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BELIEVEABILITY FUNCTIONS INDUCED BY PARTIAL GENERALIZED COMPATIBILITY RELATIONS

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ABSTRACT. An internal state s of a system is compatible with an empirical value x, if the possibility that s is the actual internal state of the system when x was observed cannot be logically avoided on the grounds of laws and rules governing the system and its environment. Such a compatibility relation can be easily generalized to a relation between sets of states and sets of empirical values. Having at our disposal a fragment of such a generalized compatibility relation, we try to re-build, or at least to approximate the original compatibility relation and the corresponding belief function. The obtained results enable to apply, at least partially, the basic ideas of the Dempster–Shafer approach to uncertainty quantification and processing also under weaker conditions than those usually requested.

1. Introduction

An observer or investigator asks, which is the actual state s_0 of an investigated system, or attempts to find correct answer s_0 to a given question or solution to a problem, supposing that she/he knows the value x, perhaps a vector one, of some empirical observation(s), measurement(s), or result(s), of appropriate experiment(s). More generally, the question can read whether $s_0 \in T$ holds for some proper subset $T \subset S$, where S is the set of all possible internal states of the system (answers to the question or solutions to a problem, resp.), if $x \in F$ holds for some subset F of the set E of all possible empirical values. Relations between observations and internal states (answers, solutions), enabling to arrive at a reasonable, even if perhaps uncertain or only partial (incomplete) answer to such a question, can be expressed either quantitatively, e.g.,

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by certain conditional probabilities, or qualitatively, e.g., by certain compatibility relations. A *compatibility relation* ρ is a total mapping taking the Cartesian product $S \times E$ into the binary set $\{0,1\}$ (in other terms, ρ is a crisp subset of $S \times E$) in such a way that, for each $s \in S$ and each $x \in F$, $\rho(s,x) = 0$ holds iff the actual internal state s_0 of the system (the correct answer to the question or the correct solution to the problem, resp.) cannot be s due to the laws governing the system itself as well as the source outputting the empirical values, supposing that the empirical value actually obtained is x. Under a more subjective interpretation, $\rho(s,x) = 0$ means that the observer (investigator) is able to avoid the possibility that $s = s_0$ on the ground of her/his knowledge and deductive abilities, appropriately combined with the fact that x is the actually observed empirical value. In the opposite case $\rho(s,x) = 1$ and the state s is called *compatible* with the empirical value x, i.e., observing x, the possibility that $s = s_0$ cannot be avoided. Given a compatibility relation ρ on $S \times E$, we can ascribe to each empirical value $x \in E$ the set $U(x) = \{s \in S \colon \rho(s,x) = 1\}$ of states compatible with x. If the empirical value is of random nature, so that $x = X(\omega)$ for a random variable $X(\cdot)$, we can define, at least when S and E are finite spaces, for each $T \subset S$ the probability with which the inclusion $U\big(X(\omega)\big) \subset T$ holds. Up to a re-normalization possibly used when the probability that $U\big(X(\omega)\big) = \emptyset$ (the empty subset of S) is positive, the probability with which the inclusion $U\big(X(\omega)\big) \subset T$ holds is the value $\mathrm{Bel}(T)$ of the belief function Bel ascribed to the subset T of S; belief functions are the main tools for uncertainty quantification and processing in the so called Dempster–Shafer theory. It may be quite reasonable, now, even if charged by a risk of error, to accept that the actual state of the system (the correct answer or solution, resp.) is in T, i.e., to take the decision that $s_0 \in T$ holds, supposing that $\mathrm{Bel}(T) \geq \alpha$ holds for an appropriate threshold value α close to one.

The notion of compatibility relation can be approached also from a different side, as a secondary-level by-product resulting from the well-known "deterministic" model of decision making under uncertainty. This model is based on the idea that everything we have considered so far, including the actual state s_0 of the investigated system and the observed value x, is determined by the actual elementary state ω of the environment (Nature, Universe) or by something which can be understood in this way. If Ω denotes the set of all possible elementary states of the environment, all descriptions of the system in question and of the observation-outputting mechanism consists in two total mappings: $\sigma \colon \Omega \to S$ ascribing to each elementary state $\omega \in \Omega$ the corresponding actual state $\sigma(\omega) \in S$ of the system, and $X \colon \Omega \to E$ ascribing to ω the corresponding observed value $X(\omega) \in E$. Hence, if ω_0 is the actual elementary state of the environment, then $s_0 = \sigma(\omega_0)$ is the actual state of the system and $X(\omega_0)$ the observed empirical value. Two features of this model are worth emphasizing:

(1) All the relations between states and observations are defined by the inverse images of the corresponding values or sets of values induced in Ω by the mappings σ and X.

(2) The elementary states are inaccessible for an immediate observation and/or decision making, they can be approached only by inverse images of the sets from E induced in Ω by X.

Hence, decision making under uncertainty corresponds, at least in this setting, to the case when we know that $X(\omega_0) = x$ ($X(\omega_0) \in F$, resp.) holds for some $x \in E$ ($F \subset E$, resp.), in other words, we know that $\omega_0 \in \{\omega \in \Omega\colon X(\omega) = x\}$ ($\omega_0 \in \{\omega \in \Omega\colon X(\omega) \in F\}$, resp.), but there are $\omega_1, \omega_2 \in \{\omega \in \Omega\colon X(\omega) = x\}$ ($\omega_1, \omega_2 \in \{\omega \in \Omega\colon X(\omega) \in F\}$, resp.) such that $\sigma(\omega_1) \neq \sigma(\omega_2)$ ($\sigma(\omega_1) \in T$, but $\sigma(\omega_2) \in S - T$ for a given subset $T \subset S$, resp.). Under still another reformulation, the question whether $\sigma(\omega_0) = s_0 \in T$ holds can be answered affirmatively, having known that $X(\omega) \in F$, if $\{\omega \in \Omega\colon X(\omega) \in F\} \subset \{\omega \in \Omega\colon \sigma(\omega) \in T\}$, it can be answered negatively, on the grounds of the same knowledge, if $\{\omega \in \Omega\colon X(\omega) \in F\} \subset \{\omega \in \Omega\colon \sigma(\omega) \in S - T\}$, and it must be left unanswered otherwise, at least at this level of decision making.

