Tatra Mountains Math. Publ. **5** (1995), 61–69

TATRA MOUNTAINS *Mathematical Publications*

ON A CHARACTERIZATION OF REES VARIETIES

BRANIMIR ŠEŠELJA — ANDREJA TEPAVČEVIĆ

Dedicated to Professor J. Jakubík on the occasion of his 70th birthday

ABSTRACT. Necessary and sufficient conditions under which the weak congruence lattice of an algebra is a sublattice of the corresponding lattice of the partitions in a set are given. The variety of such algebras turns out to be the Rees one. In particular, if those two lattices coincide, the variety of sets is obtained.

A *weak equivalence* ρ on a nonempty set A is a symmetric and transitive relation on A. The corresponding family of subsets is said to be the *partition in* A. Obviously, the partition in A is a partition of a subset of A. The lattice of weak equivalences on A, EwA, has been investigated by H. Draškovičová [3]. She proved that this lattice is semimodular, and gave necessary and sufficient conditions under which this lattice is modular, distributive, and relatively complemented.

The lattice of weak congruences of an algebra $\mathcal{A}$, $Cw\mathcal{A}$, i.e., the lattice of all the congruences on all the subalgebras of $\mathcal{A}$ was introduced in [8]. This lattice is not necessarily a sublattice of EwA, as is in the case of $\operatorname{Con}\mathcal{A}$ and EA (congruence lattice and lattice of equivalences, respectively).

We shall use the notations and some results from [5] and [6].

Recall that the diagonal relation $\Delta = \{(x,x) \mid x \in \mathcal{A}\}$ is a codistributive element in $Cw\mathcal{A}$ (for $\rho,\theta \in Cw\mathcal{A}$, $\Delta\wedge(\rho\vee\theta) = (\Delta\wedge\rho)\vee(\Delta\wedge\theta)$). The congruence lattice of $\mathcal{A}$ is the filter $[\Delta)$ in $Cw\mathcal{A}$, and $\operatorname{Sub}\mathcal{A}$ (the subalgebra lattice) is isomorphic with the ideal $(\Delta]$ (under $\mathcal{B} \longrightarrow \{(x,x) \mid x \in B\}$, $\mathcal{B} \in \operatorname{Sub}\mathcal{A}$).

An algebra $\mathcal{A}$ has the *congruence extension property* (the CEP) if every congruence on a subalgebra of $\mathcal{A}$ is a restriction of a congruence on $\mathcal{A}$.

$\mathcal{A}$ has the *congruence intersection property* (the CIP), [8], if for $\rho,\theta \in Cw\mathcal{A}$

$$(\rho\cap\theta)_A = \rho_A \cap \theta_A ,$$

where ρ_A is a minimal congruence of $\mathcal{A}$ extending ρ.

AMS Subject Classification (1991): 08A30, 06C10.
Key words: partition in a set, Rees algebras, weak congruences.

In the lattice $Cw\mathcal{A}$ $\rho_A = \rho \vee \Delta$, and thus $\mathcal{A}$ has the CIP if and only if Δ is a distributive element of $Cw\mathcal{A}$, i.e., iff for $\rho, \theta \in Cw\mathcal{A}$

$$\Delta \vee (\rho \wedge \theta) = (\Delta \vee \rho) \wedge (\Delta \vee \theta).$$

Some other lattice characterizations of the CIP as well as of the CEP were given in [5] and [6].

$\mathcal{A}$ is said to have the *CIP if for every family $\{\rho_i \mid i \in I\} \subseteq Cw\mathcal{A}$,

$$\Delta \vee \bigwedge_{i\in I} \rho_i = \bigwedge_{i\in I}(\Delta \vee \rho_i), \quad [5].$$

Summing up the notations, we have: If $\mathcal{A} = (A, F)$ is an algebra then EA, EwA, $\mathrm{Con}\,\mathcal{A}$, $\mathrm{Sub}\,\mathcal{A}$, $Cw\mathcal{A}$ are the lattices of equivalences on $\mathcal{A}$, weak equivalences on $\mathcal{A}$, congruences on $\mathcal{A}$, subalgebras on $\mathcal{A}$, and weak congruences on $\mathcal{A}$, respectively.

