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ONE APPROACH TO DECISION MAKING WITH FUZZY GOALS

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ABSTRACT. A decision making problem consisting in the choice of intensities of application of n means to reach m given fuzzy goals is formulated in two versions, which lead to solving nonstandard optimization problems. The solution procedures follow from [1], [2]. Modifications and extensions are briefly discussed.

1. Necessary concepts and notations

We shall assume that the membership functions of all fuzzy sets considered in this contribution are defined on a finite set $N = \{1, \dots, n\}$, so that for any fuzzy set A, its membership function is $m_A \colon N \to [0,1]$; $\alpha \wedge \beta \equiv \min(\alpha, \beta)$ for any reals α, β. If A, B are two fuzzy sets with membership functions m_A, m_B we set as usual:

$$m_{A \cap B}(j) = m_A(j) \wedge m_B(j) \qquad \text{for all} \quad j \in N\,.$$

If A is a fuzzy set with a membership function m_A, then $h(A) = \max_{j \in N} m_A(j)$ is called the *height* of A. Therefore

$$h(A \cap B) = \max_{j \in N} \big(m_A(j) \wedge m_B(j)\big)\,.$$

We set further for any $x^T = (x_1, \dots, x_k) \in \mathbb{R}^k$:

$$\|x\| \equiv \max_{1 \le j \le k} |x_j|\,, \quad \text{the Tshebyshev norm}\,.$$

2. Motivation

Suppose that we have at our disposal n means (e.g., medicines, remedies) to reach m goals (e.g., to cure m diseases, defects). For each goal $i \in S \equiv$

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$\{1,\ldots,m\}$, there is given a fuzzy set $A^{(i)}$ of "appropriate or effective means for the goal i" with membership function $m_{A_i}\colon N \to [0,1]$. The value $m_{A_i}(j)$ can be interpreted for instance as a given level of "appropriacy" or "effectivity" of the means (medicine) j to reach the goal i (e.g., to cure the disease or defect i). We want to find the vector $(x_1,\ldots,x_n)$ of "intensities", ($x_j \in [0,1]$ for all $j \in N$), with which the means $1,\ldots,n$ are applied. The values x_j can be interpreted as the values of membership function $m_X\colon N \to [0,1]$ of a fuzzy set X of "used" or "applied" means, so that $x_j = m_X(j)$ for all $j \in N$. It is natural to assume that the effectivity of any means j increases if the intensity x_j increases up to a certain level and then it does not increase any more with further increase of x_j, e.g., it can remain on the level $m_{A_i}(j)$. If we denote $a_{ij} \equiv m_{A_i}(j)$ for all i,j, then such behaviour of the "effectivity" $e_{ij}(x_j)$ of the jth means with respect to the ith goal depending on the intensity x_j can be described by a function

$$e_{ij}(x_j) \equiv a_{ij} \wedge x_j$$

(compare Fig. 2.1).

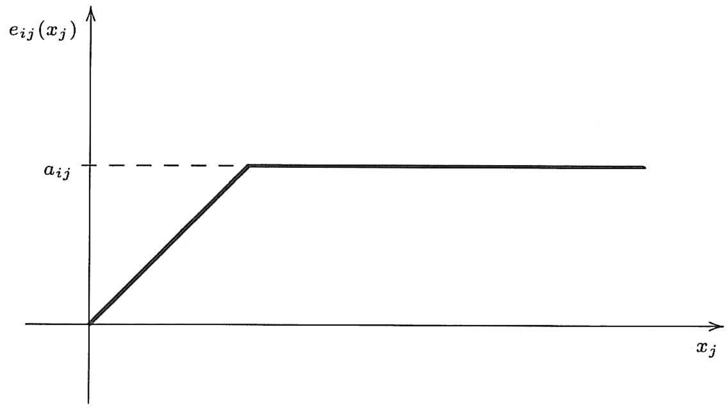

FIGURE 2.1.