As a matter of fact, the inclusion among the subsets of Ω just introduced need not be decidable by the subject in question and it is just the compatibility relation which may be useful when trying at least to approximate the three-valued decision function presented above. The definition is very simple, a state $s \in S$ is compatible with an elementary value $x \in S$, in symbols, $\rho(s,x) = 1$, if

$$\{\omega \in \Omega\colon \sigma(\omega) = s\} \cap \{\omega \in \Omega\colon X(\omega) = x\} \neq \emptyset, \tag{1.1}$$

hence, if there exists $\omega \in \Omega$ such that $\sigma(\omega) = s$ and $X(\omega) = x$. Defining, again, for each $x \in E$, by $U(x) = \{s \in S\colon \rho(s,x) = 1\}$ the set of states compatible with x, we arrive at the following three-valued decision function: decide, given $X(\omega_0)$, that $\sigma(\omega_0) \in T$ holds, if $U(X(\omega_0)) \subset T$ holds, decide that $\sigma(\omega_0) \in S-T$ holds, if $U(X(\omega)) \subset S-T$ holds, and leave the question unanswered otherwise, i.e., if $U(X(\omega_0)) \cap T \neq \emptyset$ and $U(X(\omega_0)) \cap (S-T) \neq \emptyset$. Obviously this decision function is the best approximation of the three-valued decision function introduced above, which is definable in the terms of compatibility relation and which conserves the principle that the positive as well as the negative answers are always sure, i.e., not charged by a possibility of error. Believeability and plausibility functions quantify probabilistically the three cases just introduced, namely, if $\tilde{\mathcal{A}}$ is a nonempty σ-field of subset of Ω and P is a probability measure on this σ-field, and if the mappings σ, X, and ρ are such that $\{\omega \in \Omega, U(X(\omega)) \subset T\} \in \tilde{\mathcal{A}}$ holds for each $T \subset S$ including $T = \emptyset$, where we suppose, moreover, that $P(\{\omega \in \Omega, U(X(\omega)) = \emptyset\}) < 1$, then

$$\begin{aligned}\mathrm{Bel}(T) &= P(\{\omega\colon \omega \in \Omega,\ U(X(\omega)) \subset T\} / \{\omega\colon \omega \in \Omega,\ U(X(\omega)) \neq \emptyset\}) = \\ &= \frac{P(\{\omega\colon \omega \in \Omega,\ U(X(\omega)) \subset T\})}{P(\{\omega\colon \omega \in \Omega,\ \emptyset \neq U(X(\omega))\})},\end{aligned} \tag{1.2}$$

and

$$\mathrm{Pl}(T) = 1 - \mathrm{Bel}(S-T) = \frac{P(\{\omega\colon \omega \in \Omega,\ U(X(\omega)) \cap T \neq \emptyset\})}{P(\{\omega\colon \omega \in \Omega,\ U(X(\omega)) \neq \emptyset\})}. \tag{1.3}$$

2. Partial generalized compatibility relations

A compatibility relation $\rho\colon S \times E \to \{0,1\}$ can be easily extended to a total relation $\rho^*\colon \mathcal{P}(S) \times \mathcal{P}(E)$, setting for each $T \subset S$ and each $F \subset E$ such that $T, E \neq \emptyset$,

$$\rho^*(T,F) = \max\{\rho(s,x)\colon s \in T,\ x \in F\}, \tag{2.1}$$

and setting $\rho^*(T,\emptyset) = \rho^*(\emptyset, F) = 0$ for each $T \subset S$ and each $F \subset E$. Obviously, $\rho^*(\{s\},\{x\}) = \rho(s,x)$ for each $s \in S$ and $x \in E$. Hence, $\rho^*(T,F) = 1$ iff there exist $s \in T$ and $x \in F$ such that $\rho(s,x) = 1$. If ρ is defined through the mappings σ and X by (1.1), then the relation (2.1) yields

$$\begin{aligned} \rho^*(T,F) = 1 &\iff (\exists\, s \in T)(\exists\, x \in F)\left[\{\omega \in \Omega\colon \sigma(\omega) = s\} \cap \{\omega \in \Omega\colon X(\omega) = x\} \neq \emptyset\right] \\ &\iff \Big(\bigcup_{s\in T}\{\omega \in \Omega\colon \sigma(\omega) = s\}\Big) \cap \Big(\bigcup_{x \in F}\{\omega \in \Omega\ \colon X(\omega) = x\}\Big) \neq \emptyset \\ &\iff \{\omega \in \Omega\colon \sigma(\omega) \in T\} \cap \{\omega \in \Omega\colon X(\omega) \in F\} \neq \emptyset\,. \end{aligned} \tag{2.2}$$

The extension of ρ and ρ^* defined by (2.1) and (2.2) agrees with our intuition imposed above on the notion of compatibility between states and empirical values. Or, $\rho^*(T,F) = 0$ should mean that if the observed value is in F, then the laws and rules governing the system and its environment as a whole are such that the membership of the actual state s_0 in T is impossible. In a more subjective way taken, knowing that the observed empirical value is in F, but not knowing anything more about it, we are able to prove that s_0 cannot be in T. From both these interpretations it follows immediately, that in such a case each state $s \in T$ must be incompatible with each $x \in F$, so that $\rho(s,x) = 0$ for each $s \in T$, $x \in F$, and (2.1) follows. The reasoning verifying the inverse implication, i.e., that $\rho(s,x) = 0$ for all $s \in T$ and $x \in F$ should imply $\rho^*(T,F) = 0$, is not so persuasive and immediate and is charged with a great portion of Platonistic idealization, but we shall accept it as a useful simplification for our further considerations and computations. In more detail, the case that $\rho^*(T,F) = 0$ but $\rho(s,x) = 1$ for some $s \in T$ and some $x \in F$ evidently contradicts the intuition behind and the relation (2.2), but the case when $\rho^*(T,F) = 1$ and $\rho(s,x) = 0$ for all $s \in T$ and all $x \in F$, even if also contradicts (2.2), admits an interesting interpretation. Or, consider the case when, in order to arrive at the conclusion that $\emptyset \neq T \subset S$ and $F = \{x\} \subset E$ are incompatible, we have to prove, within an appropriate deductive formalism, that $\rho(s,x) = 0$ holds for each $s \in T$ separately. If T is infinite, this cannot be sequentially done by a finite proof, so that we cannot arrive at the conclusion that $\rho^*(T,\{x\}) = 0$ and we must accept that $\rho^*(T,\{x\}) = 1$. The same situation occurs also for finite sets T supposing that only proofs not longer than a given threshold value are accepted as proofs, because of perhaps various reasons of mathematical as well as extra-mathematical nature. So, it may be also worth considering more general extensions of ρ to $\mathcal{P}(S) \times \mathcal{P}(E)$, namely, the mappings $\rho^{**}\colon \mathcal{P}(S) \times \mathcal{P}(E) \to \{0,1\}$ such that

$$\rho^{**}(T,F) \geq \max\{\rho(s,x)\colon s \in T,\ x \in E\} \tag{2.3}$$

holds for each $\emptyset \neq T \subset S$, $\emptyset \neq F \subset E$, with $\rho^{**}(T, \emptyset) = \rho^{**}(\emptyset, F) = 0$ as above.