LEMMA 1. *If $A \neq \emptyset$ and $\rho \in EwA$, then $\rho \cup \Delta \in EA$.*

Proof. Obvious. □

An element a of a bounded lattice L is *neutral* if $x \to x \wedge a$ and $x \to x \vee a$ are homomorphisms from L to $(a]$ and to $[a)$, respectively and $x \to (x \wedge a, x \vee a)$ is an embedding of L into $(a] \times [a)$.

An element a of L is neutral if and only if it is distributive, codistributive and satisfies the property:

if $x \wedge a = y \wedge a$ and $x \vee a = y \vee a$ then $x = y$. [4].

PROPOSITION 1. *The diagonal relation Δ is a neutral element in the lattice EwA.*

Proof. Δ is codistributive for $\rho, \theta \in EwA$,

$$(\Delta \wedge \rho) \vee (\Delta \wedge \theta) = (\Delta \cap \rho) \vee (\Delta \cap \theta) = (\Delta \cap \rho) \cup (\Delta \cap \theta) = \Delta \cap (\rho \cup \theta) = \Delta \wedge (\rho \vee \theta),$$

since $\rho \cup \theta$ and $\rho \vee \theta$ have the same diagonal.

Δ is distributive, since by Lemma 1

$$\Delta \vee (\rho \wedge \theta) = \Delta \cup (\rho \cap \theta) = (\Delta \cup \rho) \cap (\Delta \cup \theta) = (\Delta \vee \rho) \wedge (\Delta \vee \theta).$$

If $\rho \cap \Delta = \theta \cap \Delta$ and $\rho \cup \Delta = \theta \cup \Delta$, then ρ and θ are equivalences on the same subsets of A. Now, having the same extension to A (by Δ), they are equal.

Thus, Δ is neutral. □

It was proved in [3] that EwA can be embedded into the lattice $EA \times \{1, 2\}$. We give another characterization of that kind.

COROLLARY 1. *The mapping $\rho \to (\rho \cap \Delta, \rho \cup \Delta)$ is an embedding from EwA into $\mathcal{P}(A) \times EA$ ($\mathcal{P}(A)$ is a power set of A).*

Proof. Straightforward, since $\mathcal{P}(A) = (\Delta]$ under $B \to \{(x,x) \mid x \in B\}$ for $B \subseteq A$, and since $[\Delta) = EA$. □

COROLLARY 2. *Any lattice identity is satisfied on EwA if and only if this identity holds on EA.*

An element a of a bounded lattice L is said to be *infinitely distributive* if for every family $\{x_i \mid i \in I\} \subseteq L$,

$$a \vee \bigwedge_{i \in I} x_i = \bigwedge_{i \in I} (a \vee x_i).$$

PROPOSITION 2. *In the lattice EwA, Δ is an infinitely distributive element.*

Proof. For a family $\{\rho_i \mid i \in I\}$ of weak equivalences on A,

$$\Delta \vee \bigwedge_{i \in I} \rho_i = \Delta \cup \bigcap_{i \in I} \rho_i = \bigcap_{i \in I} (\Delta \cup \rho_i) = \bigwedge_{i \in I} (\Delta \vee \rho_i).$$

□

PROPOSITION 3. *For the lattice of weak equivalences EwA on a nonempty set A, there is a algebra $\mathcal{A} = (A, F)$, whose lattice of weak congruences $Cw\mathcal{A}$ coincides with EwA.*

Proof. An idempotent algebra $(f(x) = x)$ on A satisfies the required condition. □

An algebra $\mathcal{A}$ is said to have the *strong* CEP [1] if for every $\rho \in Cw\mathcal{A}$,

$$\rho \cup \Delta \in Cw\mathcal{A}.$$

PROPOSITION 4. *Let $\mathcal{A} = (A, F)$ be an algebra for which the lattice of weak congruences $Cw\mathcal{A}$ is a sublattice of the weak equivalence lattice EwA. Then, $\mathcal{A}$ has strong* CEP, CEP, CIP *and* *CIP.

Proof. If $Cw\mathcal{A}$ is a sublattice of EwA, then $\rho \vee \Delta = \rho \cup \Delta$ (by Lemma 1), for every $\rho \in Cw\mathcal{A}$, and $\mathcal{A}$ has the strong CEP. By Proposition 1, $\mathcal{A}$ has the CEP, and the CIP as well. The *CIP holds by Proposition 2, since $Cw\mathcal{A}$ is a complete sublattice of EwA. □

Considering subalgebras, we have the following statement.