Remark 2.1. We could consider also other types of functions $e_{ij}(x_j)$ as, e.g., $e_{ij}(x_j) \equiv \max(0, x_j - b_{ij}) \wedge a_{ij}$ (compare Fig. 2.2) or $e_{ij}(x_j)$, which is no more partially linear. We shall confine ourselves here to the simplest case $e_{ij}(x_j) = a_{ij} \wedge x_j$, but some considerations given in the sequel can be extended to more general cases with increasing e_{ij}'s.

We must describe now the conditions posed on x_j's (besides the trivial condition that $x_j \in [0,1]$ for all $j \in N$). We can require for instance that x_j must be chosen in such a way that the maximal effectivity with respect to any goal $i \in S$ is greater than or equal to a given level $b_i \in [0,1]$, i.e., $\max\limits_{j\in N} e_{ij}(x_j) \geq b_i$. We will

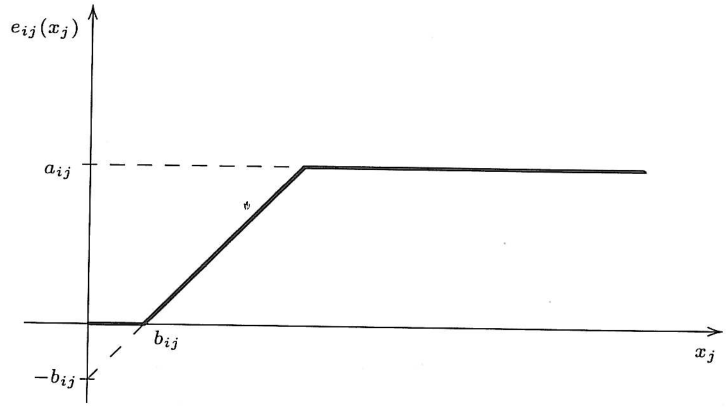

FIGURE 2.2.

require at the same time that x_j is as close as possible to some "recommended" intensities $\hat{x}_j$ in the Tshebyshev norm.

Another possibility is to choose x_j's in such a way that the vector of maximal effectivities with respect to goals $1, 2, \ldots, m$, $(\max_{j \in N} l_{1j}(x_j), \ldots, \max_{j \in N} l_{mj}(x_j))$, is as close as possible to some prescribed vector of levels $b^T = (b_1, \ldots, b_m)$ in the sense of the Tsebyshev norm. Using the usual notation of mathematical optimization and the notations introduced above with $l_{ij}(x_j) \equiv a_{ij} \wedge x_j$, we want to solve the optimization problems:

$$\|x - \hat{x}\| \to \min$$

subject to

$$h(A_i \cap [X) \equiv \max_{j \in N}(a_{ij} \wedge x_j) \geq b_i\,, \qquad i - 1, \ldots, m\,,$$
$$h_j \leq x_j \leq H_j\,, \qquad j = 1, \ldots, n\,, \tag{P1}$$

in the first case and in the second case

$$\max_{1 \leq i \leq m} \left| \max_{j \in N}(a_{ij} \wedge x_j) - b_i \right| \to \min \tag{P2}$$

subject to

$$h_j \leq x_j \leq H_j, \qquad j = 1, \ldots, n\,,$$

where instead of $0 \leq x_j \leq 1$, we require that $h_j \leq x_j \leq H_j$, where h_j, H_j are some prescribed bounds from $[0, 1]$. The given values h_j, H_j, $\hat{x}_j$ may follow for instance from some technological, medical or biological considerations.

It will be shown in the sequel that the nonstandard optimization (P1), (P2) can be solved using the results of [2], [1], respectively.