Given a (total) compatibility relation of ρ on $S \times E$, the relation ρ^* on $\mathcal{P}(S) \times \mathcal{P}(E)$, uniquely defined by (2.1), is called *the (total) generalized compatibility relation induced (on $\mathcal{P}(S) \times \mathcal{P}(E)$) by ρ*, and each relation ρ^{**} on $\mathcal{P}(S) \times \mathcal{P}(E)$ satisfying (2.3) is called a *quasi-compatibility relation induced (on $\mathcal{P}(S) \times \mathcal{P}(E)$) by ρ*. A *partial generalized compatibility relation (partial quasi-compatibility relation*, resp.) on $\mathcal{P}(S) \times \mathcal{P}(E)$ is a mapping ρ^0 defined on a subset $\mathrm{Dom}(\rho^0) \subset \mathcal{P}(S) \times \mathcal{P}(E)$, taking its values in $\{0,1\}$ and such that there exists a total generalized compatibility relation ρ^* (quasi-compatibility relation ρ^{**}, resp.) on $\mathcal{P}(S) \times \mathcal{P}(E)$ such that ρ^0 is the restriction of ρ^* (of ρ^{**}, resp.) to $\mathrm{Dom}(\rho^0)$, in symbols $\rho^0 = \rho^* \upharpoonright \mathrm{Dom}(\rho^0)$ ($\rho^0 = \rho^{**} \upharpoonright \mathrm{Dom}(\rho^0)$, resp.).

Obviously, not every partial or total mapping $\rho^0 \colon \mathcal{P}(S) \times \mathcal{P}(E) \to \{0,1\}$ is a partial generalized compatibility relation or a partial quasi-compatibility relation on $\mathcal{P}(S) \times \mathcal{P}(E)$; as a counterexample can serve each mapping ρ^0 such that for some $T_1 \subset T_2 \subset S$ and for some $F_1 \subset F_2 \subset E$, $\{\langle T_1, F_1\rangle, \langle T_2, F_2\rangle\} \subset \mathrm{Dom}(\rho^0)$ and $\rho^0(T_1, F_1) > \rho^0(T_2, F_2)$ holds. In this paper we shall investigate, first of all, under which conditions a (partial) mapping $\rho^0 \colon \mathcal{P}(S) \times \mathcal{P}(E) \to \{0,1\}$ is a partial generalized compatibility relation and when the corresponding total generalized compatibility relation is defined unambiguously. Consequently, we shall focus our attention on the cases when a partial generalized compatibility relation is the only knowledge about the investigated system and its environment being at hand. Then, we shall try to deduce, or at least to approximate, the original compatibility function on $S \times E$ and to use this approximation in order to obtain reasonable approximations of the belief and plausibility functions defined by the original compatibility relation.

Given a partial mapping $\rho^0 \colon \mathcal{P}(S) \times \mathcal{P}(E) \to \{0,1\}$ with the domain $\mathrm{Dom}(\rho^0) \subset \mathcal{P}(S) \times \mathcal{P}(E)$ we define, for each $s \in S$ and each $x \in E$,

$$\overline{\rho}(s, x) = \min\{\rho^0(T, F) \colon \langle T, F\rangle \in \mathrm{Dom}(\rho^0),\ s \in T,\ x \in F\}, \qquad (2.4)$$

if $\{\langle T, F\rangle \in \mathrm{Dom}(\rho^0) \colon s \in T,\ x \in F\} \neq \emptyset$, $\overline{\rho}(s, x) = 1$ otherwise. We also define, for each $T \subset S$, $F \subset E$,

$$\overline{\rho}^*(T, F) = \max\{\overline{\rho}(s, x) \colon s \in T,\ x \in F\}, \qquad (2.5)$$

in other words, $\overline{\rho}^*$ is a total mapping taking $\mathcal{P}(S) \times \mathcal{P}(E)$ into $\{0,1\}$ and defined by $\overline{\rho}^* = (\overline{\rho})^*$.

THEOREM 2.1. *Let $\rho^0 \colon \mathcal{P}(S) \times \mathcal{P}(E) \to \{0,1\}$ be a partial mapping with the domain $\mathrm{Dom}(\rho^0)$, let $\overline{\rho}$ and $\overline{\rho}^*$ be defined by (2.4) and (2.5).*

(i) For each $\langle T, F\rangle \in \mathrm{Dom}(\rho^0)$ the inequality $\overline{\rho}^*(T, F) \leq \rho^0(T, F)$ holds.

(ii) Let ρ^0 be such that

(a) for each $\langle T, F\rangle \in \mathrm{Dom}(\rho^0)$ such that $\rho^0(T, F) = 0$ and each $\langle T_1, F_1\rangle \in \mathcal{P}(S) \times \mathcal{P}(E)$ such that $T_1 \subset T$ and $F_1 \subset F$, $\langle T_1, F_1\rangle \in \mathrm{Dom}(\rho^0)$ holds and $\rho^0(T_1, F_1) = 0$.

(b) *for each nonempty parametric set* Λ *and for each* $\{\langle T_\lambda, F_\lambda\rangle : \lambda \in \Lambda\} \subset \mathrm{Dom}(\rho^0)$, *if* $\langle \bigcup_{\lambda\in\Lambda} T_\lambda, \bigcup_{\lambda\in\Lambda} F_\lambda\rangle \in \mathrm{Dom}(\rho^0)$ *holds, then* $\rho^0(\bigcup_{\lambda\in\Lambda} T_\lambda, \bigcup_{\lambda\in\Lambda} F_\lambda) = \max\{\rho^0(T_\lambda, F_\lambda) : \lambda \in \Lambda\}$. *Then* $\overline{\rho}^*(T,F) = \rho^0(T,F)$ *for each* $\langle T,F\rangle \in \mathrm{Dom}(\rho^0)$.

(iii) *If* $\rho^0 = \rho^*$ *for a compatibility relation* $\rho: S \times E \to \{0,1\}$, *then* $\rho \equiv \overline{\rho}$, *i.e.,* $\rho(s,x) = \overline{\rho}(s,x)$ *for each* $s \in S$ *and each* $x \in E$.

