset X^{\mathbb{N}}\times X$ to be the set of all pairs (S,x) such that

$$S(n) = S_1(n) + S_2(n)\otimes\pi + q_1b_1 + \cdots + q_kb_k\,,\quad n\in\mathbb{N},$$

where $k\in\mathbb{N}$, $b_i\in B$, $q_i\in\mathbb{Q}$, $i=1,\dots,k$, S_1 is a Cauchy sequence of rational numbers, S_2 is an $\mathscr{L}_1$-convergent sequence of real numbers, and

$$x = x_1 + x_2\otimes\pi + q_1b_1 + \cdots + q_kb_k\,,$$

where $(S_1,x_1)\in\mathscr{M}$, $(S_2,x_2)\in\mathscr{L}_1$. Clearly, $\mathscr{L}_F$ is an $\mathcal{L}_0$-group convergence on F and the restriction of $\mathscr{L}_F$ to $\mathbb{R}$ is equal to $\mathscr{L}_{\mathbb{R}}$. Let $\mathscr{L}_F^*$ be the Urysohn modification of $\mathscr{L}_F$. We have to show that $(F,\mathscr{L}_F^*)$ is complete. Let S be an $\mathscr{L}_F^*$-Cauchy sequence. Straightforwardly (cf. Lemma 2.2), there are $k\in\mathbb{N}$, $q_i\in\mathbb{Q}$, $b_i\in B$, $i=1,\dots,k$, $S_1\in\mathbb{Q}^{\mathbb{N}}$, $S_2\in\mathbb{R}^{\mathbb{N}}$ such that, barred finitely many $n\in\mathbb{N}$, we have $S(n)=S_1(n)+S_2(n)\otimes\pi+q_1b_1+\cdots+q_kb_k$. Further, for each $s\in MON$ there exists $t\in MON$ such that $s_1\circ s\circ t$ is $\mathscr{M}_{\mathbb{Q}}$-Cauchy (hence $\mathscr{L}_F^*$-convergent) and $S_2\circ s\circ t$ is $\mathscr{L}_1$-convergent (hence $\mathscr{L}_F^*$-convergent). Consequently S is $\mathscr{L}_F^*$-convergent, too. Thus $(F,\mathscr{L}_F^*)$ is complete. It will be called the *Ferenczi completion* of $(\mathbb{Q},\mathscr{M}_{\mathbb{Q}})$. It is easy to see that $\mathscr{L}_F^*$ is the finest $\mathcal{L}_0^*$-group convergence on F such that the restriction of $\mathscr{L}_F$ to $\mathbb{R}$ is equal to $\mathscr{L}_{\mathbb{R}}$ and each $\mathscr{L}_{\mathbb{R}}$-Cauchy sequence S is $\mathscr{L}_F$-convergent to $[S]$. Since the Urysohn modification commutes with the operation of restriction to a subspace, it follows that $(F,\mathscr{L}_F^*)$ is the categorical $\mathcal{L}_0^*$-group completion of $(\mathbb{R},\mathscr{L}_{\mathbb{R}}^*)$. Indeed, the categorical completion is (up to an equivalence) uniquely determined.

EXAMPLE 3.2. Consider any $\mathcal{L}_0^*$-group completion $(\mathbb{R},\mathscr{L}^*)$ of $(\mathbb{Q},\mathscr{M}_{\mathbb{Q}})$. E.q., take any $(\mathbb{R},\mathscr{L}_A^*)$, $A\subset B$, from [FKP], or any $\mathcal{L}_0^*$-field completion of $(\mathbb{Q},\mathscr{M}_{\mathbb{Q}})$ from [FCB], or simply $(\mathbb{R},\mathscr{M})$. As in the previous example, define an $\mathcal{L}_0$-group convergence on $\mathbb{R}\otimes\pi$ via $\mathscr{L}$, then extend it canonically to F and take its Urysohn modification. Again, the resulting object is an $\mathcal{L}_0^*$-group completion of $(\mathbb{R},\mathscr{L}_{\mathbb{R}}^*)$ and hence of $(\mathbb{Q},\mathscr{M}_{\mathbb{Q}})$. This way we obtain $\exp\exp\omega$ nonequivalent $\mathcal{L}_0^*$-group completions of $(\mathbb{Q},\mathscr{M}_{\mathbb{Q}})$.