3. Solution Method for (P1)

In order to solve problem (P1) we may use the method described in [2]. If we set

$$r_{ij}(x_j) \equiv a_{ij} \wedge x_j\,, \quad f_j(x_j) \equiv |x_j - \hat{x}_j| \qquad \text{and}$$
$$V_{ij} \equiv \{x_j \mid h_j \leq x_j \leq H_j\,, \quad r_{ij}(x_j) \geq b_i\} \qquad \text{for all} \quad i, j\,,$$

then all assumptions of Theorem 2 of [2] (see [2], p. 360) are satisfied, if the set of feasible solutions of (P1) is nonempty, because r_{ij}'s are continuous and nondecreasing and $f_j's$ are continuous. We bring in the sequel a specialized version of the algorithm from [2] for the solution of (P1).

It holds in our case:

$$V_{ij} = \{x_j \mid h_j \leq x_j \leq H_j,\ a_{ij} \wedge x_j \geq b_i\} = \begin{cases} [b_i, H_j]\,, & \text{if } a_{ij} \leq b_i \text{ and } b_i \leq H_j\,, \\ \emptyset\,, & \text{otherwise}\,. \end{cases}$$

Therefore if R is an arbitrary subset of S and $V_{ij} \neq \emptyset$ for all $i \in R$, then

$$\bigcap_{i \in R} = [\max_{i \in R} b_i, H_j] = [b_l, H_j] \qquad \text{for some} \quad l \in R\,. \tag{3.1}$$

Lemma 3.1. *Let M be the set of feasible solutions of* (P1)*. It holds*

$$M \neq \emptyset \iff \forall\, i \in S\ \exists\, j(i) \in N \quad \text{such that} \quad V_{ij(i)} \neq \emptyset\,.$$

Proof. Let x be a feasible solution of (P1). Then for any $i \in S$ it is

$$\max_{j \in N} a_{ij} \wedge x_j = a_{j(i)} \wedge x_{j(i)} \geq b_i \quad \text{so that} \quad V_{ij(i)} \neq \emptyset\,.$$

If the latter condition on the left in the $\iff$-relation is satisfied, we have

$$\max_{j \in N} a_{ij} \wedge x_j \geq a_{ij(i)} \wedge x_{j(i)} \geq b_i\,,$$

so that $x = (x_1, \ldots, x_n)$ is a feasible solution for (P1). □

The following theorem follows immediately from a more general Theorem 2 in [2].

Theorem 3.1. *Let the set of feasible solutions of* (P1) *be nonempty. Let $f_j(x_j) \equiv |x_j - \hat{x}_j|$, $L_i \equiv \{j \mid V_{ij} \neq \emptyset\}$ and $\bar{x}_{j(i)}$ for any i be defined as follows:*

$$f_{j(i)}(\bar{x}_{j(i)}) = \min_{j \in L_i} \min_{x_j \in V_{ij}} f_j(x_j)\,.$$

Let us set further $S^{(j)} \equiv \{i \mid j(i) = j\}$ *for all* $j \in N$,

$$Z^{(j)} = \bigcap_{i \in S^{(j)}} V_{ij} \quad \text{and}$$

$$f_k(x_k^{\text{opt}}) = \begin{cases} \min\limits_{x_k \in Z^{(k)}} f_k(x_k)\,, & \text{if } S^{(k)} \neq \emptyset, \\ \min\limits_{x_k \in [h_k, H_k]} f_k(x_k)\,, & \text{otherwise}. \end{cases}$$

Then $x^{\text{opt}} = (x_1^{\text{opt}}, \dots, x_n^{\text{opt}})$ *is the optimum solution of (P1).*

Remark 3.1. It can be easily shown that Theorem 3.1 can be extended to the case, where $e_{ij}(x_j)$ are arbitrary nondecreasing continuous functions.

Although this theorem is the consequence of Theorem 2 in [2], for the sake of completeness we bring the proof of this theorem, which is independent of the results from [2] in the Appendix.

EXAMPLE 3.1.