Proof. Let $\mathrm{Dom}(\rho^0) = \emptyset$, then the equality $\rho^0 = \overline{\rho}^*$ on $\mathrm{Dom}(\rho^0)$ holds trivially. Let $\rho^0(T,F) = 1$ for each $\langle T,F\rangle \in \mathrm{Dom}(\rho^0)$, then $\overline{\rho}(s,x) = 1$ for each $s \in S$ and each $x \in E$, hence, $\overline{\rho}^*(T,F) = 1$ for each $T \subset S$ and each $F \subset E$, and the equality of ρ^0 and $\overline{\rho}^*$ on $\mathrm{Dom}(\rho^0)$ again immediately follows. So, let there exist $\langle T,F\rangle \in \mathrm{Dom}(\rho^0)$ such that $\rho^0(T,F) = 0$. Relation (2.4) yields, then, for each $s \in T$ and each $x \in F$, that

$$\overline{\rho}(s,x) = \min\{\rho^0(T_1,F_1) : \langle T_1,F_1\rangle \in \mathrm{Dom}(\rho^0),\ s \in T_1,\ x \in F_1\} \leq \\ \leq \rho^0(T,F) = 0. \tag{2.6}$$

Consequently, by (2.5),

$$\overline{\rho}^*(T,F) = \max\{\rho(s,x) : s \in S,\ x \in F\} = 0, \tag{2.7}$$

so that the inequality $\overline{\rho}^*(T,F) \leq \rho^0(T,F)$ for each $\langle T,F\rangle \in \mathrm{Dom}(\rho^0)$ immediately follows and (i) is proved.

Let the conditions of (ii) hold, let $\langle T,F\rangle \in \mathrm{Dom}(\rho^0)$ be such that $\overline{\rho}^*(T,F) = 0$. So, by (2.5), $\max\{\overline{\rho}(s,x) : s \in T,\ x \in F\} = 0$, consequently, by (2.4),

$$\min\{\rho^0(T_1,F_1) : \langle T_1,F_1\rangle \in \mathrm{Dom}(\rho^0),\ s \in T_1,\ x \in F_1\} = 0 \tag{2.8}$$

holds for each $s \in T$ and each $x \in F$. Hence, for each pair $\langle s,x\rangle \in T \times F$ there exists $\langle T_{\langle s,x\rangle}, F_{\langle s,x\rangle}\rangle \in \mathrm{Dom}(\rho^0)$ such that $s \in T_{\langle s,x\rangle}$, $x \in F_{\langle s,x\rangle}$, and $\rho(T_{\langle s,x\rangle} F_{\langle s,x\rangle}) = 0$; let us choose just one such $\langle T_{\langle s,x\rangle}, F_{\langle s,x\rangle}\rangle$ for each $\langle s,x\rangle \in T \times F$ using the axiom of choice. Set, for each $\langle s,x\rangle \in T \times F$, $T^0_{\langle s,x\rangle} = T \cap T_{\langle s,x\rangle}$, $F^0_{\langle s,x\rangle} = F \cap F_{\langle s,x\rangle}$, then $s \in T^0_{\langle s,x\rangle}$ and $x \in F^0_{\langle s,x\rangle}$ hold for each $s \in T$ and each $x \in F$, moreover, $\bigcup_{\langle s,x\rangle \in T\times F} T^0_{\langle s,x\rangle} = T$ and $\bigcup_{\langle s,x\rangle \in T\times F} F^0_{\langle s,x\rangle} = F$. By (ii) (a), $\langle T^0_{\langle s,x\rangle}, F^0_{\langle s,x\rangle}\rangle \in \mathrm{Dom}(\rho^0)$ and $\rho^0(T^0_{\langle s,x\rangle}, F^0_{\langle s,x\rangle}) = 0$ for each $s \in T$ and each $x \in F$, so that, by (ii) (b), $\langle \bigcup_{\langle s,x\rangle \in T\times F} T^0_{\langle s,x\rangle}, \bigcup_{\langle s,x\rangle \in T\times F} F^0_{\langle s,x\rangle}\rangle \in \mathrm{Dom}(\rho^0)$ and

$$\rho^0\Big(\bigcup_{\langle s,x\rangle \in T\times F} T^0_{\langle s,x\rangle}, \bigcup_{\langle s,x\rangle \in T\times F} F^0_{\langle s,x\rangle}\Big) = \rho^0(T,F) = \\ = \max\{\rho^0(T^0_{\langle s,x\rangle}, F^0_{\langle s,x\rangle}) : \langle s,x\rangle \in T \times F\} = 0. \tag{2.9}$$

Consequently, $\overline{\rho}^*(T,F) = 0$ implies $\rho^0(T,F) = 0$ and this implication, combined with (i), yields that $\overline{\rho}^*(T,F) = \rho^0(T,F)$ for each $\langle T,F\rangle \in \mathrm{Dom}(\rho^*)$ and (ii) is proved.

Let $\rho^0 = \rho^*$ for a compatibility relation $\rho\colon S\times E \to \{0,1\}$, so that $\rho^0(T,F) = \max\{\rho(s,x)\colon s \in T,\ x \in F\}$ for each $T \subset S$ and each $F \subset E$. If $T_1 \subset T$ and $F_1 \subset F$, then, obviously, $\rho^0(T_1,F_1) \le \rho^0(T,F)$, so that $\rho^0(T,F) \ge \rho^0(\{s\},\{x\})$ holds for each $s \in T$ and each $x \in F$. Consequently,

$$\begin{aligned}\overline{\rho}(s,x) &= \min\{\rho^0(T,F)\colon s \in T\,,\ x \in F\} = \rho^0(\{s\},\{x\}) = \rho^*(\{s\},\{x\}) = \\ &= \max\{\rho(s_1,x_1)\colon s_1 \in \{s\}\,,\ x_1 \in \{x\}\} = \rho(s,x)\end{aligned} \tag{2.10}$$

holds for each $s \in T$ and each $x \in F$, so that (iii) is proved. The proof of Theorem 2.1 is completed. □

THEOREM 2.2. *Let $\rho^0\colon \mathcal{P}(S)\times\mathcal{P}(E) \to \{0,1\}$ be a partial mapping with the domain $\mathrm{Dom}(\rho^0)$ such that $\rho^0(T_1,F_1) \ge \rho^0(T_2,F_2)$ holds for each $\{\langle T_1,F_1\rangle, \langle T_2,F_2\rangle\} \subset \mathrm{Dom}(\rho^0)$ such that $T_1 \supset T_2$ and $F_1 \supset F_2$, let $s \in S$ and $x \in E$ be such that $\langle\{s\},\{x\}\rangle \in \mathrm{Dom}(\rho^0)$, let $\overline{\rho}$ be defined by (2.4). Then $\overline{\rho}(s,x) = \rho^0(\{s\},\{x\})$.*