Lemma 2. *If $Cw\mathcal{A}$ is a sublattice of EwA, then $\mathcal{A}$ is a U–algebra (i.e., its subalgebras are closed under the set union).*

Proof. Recall that $\mathrm{Sub}\,\mathcal{A}$ is, up to the isomorphism, a sublattice of $Cw\mathcal{A}$. □

Following [1], $\mathcal{A}$ is said to be a *Rees algebra* if $B^2 \cup \Delta$ is a congruence on $\mathcal{A}$, for every subalgebra $\mathcal{B}$ of $\mathcal{A}$.

If an algebra has the strong CEP, then obviously it has the CEP as well. The converse holds for Rees algebras.

Proposition 5. *For a Rees algebra, the* CEP *implies the strong* CEP.

Proof. Let $\rho \in Cw\mathcal{A}$, $\rho \in \mathrm{Con}\,\mathcal{B}$. Then,

$$(B^2 \wedge (\rho \vee \Delta)) \cup \Delta = \rho \cup \Delta\,,$$

since, by CEP (see [5]):

$$B^2 \wedge (\rho \vee \Delta) = \rho\,.$$

On the other hand, $\mathcal{A}$ is a Rees algebra and

$$\begin{aligned}(B^2 \cup \Delta) \wedge \big((\rho \vee \Delta) \cup \Delta\big) &= (B^2 \cup \Delta) \wedge (\rho \vee \Delta)\\ &= (B^2 \vee \Delta) \wedge (\rho \vee \Delta) = \rho \vee \Delta\,,\end{aligned}$$

since $B^2 \vee \Delta \geq \rho \vee \Delta$.

Δ is distributive in EwA and thus

$$\rho \cup \Delta = \big(B^2 \wedge (\rho \vee \Delta)\big) \cup \Delta = (B^2 \cup \Delta) \wedge ((\rho \vee \Delta) \cup \Delta) = \rho \vee \Delta\,,$$

and $\mathcal{A}$ has the strong CEP. □

Theorem 1. *If $\mathcal{A}$ is a Rees U–algebra having the* CEP, *then $Cw\mathcal{A}$ is a sublattice of EwA.*

Proof. Denote the join in EwA by $+$, and the one in $Cw\mathcal{A}$ by $\vee$ ($\cup$ is the set theoretic union, and $\wedge$ is the intersection, or the meet in both lattices). Let $\rho \in \mathrm{Con}\,\mathcal{B}$, $\theta \in \mathrm{Con}\,\mathcal{C}$, $\mathcal{B}, \mathcal{C} \in \mathrm{Sub}\,\mathcal{A}$.

$$(\rho + \theta) \wedge \Delta = (\rho \vee \theta) \wedge \Delta\,, \tag{1}$$

since $\rho + \theta \in E(B \cup C)$, $\rho \vee \theta \in \mathrm{Con}(\mathcal{B} \vee \mathcal{C})$, and $\mathcal{B} \cup \mathcal{C} = \mathcal{B} \vee \mathcal{C}$.

$$(\rho + \theta) + \Delta = (\rho + \Delta) + (\theta + \Delta) = (\rho \cup \Delta) + (\theta \cup \Delta) = \tag{2}$$

(by Proposition 5, i.e., by the strong CEP)

$$= (\rho \vee \Delta) + (\theta \vee \Delta) =$$

(since Con $\mathcal{A}$ is a sublattice of EA)

$$= (\rho \vee \Delta) \vee (\theta \vee \Delta) = (\rho \vee \theta) \vee \Delta = (\rho \vee \theta) + \Delta\,,$$

again by the strong CEP.

By Proposition 1, (1) and (2) imply

$$\rho + \theta = \rho \vee \theta\,.$$

□

Summing up Proposition 4, Lemma 2 and Theorem 1, we have the following propositions.

THEOREM 2. *For an algebra $\mathcal{A}$, $Cw\mathcal{A}$ is a sublattice of EwA if and only if $\mathcal{A}$ is a Rees U–algebra satisfying the* CEP.