COROLLARY 3.3. *$(\mathbb{Q},\mathscr{M}_{\mathbb{Q}})$ has exactly $\exp\exp\omega$ nonequivalent $\mathcal{L}_0^*$-group completions.*

Proof. On the one hand, according to Example 3.2, $(\mathbb{Q},\mathscr{M}_{\mathbb{Q}})$ has at least $\exp\exp\omega$ such completions. On the other hand, if $(X,\mathscr{L}_X)$ is an $\mathcal{L}_0^*$-group completion of $(\mathbb{Q},\mathscr{M}_{\mathbb{Q}})$, then the cardinality of X is $\exp\omega$ (remember $X=\omega_1-\operatorname{cl}\mathbb{Q}$) and there are at most $\exp\exp\omega$ $\mathcal{L}$-structures on X and $\exp\exp\omega$ group structures on X. Then the number of nonequivalent $\mathcal{L}_0^*$-group completions of $(\mathbb{Q},\mathscr{M}_{\mathbb{Q}})$ cannot exceed $\exp\exp\omega$. □

EXAMPLE 3.4. Observe that if in the construction described in Example 3.1 we take an incomplete $\mathcal{L}_0^*$-group precompletion of $(\mathbb{Q}, \mathscr{M}_{\mathbb{Q}})$, i.e., if the convergence on $\mathbb{R} \otimes \pi$ is incomplete, then the resulting precompletion of $(\mathbb{R}, \mathscr{L}_{\mathbb{R}}^*)$ is incomplete. Hence we can repeat the process of completion. The process can be iterated at most ω_1 times, but can stop at any iterative step.

As shown in [FCB], $(\mathbb{Q}, \mathscr{M}_{\mathbb{Q}})$ has $\exp\exp\omega$ nonequivalent $\mathcal{L}_0^*$-field completions. It follows from Example 3.2 that each of the completions yields an $\mathcal{L}_0^*$-group completion of $(\mathbb{Q}, \mathscr{M}_{\mathbb{Q}})$ the underlying group of which is F. Denote them $(F, \mathscr{L}_\alpha^*)$, $\alpha \in \exp\exp\omega$.

PROPOSITION 3.5. *For each $\alpha \in \exp\exp\omega$, the sequential order of $(F, \mathscr{L}_\alpha^*)$ is ω_1.*

Proof. According to Corollary 3.2 in [FCB], the sequential order of each of the $\mathcal{L}_0^*$-field completions of $(\mathbb{Q}, \mathscr{M}_{\mathbb{Q}})$ which yields $(F, \mathscr{L}_\alpha^*)$ is ω_1. Since the sequentially closed subgroup $\mathbb{R} \otimes \pi$ of $(F, \mathscr{L}_\alpha^*)$ has sequential order ω_1 (being homeomorphic to the corresponding $\mathcal{L}_0^*$-field completion), the sequential order of $(F, \mathscr{L}_\alpha^*)$ is ω_1, too. □

Remark 3.6. It is known (cf. [FKP], [FCB]) that the completion theory for $\mathcal{L}_0^*$-rings and $\mathcal{L}_0^*$-fields also differs from the completion theory of topological rings and fields. E.g., since in an $\mathcal{L}_0^*$-field the product ST of a Cauchy sequence S and a sequence T converging to zero need not converge to zero, the field in question does not have any $\mathcal{L}_0^*$-ring completion. Consequently, to construct iterated $\mathcal{L}_0$-ring or $\mathcal{L}_0$-field completions of $(\mathbb{Q}, \mathscr{M}_{\mathbb{Q}})$ seems to be challenging.

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