P r o o f. By (2.4),

$$\begin{aligned}\overline{\rho}(s,x) &= \min\{\rho^0(T,F)\colon \langle T,F\rangle \in \mathrm{Dom}(\rho^0)\,,\ s \in T\,,\ x \in F\} \le \\ &\le \rho^0(\{s\},\{x\})\,,\end{aligned} \tag{2.11}$$

as $\langle\{s\},\{x\}\rangle \in \mathrm{Dom}(\rho^0)$, $s \in \{s\}$, and $x \in \{x\}$. However, $\rho^0(T,F) \ge \rho^0(\{s\},\{x\})$ holds for each $\langle T,F\rangle \in \mathrm{Dom}(\rho^0)$ such that $s \in T$ and $x \in F$ due to the conditions of Theorem 2.2. Hence, $\overline{\rho}(s,x) \ge \rho^0(\{s\},\{x\})$ immediately follows and the proof is completed. □

THEOREM 2.3. *Let $\rho^0\colon \mathcal{P}(S)\times\mathcal{P}(E) \to \{0,1\}$ be a partial generalized compatibility relation such that $\rho^0 = \rho^* \restriction \mathrm{Dom}(\rho^0)$ for a compatibility relation ρ on $S\times E$, let $\overline{\rho}$ be defined by ρ^0 using (2.4). Then $\overline{\rho}(s,x) \ge \rho(s,x)$ holds for each $s \in S$ and each $x \in E$.*

P r o o f. An easy calculation yields that

$$\begin{aligned}\overline{\rho}(s,x) &= \min\{\rho^0(T,F)\colon \langle T\,,F\rangle \in \mathrm{Dom}(\rho^0)\,,\ s \in T\,,\ x \in F\} \ge \\ &\ge \min\{\rho^*(T,F)\colon s \in T \subset S\,,\ x \in F \subset E\} = \\ &= \rho^*(\{s\}\,,\{x\}) = \rho(s,x)\,,\end{aligned} \tag{2.12}$$

as the inequality $\rho^*(T,F) \ge \rho^*(\{s\},\{x\})$ obviously holds for each $T \subset S$ and $F \subset E$ such that $s \in T$ and $x \in F$. □

As can be easily proved, the inequality in the assertion of Theorem 2.3 cannot be, in general, replaced by equality. Or, let $f\colon E \to S$ be a total function such that $\rho(s,x) = 1$ iff $s = f(x)$, $\rho(s,x) = 0$ otherwise. So,

$$\begin{aligned} \rho^*(S,E) &= \max\{\rho(s,x) \colon s \in S\,,\ x \in E\} \\ &= \max\{\rho(f(x),x) \colon x \in E\} = 1\,. \end{aligned} \tag{2.13}$$

Consequently, for each $s \in S$ and each $x \in E$, if $\mathrm{Dom}(\rho^0) = \{\langle S,E\rangle\}$, then

$$\begin{aligned} \overline{\rho}(s,x) &= \min\{\rho^0(T,F) \colon \langle T,F\rangle \in \mathrm{Dom}(\rho^0)\,,\ s \in T\,,\ x \in F\} = \\ &= \rho^0(S,E) = \rho^*(S,E) = 1\,, \end{aligned} \tag{2.14}$$

so that $\overline{\rho}(s,x) > \rho(s,x)$ holds for each $s \in S$, $s \in E$ such that $s \neq f(x)$.

Before focusing our attention on a more detailed investigation of partial generalized compatibility relations we take as worth saying explicitly, that compatibility relations on $\mathcal{P}(S)\times\mathcal{P}(E)$ can be defined not only by extending compatibility relations defined on $S \times E$ to $\mathcal{P}(S) \times \mathcal{P}(E)$ by (2.1), but also directly, taking $S_0 = \mathcal{P}(S)$ instead of S and $E_0 = \mathcal{P}(E)$ instead of E in the general definition of compatibility relation. Such a compatibility relation $\rho^0 \colon S_0 \times E_0 \to \{0,1\}$ cannot be, in general, defined by an extension of a compatibility relation defined on $S \times E$, or as a fragment of such an extension, in the case when ρ^0 is partial, as it is possible that $\rho^0(T,F) = 0$, but $\rho^0(T_1,F_1) = 1$ for some $T_1 \subset T \subset S$, $F_1 \subset F \subset E$. So, such a compatibility relation on $\mathcal{P}(S) \times \mathcal{P}(E)$ can be taken as a relation between a metasystem the states of which are sets of states of the original system, and an enriched observation space the elements of which are sets of original empirical values. A more detailed mathematical investigation and interpretations of such meta-systems, meta-observations and corresponding compatibility relations would be interesting and perhaps useful, but it would exceed the intended scope of this paper and will be postponed till some other occasion.

3. Believeability functions defined by partial generalized compatibility relations

In order to simplify our further reasonings by avoiding technical difficulties we shall suppose, unless stated otherwise, that both the spaces S and E are finite. We shall consider the case when the empirical results (values from E) being at the subject's disposal, when she/he tries to identify or at least to specify the actual state of the investigated system, are of random (stochastic, statistical) nature and can be described, quantified and processed using the tools of the classical (Kolmogorov axiomatic) probability theory. To describe this situation within the framework of an appropriate mathematical apparatus, namely, that of the "deterministic" model of decision making under uncertainty as already mentioned above, we shall suppose that we have at our disposal, first of all, an abstract probability space $\langle \Omega, \tilde{\mathcal{A}}, P \rangle$. Here Ω is a nonempty set, $\tilde{\mathcal{A}}$ is a nonempty σ-field of subsets of Ω, and P is a probability (measure), i.e., a non-negative, normalized and σ-additive measure, defined on $\tilde{\mathcal{A}}$. Two mappings, $\sigma\colon \Omega \to S$ and $X\colon \Omega \to E$, are supposed to be defined on Ω and they are supposed to be measurable with respect to the σ-fields $\mathcal{P}(S)$ and $\mathcal{P}(E)$, hence, the inclusions

$$\big\{\{\omega \in \Omega\colon \sigma(\omega) \in T\}\colon T \subset S\big\} \subset \tilde{\mathcal{A}}, \quad \big\{\{\omega \in \Omega\colon X(\omega) \in F\}\colon F \subset E\big\} \subset \tilde{\mathcal{A}} \tag{3.1}$$

are supposed to be valid. For finite S and E, (3.1) reduces to

$$\big\{\{\omega \in \Omega\colon \sigma(\omega) = s\}\colon s \in S\big\} \subset \tilde{\mathcal{A}}, \quad \big\{\{\omega \in \Omega\colon X(\omega) = x\}\colon x \in E\big\} \subset \tilde{\mathcal{A}}. \tag{3.2}$$