THEOREM 3. *$Cw\mathcal{A}$ is a subalgebra of EwA if and only if $\mathcal{A}$ is a U–algebra having the strong* CEP.

In [1], some characterizations of Rees varieties were given. Using the properties of the weak congruence lattice we give another characterization of these varieties.

THEOREM 4. *For a variety $\mathcal{V}$, the following conditions are equivalent:*

(i) *$\mathcal{V}$ is a Rees variety;*
(ii) *For every $\mathcal{A} \in \mathcal{V}$, $Cw\mathcal{A}$ is a sublattice of EwA.*

P r o o f. (ii) $\Longrightarrow$ (i) by Theorem 2.

(i) $\Longrightarrow$ (ii). If $\mathcal{V}$ is a Rees variety, then subalgebras of each $\mathcal{A} \in \mathcal{V}$ are closed under the set union and satisfy the strong CEP ([1]). Thus, by Theorem 3, $Cw\mathcal{A}$ is a sublattice of EwA, for every $\mathcal{A} \in \mathcal{V}$. □

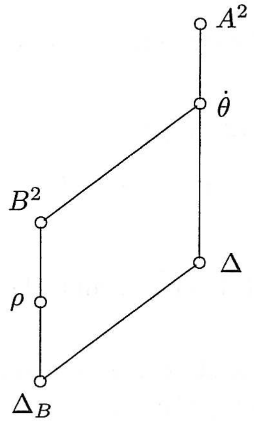
For a single algebra $\mathcal{A}$, being a Rees one does not mean that $Cw\mathcal{A}$ is a sublattice of EwA, as shown by the following example.

EXAMPLE.

$\mathcal{A}$ is a 4-element Rees groupoid with nullary operations b and c:

	a	b	c	d
a	b	a	a	c
b	b	b	a	b
c	a	a	a	a
d	d	d	d	c

The lattice of weak congruences of $\mathcal{A}$ is shown in the following figure.



Congruences on $\mathcal{A}$: A^2, $\theta = \{\{a,b,c\},\ \{d\}\}$, Δ.
The only subalgebra: $\mathcal{B} = \{a,b,c\}$.
Congruences on $\mathcal{B}$: B^2, $\rho = \{\{a,b\},\ \{c\}\}$, Δ_B.

$Cw\mathcal{A}$ is not a sublattice of EwA, since the equivalence $\rho \cup \Delta$ is not a congruence on $\mathcal{A}$.

In the following, we characterize algebras for which $Cw\mathcal{A} = EwA$.

LEMMA 3. *If $Cw\mathcal{A} = EwA$, and $|A| > 1$, then $\mathcal{A}$ has no nullary operations.*

P r o o f . Straightforward, since otherwise the least element of Sub $\mathcal{A}$ would be nonvoid. □

LEMMA 4. *Every subset of A is a subalgebra of $\mathcal{A}$ if and only if for each n-ary operation f of $\mathcal{A}$, and $x_1, \dots, x_n \in A$*

$$f(x_1, \dots, x_n) \in \{x_1, \dots, x_n\}.$$

P r o o f . Obvious. □

LEMMA 5. *If $|\ A\ | = 2$, then $Cw\mathcal{A} = EwA$ if and only if $f(x, \dots, x) = x$ for every n-ary operation f of $\mathcal{A}$.*

P r o o f . Obviously, Con $\mathcal{A}$ $=$ EA. By Lemma 4, if $A = \{a,b\}$, then $f(a, \dots, a) = a$ and $f(b, \dots, b) = b$ for every n-ary operation f of $\mathcal{A}$. □

LEMMA 6. *If* $|A| = 3$, *and* $Cw\mathcal{A} = EwA$, *then the only binary operations on* A *are projections.*

P r o o f. Let $A = \{a, b, c\}$, $\mathcal{A} = (A, f)$. If f is a binary operation of $\mathcal{A}$, then by Lemma 4, $f(x, x) = x$ for every $x \in A$, and $f(a, b) \in \{a, b\}$. Let $f(a, b) = a$. The partition $\{\{a\}, \{b, c\}\}$ induces a congruence ρ on $\mathcal{A}$, and thus (a, a), $(b, c) \in \rho$. Hence $f(a, c) = a$. Using another partition of A and the corresponding congruences, we obtain that $f(x, y) = x$ for all $x, y \in A$.