Let $m = n = 4$ so that $S = N = \{1, 2, 3, 4\}$, $b^T = (0.4, 1, 0.2, 0)$,

$$A = (a_{ij}) \equiv \begin{pmatrix} 1 & 0.4 & 0.5 & 0.7 \\ 0.7 & 0.5 & 0.3 & 1 \\ 0.2 & 1 & 1 & 0.6 \\ 0.4 & 0.5 & 0.5 & 0.8 \end{pmatrix}, \quad \hat{x}^T = (0.5, 0.5, 0.5, 0.5)\,,$$

$$h^T = (0, 0 \dots, 0)\,, \qquad H^T = (1, 1, 1, 1)\,.$$

We shall solve the problem

$$\max_{j \in N} |x_j - 0.5| \to \min$$

subject to

$$\max_{j \in N} (a_{ij} \wedge x_j) \geq b_i\,, \quad i = 1, 2, 3, 4\,,$$

$$0 \leq x_j \geq 1, \quad j = 1, 2, 3, 4\,.$$

The sets V_{ij} are now the following:

$$\begin{aligned} &V_{1j} = [0.4, 1] && \text{for } j = 1, 2, 3, 4\,; \\ &V_{2j} = \emptyset && \text{for } j = 1, 2, 3 \ \text{ and } V_{24} = [1, 1]\,; \\ &V_{3j} = [0.2, 1] && \text{for } j = 1, 2, 3, 4\,; \\ &V_{4j} = [0, 1] && \text{for } j = 1, 2, 3, 4\,. \end{aligned}$$

Therefore, the set of feasible solutions of the problem is nonempty (compare Lemma 3.1).

Let $x_j^{(i)} \equiv \operatorname{argmin}\{|x_j - 0.5| \mid x_j \in V_{ij}\}$ for $V_{ij} \neq \emptyset$. It is then

$$(x_j^{(i)}) = \begin{pmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ V_{21} = \emptyset & V_{22} = \emptyset & V_{23} = \emptyset & 1 \\ 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0.5 \end{pmatrix}.$$

Let $L_i = \{j \in N \mid V_{ij} \neq \emptyset\}$. It is then $L_1 = N$, $L_2 = \{4\}$, $L_4 = L_5 = N$. Let $|x_{j(i)}{}^{(i)} - 0.5| = \min_{j \in L_i} |x_j{}^{(i)} - 0.5|$ for all $i \in S$. Then

$$x_{j(1)}{}^{(1)} = x_1{}^{(1)} = 0.5\,, \quad x_{j(2)}{}^{(2)} = x_4{}^{(2)} = 1\,, \quad x_{j(3)}{}^{(3)} = x_1(3) = 0.5\,,$$
$$x_{j(4)}{}^{(4)} = x_1{}^{(4)} = 0.5\,, \quad \text{so that} \quad S^{(1)} = \{1,3,4\}\,, \quad S^{(2)} = S^{(3)} = \emptyset\,,$$
$$S^{(4)} = \{2\}\,, \quad Z^{(1)} = V_{1j} \cap V_{3j} \cap V_{4j} = [0.4, 1]\,, \quad Z^{(4)} = V_{24} = [1, 1]\,.$$

Therefore $x^{\text{opt}} = (0.5, 0.5, 0.5, 1)^T$ and $\|x^{\text{opt}} - \hat{x}\| = 0.5$.

4. Reference on solution method for (P2)

Using the fuzzy algebra notation $(A \otimes x)_i \equiv \max_{j \in N}(a_{ij} \wedge x_j)$ for all $i \in S$, it can be easily shown that our problem can be formulated as follows:

$$\|A \otimes x - b\| \to \min$$

subject to

$$h_j \le x_j \le H_j \qquad \text{for } j \in N\,.$$

This problem is the same as the inverse problem solved in [1], so that the same method as in [1] can be made use of to solve (P2). Let us remark that in [1] the authors assume that $h_j = 0$, $H_j = 1$ for all $j \in N$. Nevertheless the extension to general bounds h_j, H_j is a purely technical problem.