Let $\rho\colon S \times E \to \{0,1\}$ be a compatibility relation, let $U(x) = \big\{s \in S\colon \rho(s,x) = 1\big\}$ denote the set of states compatible with the empirical value $x \in E$. Setting $X(\omega)$ instead of x we obtain $U\big(X(\omega)\big)$, i.e., a mapping $U(X)\colon \Omega \to \mathcal{P}(S)$, and we can define, for each given $T \subset S$, the set $\big\{\omega \in \Omega\colon U\big(X(\omega)\big) = T\big\}$. An easy calculation yields that

$$\big\{\omega \in \Omega\colon U\big(X(\omega)\big) = T\big\} = \bigcup_{x \in E\,,\ U(x) = T} \big\{\omega \in \Omega\colon X(\omega) = x\big\}, \tag{3.3}$$

and this set is in the σ-field $\tilde{\mathcal{A}}$ due to the assumption of measurability of X and due to the finiteness of E. Consequently, also the inclusion

$$\big\{\omega \in \Omega\colon U\big(X(\omega)\big) \subset T\big\} = \bigcup_{A \subset T} \big\{\omega \in \Omega\colon U\big(X(\omega)\big) = A\big\} \in \tilde{\mathcal{A}} \tag{3.4}$$

obviously holds, so that we can define the conditional probability

$$\begin{aligned} \mathrm{Bel}(T) &= P(\{\omega \in \Omega\colon U\big(X(\omega)\big) \subset T\} / \{\omega \in \Omega\colon U\big(X(\omega)\big) \neq \emptyset\}) \\ &= \frac{P(\{\omega \in \Omega\colon \emptyset \neq U\big(X(\omega)\big) \subset T\})}{P(\{\omega \in \Omega\colon U\big(X(\omega)\big) \neq \emptyset\})}, \end{aligned} \tag{3.5}$$

if $P(\{\omega \in \Omega\colon U\big(X(\omega)\big) \neq \emptyset\})$ is positive. In order to express explicitly the role of the compatibility function ρ in the definition of Bel, to which our attention in this paper will be focused, we shall write $\mathrm{Bel}_\rho(T)$ or $\mathrm{Bel}(\rho, T)$ instead of Bel. We shall also adopt the simplifying notation omitting the symbols $\ldots\omega\colon \omega \in \Omega\ldots$ or $\ldots\omega \in \Omega\ldots$ in expressions like (3.3), (3.4) or (3.5). The value $\mathrm{Bel}_\rho(T)$ is called (*the degree*) *of belief of* (or: *ascribed to*) *the subset T of S with respect to the compatibility function ρ and probability measure P defined on $\langle \Omega, \tilde{\mathcal{A}} \rangle$*. The dual value $\mathcal{P}l(T)$ or $\mathcal{P}l_\rho(T)$, defined by

$$\mathcal{P}l_\rho(T) = 1 - \mathrm{Bel}_\rho(S - T), \tag{3.6}$$

is called the *plausibility* of T.

Let $\rho^0\colon \mathcal{P}(S) \times \mathcal{P}(E) \to \{0,1\}$ be a partial mapping, let ρ be defined by (2.4). Then we set $\mathrm{Bel}_{\rho^0}(T) = \mathrm{Bel}_{\overline{\rho}}(T)$ for each $T \subset S$.

DEFINITION 3.1. A compatibility relation ρ defined on $S \times E$ is called *consistent*, if for each $x \in E$ there exists $s \in S$ such that $\rho(s,x) = 1$.

Before discussing this notion in more detail we shall introduce some of its almost trivial consequences. If ρ is consistent, then, obviously, $U_\rho(x) \neq \emptyset$ holds for each $x \in E$, so $P(\{U_\rho(X(\omega)) \neq \emptyset\}) = 1$ and the equality $\mathrm{Bel}_\rho(T) = P(\{U_\rho(X(\omega)) \subset T\})$ holds for each $T \subset S$. Hence, the following assertion is almost obvious.

THEOREM 3.1. *Let the notations and conditions of Theorem 2.3 hold, let ρ be consistent. Then $\mathrm{Bel}_{\overline{\rho}}(T) \leq \mathrm{Bel}_\rho(T)$ and $\mathcal{P}l_{\overline{\rho}}(T) \geq \mathcal{P}l_\rho(T)$ hold for each $T \subset S$.*

Proof. By Theorem 2.3, $\overline{\rho}(s,x) \geq \rho(s,x)$ holds for each $s \in S$, $x \in E$, so that

$$U_{\overline{\rho}}(x) = \{s \in S \colon \overline{\rho}(s,x) = 1\} \supset \{s \in S \colon \rho(s,x) = 1\} = U_\rho(x). \tag{3.7}$$

Hence, $U_{\overline{\rho}}(x) \neq \emptyset$ holds for each $x \in E$, $\mathrm{Bel}_{\overline{\rho}}(T) = P(\{U_{\overline{\rho}}(X(\omega)) \subset T\})$, moreover, $U_{\overline{\rho}}(X(\omega)) \supset U_\rho(X(\omega))$ is valid for each $\omega \in \Omega$. Consequently, for each $T \subset S$, if $U_{\overline{\rho}}(X(\omega)) \subset T$, then $U_\rho(X(\omega)) \subset T$, in other terms,

$$\{\omega \in \Omega \colon U_{\overline{\rho}}(X(\omega)) \subset T\} \subset \{\omega \in \Omega \colon U_\rho(X(\omega)) \subset T\}, \tag{3.8}$$

and this inclusion immediately yields that

$$\mathrm{Bel}_{\overline{\rho}}(T) = P(\{U_{\overline{\rho}}(X(\omega)) \subset T\}) \leq P(\{U_\rho(X(\omega)) \subset T\}) = \mathrm{Bel}_\rho(T). \tag{3.9}$$

The dual assertion for plausibility functions is obvious so that the assertion is proved. □

It is perhaps worth saying explicitly that if the basic compatibility relation ρ is not consistent, then the inequality (3.9) need not hold, as the following example illustrates.