If $f(a, b) = b$, then similarly $f(x, y) = y$. □

LEMMA 7. *Let* $\mathcal{A} = (A, f)$ *be an algebra for which* $Cw\mathcal{A} = EwA$. *Now, if for an* n*-ary operation* f *of* $\mathcal{A}$ *and* $a_1, \ldots, a_n \in A$

$$f(a_1, \ldots, a_i, \ldots, a_n) = a_i ,$$

then

$$f(x_1, \ldots, x_n) = a_i$$

for all $x_1, \ldots, x_n \in A$ *such that* $x_j \doteq a_i$ *iff* $a_j = a_i$, $j \in \{1, \ldots, n\}$.

P r o o f. The equivalence ρ induced by the partition $\{\{a_i\}, A \setminus \{a_i\}\}$ yields $(a_k, x_k) \in \rho$, $k \in \{1, \ldots, n\}$. Thus

$$\big(f(a_1, \ldots, a_n), f(x_1, \ldots, x_n)\big) \in \rho ,$$

i.e., $\big(a_i, f(x_1, \ldots, x_n)\big) \in \rho$, and $f(x_1, \ldots, x_n) = a_i$. □

THEOREM 5. $\mathcal{A} = (A, f)$, $|A| \geq 3$, *is an algebra for which* $Cw\mathcal{A} = EwA$ *if and only if there are no operations in* $\mathcal{A}$ *other than projections.*

P r o o f. (i) Let all the operations in $\mathcal{A}$ be projections. Obviously $\mathrm{Sub}\,\mathcal{A} = P(A)$.

Let ρ be an equivalence on a subset B of A. We have to prove that ρ is a weak congruence on A. Let $f \in F_n \subseteq F$, and for $x_1, \ldots, x_n$, $y_1, \ldots, y_n \in B$, $(x_i, y_i) \in \rho$. Since a restriction to B of a projection remains the projection, we have that

$$\big(f(x_1, \ldots, x_n),\ f(y_1, \ldots, y_n)\big) \in \rho .$$

(ii) Suppose that $Cw\mathcal{A} = EwA$. Then $\mathrm{Sub}\,\mathcal{A} = \mathcal{P}(A)$, since $[\Delta)$ is isomorphic with $\mathrm{Sub}\,\mathcal{A}$ in $Cw\mathcal{A}$ and with $\mathcal{P}(A)$ in EwA. Suppose that there is an operation f on $\mathcal{A}$ which is not a projection. By Lemma 4, there are $a_1, \ldots, a_n$, $b_1, \ldots, b_n \in A$, such that for some $i, j \in \{1, \ldots, n\}$, $i \neq j$,

$$f(a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n) = a_i , \tag{3}$$

$$f(b_1, \ldots, b_i, \ldots, b_j, \ldots, b_n) = b_j , \tag{4}$$

and

$$a_k \neq a_i \quad \text{or} \quad b_k \neq b_j\,, \quad \text{for each} \quad k = 1, \ldots, n\,. \tag{5}$$

Now, we shall consider the following two cases:

$$\text{a) } a_i \neq b_j \qquad \text{and} \qquad \text{b) } a_i = b_j\,.$$

a) $a_i \neq b_j$. Consider $x_1, \ldots, x_n \in A$ such that $x_k = a_i$ iff $a_k = a_i$ and $x_k = b_j$ iff $b_k = b_j$, $k = 1, \ldots, n$. By Lemma 7, (3) implies that $f(x_1, \ldots, x_n) = a_i$ and by (4) $f(x_1, \ldots, x_n) = b_j$, which is a contradiction.

b) $a_i = b_j = a$. Now, by (3), (4) and (5)

$$f(a_1, \ldots, a_n) = f(b_1, \ldots, b_n) = a\,, \quad \text{where} \quad a_k = b_k \quad \text{implies} \quad a_k \neq a\,.$$

Since $|A| \geq 3$ there are $x, y \in A$ such that a, x and y are all different. By Lemma 7, (3) implies

$$f(x_1, \ldots, x_n) = a\,, \tag{6}$$

where

$$x_k = \begin{cases} a\,, & \text{for } a_k = a\,, \\ y\,, & \text{for } b_k = a\,, \\ x\,, & \text{otherwise}\,. \end{cases}$$