R e m a r k 4.1. Another approach to solving (P2) consists in solving a parametrized equivalent problem

$$t \to \min$$

subject to

$$M(t) \equiv \{x \mid h_j \le x_j \le H_j \ \ \forall j \in N, \ \ \|A \otimes x - \hat{x}\| \le t\} \neq \emptyset\,.$$

If $(t^{\text{opt}}, x^{\text{opt}})$ is the optimum solution of the problem, then x^{opt} is the optimum solution of (P2) with the optimal value of the objective function equal to t^{opt}. We do not describe the latter approach in this contribution. Let us only remark that unlike the method from [1], this latter approach can be relatively easily extended to other functions $e_{ij(x_j)}$ different from $a_{ij} \wedge x_j$ (compare Remark 2.1).

Appendix

According to Lemma 3.1 $L_i \neq \emptyset$, for all i. Further, it holds $i \in S^{(j(i))}$ for any $i \in S$. It is further $x_{j(i)}^{\text{opt}} \in Z^{(j(i))} = [\max_{j \in S^{(j(i))}} b_i\,,\ H_{j(i)}] \subset [b_i, H_{j(i)}] = V_{ij(i)}$ for any i, so that

$$a_{ij(i)} \wedge x_{j(i)}^{\text{opt}} \geq b_i\,,$$

and thus

$$\max_{j \in N}(a_{ij} \wedge x_j^{\text{opt}}) \geq a_{ij(i)} \wedge x_{j(i)}^{\text{opt}} \geq b_i\,,$$

so that x^{opt} is a feasible solution of (P1). It remains to prove that

$$\|x - \hat{x}\| \geq \|x^{\text{opt}} - \hat{x}\| = f_p(x_p^{\text{opt}}) = |x_p^{\text{opt}} - \hat{x}_p|$$

for any feasible solution x of (P1).

If $S^{(p)} = \emptyset$ or $S^{(p)} \neq \emptyset$ and at the same time $f_p(x_p) \geq f_p(x^{\text{opt}})$, we obtain

$$\|x - \hat{x}\| \geq f_p(x_p) \geq f_p(x_p^{\text{opt}}) = \|x^{\text{opt}} - \hat{x}\|\,.$$

It remains to investigate the case that $S^{(p)} \neq \emptyset$ and at the same time $f_p(x_p) < f_p(x_p^{\text{opt}})$.

According to (3.1) there exists an index $\bar{\imath} \in S$ such that $x^{(p)} = [b_{\bar{\imath}}, H_p] = V_{\bar{\imath}p}$ and thus

$$f_p(x_p^{\text{opt}}) = \min_{y_p \in V_{\bar{\imath}p}} f_p(y_p) = \min_{j \in L_{\bar{\imath}}} \min_{y_j \in V_{\bar{\imath}j}} f_j(y_j)\,. \qquad (*)$$

Since $x \in M$ (M is the set of feasible solutions of (P1)), it must exist for this index $\bar{\imath}$ and index $j(\bar{\imath}) \in N$ such that $x_{j(\bar{\imath})} \in V_{\bar{\imath}j(\bar{\imath})}$ (otherwise it would be $a_{ij} \wedge x_j < b_i$ for all $j \in N$ so that also $\max_{j \in N}(a_{ij} \wedge x_j) < b_i$ and thus $x \notin M$). Then

$$f_{j(i)}(x_{j(\bar{\imath})}) \geq \min_{y_{j(\bar{\imath})} \in V_{\bar{\imath}j(\bar{\imath})}} f_{j(\bar{\imath})}(y_{j(\bar{\imath})}) \geq \min_{j \in L_{\bar{\imath}}} \min_{y_j \in V_{\bar{\imath}j}} f_j(y_j) =$$
$$= \min_{y_p \in V_{\bar{\imath}p}} f_p(y_p) = f_p(x_p^{\text{opt}})\,,$$

where the last two equalities hold according to $(*)$. Therefore, we have again

$$\|x - \hat{x}\| \geq f_{j(\bar{\imath})}(x_{j(\bar{\imath})}) \geq f_p(x_p^{\text{opt}}) = \|x^{\text{opt}} - \hat{x}\|\,,$$

□

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