Let $S = \{s_1, s_2, s_3\}$, let $E = \{x_1, x_2, x_3\}$, let $p(x_i) = P(\{X(\omega) = x_i\}) = 1/3$ for each $i = 1,2,3$. Let the compatibility relation ρ on $S \times E$ be defined as follows: $\rho(s_1, x_1) = 1$, $\rho(s_i, x_3) = 1$ for each $i = 1,2,3$, $\rho(s_i, x_j) = 0$ otherwise. Hence ρ is not consistent, as there is no state s_i compatible with the empirical value x_2. Recalling that $U_\rho(x_i) = \{s \in S \colon \rho(s, x_i) = 1\}$ we obtain easily that $U_\rho(x_1) = \{s_1\}$, $U_\rho(x_2) = \emptyset$, $U_\rho(x_3) = S = \{s_1, s_2, s_3\}$. Set $T_0 = \{s_1, s_2\} \subset S$, $F_0 = \{x_1, x_2\} \subset E$, an easy calculation yields that

$$\begin{aligned} \mathrm{Bel}_\rho(T_0) &= P(\{U_\rho(X(\omega)) \subset T_0\} / \{U_\rho(X(\omega)) \neq \emptyset\}) = \\ &= \frac{P(\{\emptyset \neq U_\rho(X(\omega)) \subset T_0\})}{P(\{\emptyset \neq U_\rho(X(\omega))\})} = \frac{\sum_{x \in E,\ \emptyset \neq U(x) \subset T_0} p(x)}{\sum_{x \in E,\ \emptyset \neq U(x)} p(x)} = \\ &= \frac{p(x_1)}{p(x_1) + p(x_3)} = \frac{1/3}{1/3 + 1/3} = 1/2. \end{aligned} \tag{3.10}$$

For the generalized compatibility relation ρ^* induced by ρ we obtain that

$$\rho^*(T_0, F_0) = \max\{\rho(s,x)\colon s \in T_0\,,\ x \in F_0\} \geq \rho(s_1, x_1) = 1\,, \tag{3.11}$$

as $s_1 \in T_0$ and $x_1 \in F_0$. Moreover,

$$\rho^*(T, \{x_3\}) = \max\{\rho(s,x)\colon s \in T\,,\ x \in \{x_3\}\} = 1 \tag{3.12}$$

for each T, $\emptyset \neq T \subset S$, and

$$\rho^*(\{s_3\}, \{x_1\}) = \rho(s_3, x_1) = 0 = \rho(s_3, x_2) = \rho^*(\{s_3\}, \{x_2\})\,. \tag{3.13}$$

Let $\rho^0 = \rho^* \restriction \mathrm{Dom}(\rho^0)$, where

$$\begin{aligned}\mathrm{Dom}(\rho^0) = \{\langle T_0, F_0\rangle\,,\ \langle\{s_3\}, \{x_1\}\rangle\,,\ \langle\{s_3\}\,,\ \{x_2\}\rangle\} \cup \\ \cup \{\langle T, \{x_3\}\rangle\colon \emptyset \neq T \subset S\}\,.\end{aligned} \tag{3.14}$$

We obtain easily that for both $i = 1, 2$, $j = 1, 2$,

$$\begin{aligned}\overline{\rho}(s_i, x_j) &= \min\{\rho^0(T,F)\colon \langle T,F\rangle \in \mathrm{Dom}(\rho^0)\,,\ s_i \in T\,,\ x_j \in F\} = \\ &= \rho^0(T_0, F_0) = \rho^*(T_0, F_0) = 1\,,\end{aligned} \tag{3.15}$$

as $\langle T_0, F_0\rangle$ is the only pair $\langle T, F\rangle$ in $\mathrm{Dom}(\rho^0)$ such that $s_i \in T$ and $x_j \in F$ hold simultaneously for $i = 1$ or 2 and $j = 1$ or 2. Moreover, for $i = 1, 2, 3$

$$\begin{aligned}\overline{\rho}(s_i, x_3) &= \min\{\rho^0(T,F)\colon \langle T,F\rangle \in \mathrm{Dom}(\rho^0)\,,\ s_i \in T\,,\ x_3 \in F\} = \\ &= \min\{\rho^0(T, \{x_3\})\colon s_i \in T\} = \min\{\rho^*(T, \{x_3\})\colon s_i \in T\} = 1\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}\overline{\rho}(s_3, x_1) &= \min\{\rho^0(T,F)\colon \langle T,F\rangle \in \mathrm{Dom}(\rho^0)\,,\ s_3 \in T\,,\ x_1 \in F\} = \\ &= \rho^0(\{s_3\}, \{x_1\}) = \rho^*(\{s_3\}, \{x_1\}) = \rho(s_3, x_1) = 0\,,\end{aligned} \tag{3.17}$$

as well as

$$\begin{aligned}\overline{\rho}(s_3, x_2) &= \min\{\rho^0(T,F)\colon \langle T,F\rangle \in \mathrm{Dom}(\rho^0)\,,\ s_3 \in T\,,\ x_2 \in F\} = \\ &= \rho^0(\{s_3\}, \{x_2\}) = \rho^*(\{s_3\}, \{x_2\}) = \rho(s_3, x_2) = 0\,.\end{aligned} \tag{3.18}$$

So

$$U_{\overline{\rho}}(x_1) = \{s \in S\colon \overline{\rho}(s, x_1) = 1\} = \{s_1, s_2\} \quad \text{by (3.15) and (3.17)}\,, \tag{3.19}$$

analogously, by (3.15) and (3.18), we obtain that

$$U_{\overline{\rho}}(x_2) = \{s_1, s_2\}\,. \tag{3.20}$$

Finally, (3.16) yields that

$$U_{\overline{\rho}}(x_3) = S = \{s_1, s_2, s_3\}\,. \tag{3.21}$$

So, $U_{\overline{\rho}}(x) \neq \emptyset$ for all $x \in E$, and an easy calculation yields that

$$\begin{aligned} \mathrm{Bel}_{\overline{\rho}}(T_0) &= P\big(\{U_{\overline{\rho}}(X(\omega)) \subset T_0\} \,/\, \{U_{\overline{\rho}}(X(\omega)) \neq \emptyset\}\big) \\ &= P\big(\{U_{\overline{\rho}}(X(\omega)) \subset T_0\}\big) \\ &= \sum_{x \in E\,,\ U_{\overline{\rho}}(x) \subset T_0} p(x) = p(x_1) + p(x_2) \\ &= 1/3 + 1/3 = 2/3 > 1/2 = \mathrm{Bel}_{\rho}(T_0) \quad \text{by (3.10)}\,. \end{aligned} \tag{3.22}$$

Hence, the inequality (3.9) does not hold. □

As the example just presented shows, if the basic compatibility relation ρ on $S \times E$ is not consistent, then its behaviour and the properties of the corresponding belief functions are rather counter-intuitive. Namely, having at our disposal only a partial knowledge about the compatibility relation ρ, i.e., the knowledge encoded by a fragment of the induced generalized compatibility relation, we can arrive at *higher* values of the belief function for some subsets of S. This fact follows from a more general paradoxical property of belief functions according to which enriching the database by new items which are inconsistent with the former ones can augment degree of belief for some sets of states. It is just this strange property which, together with the technical difficulties involved by the apparatus of conditional probabilities, makes a great portion of specialists dealing with the D.–S. theory to consider just the case of consistent compatibility relations. Another solution may be, to abandon the assumption of closed world, i.e., to admit that there are also some possible states of the system which are not in S and to take the case when the data are inconsistent as the indication that the actual state of the system is beyond the set S. However, because of the limited extent of this paper there is no time to go on with this discussion; let us refer to [2], [4], and [5] for some details.