By (4),

$$f(y_1, \ldots, y_n) = a\,, \tag{7}$$

where

$$y_k = \begin{cases} a\,, & \text{for } b_k = a\,, \\ x\,, & \text{otherwise}\,. \end{cases}$$

The congruence ρ induced by the partition $\{\{a, x\},\ A \setminus \{a, x\}\}$ implies by (6) that

$$\big(f(x_1, \ldots, x_n)\,,\ f(z_1, \ldots, z_n)\big) \in \rho\,,$$

where

$$z_k = \begin{cases} x\,, & \text{for } x_k = a\,, \\ x_k\,, & \text{otherwise}\,. \end{cases}$$

Hence, since $f(x_1, \ldots, x_n) = a$, $f(z_1, \ldots, z_n) \in \{x, y\}$, and since $(a, x) \in \rho$, but $(a, y) \notin \rho$, we have that

$$f(z_1, \ldots, z_n) = x\,. \tag{8}$$

On the other hand, the partition $\{\{a, y\},\ A \setminus \{a, y\}\}$ and the corresponding congruence θ, yield by (7) that

$$\big(f(y_1, \ldots, y_n)\,,\ f(z_1, \ldots, z_n)\big) \in \theta\,.$$

Since $f(y_1, \ldots, y_n) = a$, and $f(z_1, \ldots, z_n) \in \{x, y\}$, and $(a, y) \in \theta$ but $(a, x) \notin \theta$, it follows that

$$f(z_1, \ldots, z_n) = y\,,$$

contradicting to (8).

By (i) and (ii), the proof is complete. □

COROLLARY 3. *For an algebra* $\mathcal{A}$, $Cw\mathcal{A} = Ew A$ *if and only if* $\operatorname{Con}\mathcal{A} = EA$ *and* $\operatorname{Sub}\mathcal{A} = P(A)$.

P r o o f. The "only if" part follows from the previous theorem, since the equalities $\operatorname{Sub}\mathcal{A} = P(A)$ and $\operatorname{Con}\mathcal{A} = EA$ imply that every $f \in F$ is a projection. □

An obvious consequence of Theorem 5 is also the following proposition.

THEOREM 6. *For a variety* $\mathcal{V}$, *the following conditions are equivalent:*

(i) *for every* $\mathcal{A} \in \mathcal{V}$, $Cw\mathcal{A} = Ew A$;

(ii) $\mathcal{V}$ *is equivalent to the variety of sets.*

(Recall that (ii) means that $\mathcal{V}$ has no operations other than projections). □

REFERENCES

[1] CHAJDA, I.—DUDA, J.: *Rees algebras and their varieties*, Publ. Math. Debrecen **32** (1985), 17–22.

[2] CSÁKANY, B.: *Congruences and subalgebras*, Ann. Univ. Sci. Budapest, Eötvös Sect. Math. **18** (1975), 37–44.

[3] DRAŠKOVIČOVÁ, H.: *The lattice of partitions in a set*, Acta Fac. Rerum Natur. Univ. Comenian. Math. **24** (1970), 37–65.

[4] GRÄTZER, G.: *General Lattice Theory*, Academie Verlag, Berlin, 1978.

[5] ŠEŠELJA, B.—TEPAVČEVIĆ, A.: *Infinitely distributive elements in the lattice of weak congruences*, in: General algebra 1988 North-Holland, Amsterdam, 1990, pp. 241–253.

[6] ŠEŠELJA, B.—TEPAVČEVIĆ, A.: *On CEP and semimodularity in the lattice of weak congruences*, Review of Research, Fac. of Sci, Univ. of Novi Sad **22** (1992), 95–106.

[7] TICHÝ, R.: *The Rees congruence in universal algebras*, Publ. Inst. Rech. Math. Av. **29** 43 (1981), 229–239.

[8] VOJVODIĆ, G.—ŠEŠELJA, B.: *On the lattice of weak congruence relations*, Algebra Universalis **25** (1988), 121–130.

Received December 20, 1993

Institute of Mathematics
University of Novi Sad
Trg D. Obradovića 4
YU–21000 Novi Sad
YUGOSLAVIA

E-mail: andreja@unsim.ns.ac.yu