The next assertion generalizes Theorem 3.1 in the sense that two partial generalized compatibility relations induced by the same compatibility relation on $S \times E$ and with domains ordered by set-theoretical inclusion can be compared as far as the corresponding belief functions are concerned.

THEOREM 3.2. *Let $\rho^1, \rho^2 \colon \mathcal{P}(S) \times \mathcal{P}(E) \to \{0,1\}$ be two partial generalized compatibility relations such that $\rho^i = \rho^* \restriction \mathrm{Dom}(\rho^i)$ for both $i = 1,2$, and for a consistent compatibility relation ρ on $S \times E$, let $\mathrm{Dom}(\rho^1) \subset \mathrm{Dom}(\rho^2) \subset \mathcal{P}(S) \times \mathcal{P}(E)$ hold. Let*

$$\overline{\rho}^i(s,x) = \min\{\rho^i(T,F) \colon \langle T\,,\ F\rangle \in \mathrm{Dom}(\rho^i)\,,\ s \in T\,,\ x \in F\} \tag{3.23}$$

for both $i = 1,2$ and for all $s \in T$, $x \in E$, for which this value is defined, let $\overline{\rho}^i(s,x) = 1$ otherwise. Then the inequalities $\mathrm{Bel}_{\overline{\rho}^1}(T) \leq \mathrm{Bel}_{\overline{\rho}^2}(T)$ and $\mathcal{P}l_{\overline{\rho}^1}(T) \geq \mathcal{P}l_{\overline{\rho}^2}(T)$ hold for each $T \subset S$.

P r o o f . For each $\langle T,\, F\rangle$, $T \subset S$, $F \subset E$, if $\langle T,\, F\rangle \in \mathrm{Dom}(\rho^1)$, then $\langle T,\, F\rangle \in \mathrm{Dom}(\rho^2)$ and, moreover, $\rho^1(T,F) = \rho^*(T,F) = \rho^2(T,F)$, as both ρ^1

and ρ^2 result from restrictions of the same generalized compatibility function ρ^2 to various domains. Hence, for each $s \in S$, $x \in E$ such that $\rho^1(s,x)$ is defined by (3.23),

$$\begin{aligned} \overline{\rho}^1(s,x) &= \min\{\rho^1(T,F)\colon \langle T,F\rangle \in \mathrm{Dom}(\rho^1),\ s \in T,\ x \in F\} \\ &\geq \min\{\rho^2(T,F)\colon \langle T,F\rangle \in \mathrm{Dom}(\rho^2),\ s \in T,\ x \in F\} \\ &= \overline{\rho}^2(s,x)\,. \end{aligned} \tag{3.24}$$

If $\overline{\rho}^1(s,x)$ is not defined by (3.23), then $\overline{\rho}^1(s,x) = 1$ and the inequality $\overline{\rho}^1(s,x) \geq \overline{\rho}^2(s,x)$ holds trivially. Setting, for both $i = 1,2$ and for each $x \in E$,

$$U_{\overline{\rho}^i}(x) = \{s \in S\colon \overline{\rho}^i(s,x) = 1\}\,, \tag{3.25}$$

we obtain easily that, for each $x \in E$, $U_{\overline{\rho}^1}(x) \supset U_{\overline{\rho}^2}(x)$, and both these sets are nonempty (both of them contain $U_\rho(s,x) \neq \emptyset$, as the relation ρ is supposed to be consistent). As in the proof of Theorem 3.1 we obtain that the inclusion $U_{\overline{\rho}^1}(X(\omega)) \supset U_{\overline{\rho}^2}(X(\omega))$ holds for each $\omega \in \Omega$, consequently, for each $T \subset S$,

$$\{\omega \in \Omega\colon U_{\overline{\rho}^1}(X(\omega)) \subset T\} \subset \{\omega \in \Omega\colon U_{\overline{\rho}^2}(X(\omega)) \subset T\}\,, \tag{3.26}$$

which immediately yields that

$$\begin{aligned} \mathrm{Bel}_{\overline{\rho}^1}(T) &= P(\{\omega \in \Omega\colon U_{\overline{\rho}^1}(X(\omega)) \subset T\}) \\ &\leq P(\{\omega \in \Omega\colon U_{\overline{\rho}^2}(X(\omega)) \subset T\}) = \mathrm{Bel}_{\overline{\rho}^2}(T)\,. \end{aligned} \tag{3.27}$$

The dual inequality for the plausibility functions follows trivially, so that the assertion is proved. □

4. Conclusion

As follows from Theorems 3.1 and 3.2, belief function $\mathrm{Bel}_{\overline{\rho}}$, defined by fragments of the generalized belief function generated by an original compatibility relation ρ, is a lower approximation of the original belief function Bel_ρ. This approximation can be improved, i.e., Bel_ρ can be approximated more closely, if the fragments being at our disposal are enriched by a new part. Consequently, when using the original belief function Bel_ρ in decision rules according to which the hypothesis that the actual state of the system is in $T \subset S$ is accepted supposing that $\mathrm{Bel}_\rho(T) \geq \alpha$ holds for some threshold value α close enough to one, this decision rule can be replaced, conserving the pessimistic worst-case principle typical for the Dempster–Shafer way of reasoning, by a more severe rule which accepts the same hypothesis when $\mathrm{Bel}_{\overline{\rho}}(T) \geq \alpha$ holds. On the other hand, knowing that the last inequality holds, we do not need to compute the value $\mathrm{Bel}_\rho(T)$, which may be much more time and space consuming, to be able to decide that $\mathrm{Bel}_\rho(T) \geq \alpha$ holds. At least the two following ways of further development are worth considering. (1) to apply our reasonings to particular partial generalized compatibility relations, e.g., to those generated by appropriate equivalence relations on the spaces S and E, to arrive at more detailed results than those introduced above, and (2) to compute the time and/or space computational savings achieved when replacing Bel_ρ by $\mathrm{Bel}_{\overline{\rho}}$ in decision rules like that one mentioned above. However, a more detailed investigation in some of these direction would bring us far beyond the limited extent of this contribution and must be, therefore, postponed till another occasion.

Because of the fact that the reasonings presented above are very simple, this paper is almost self-explanatory and so we do not need to take profit of already very numerous books, papers and other items dealing with the Dempster–Shafer approach to uncertainty quantification and processing and with the related topics. Therefore we limit our references to monographs [1] and [3], which can serve as a useful introduction for anybody wanting to get familiar with this domain.

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