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THE MEASURE EXTENSION THEOREM ON MV σ–ALGEBRAS

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ABSTRACT. The aim of this paper is to provide some results regarding the measure extension on MV σ-algebras.

1. Introduction

The problem of measure extension was solved by Piasecki [6] and Riečan [7] in soft σ-algebra and in orthomodular σ-continuous lattices and by Chovanec and Kôpka in quasi-orthocomplemented lattices [3]. In [2] Chovanec defined a state on MV σ-algebras. This was motivated by the state on D–σ-posets [4]; his definition of the state on MV σ-algebras is different from definitions in [6], [7], [3].

These results have led us to a measure extension theorem on MV σ-algebras.

2. MV σ-algebras

In [5] an MV algebra is defined as follows:

An *MV algebra* is an algebra $(\mathcal{F}, \oplus, \odot, *, 0, 1)$, where $\mathcal{F}$ is a non-empty set, 0 and 1 are constant elements of $\mathcal{F}$, $\oplus$ and $\odot$ are binary operations, and $*$ is a unary operation, satisfying the following axioms:

[1.1] $a \oplus b = b \oplus a$,

[1.2] $(a \oplus b) \oplus c = a \oplus (b \oplus c)$,

[1.3] $a \oplus 0 = a$,

[1.4] $a \oplus 1 = 1$,

[1.5] $(a^*)^* = a$,

[1.6] $0^* = 1$,

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[1.7] $a \oplus a^* = 1$,

[1.8] $(a^* \oplus b)^* \oplus b = (a \oplus b^*)^* \oplus a$,

[1.9] $a \odot b = (a^* \oplus b^*)^*$.

The lattice operations $\vee$ and $\wedge$ are defined by the formulas

[1.10] $a \vee b = (a \odot b^*) \oplus b$,

[1.11] $a \wedge b = (a \oplus b^*) \odot b$.

We write $a \leq b$ iff $a \vee b = b$. The relation $\leq$ is a partial ordering over $\mathcal{F}$ and $0 \leq a \leq 1$, for every $a \in \mathcal{F}$. An MV algebra is a distributive lattice with respect to the operations $\vee$, $\wedge$.

In [1] the following assertions have been proved:

[1.12] $a \odot b \leq a \wedge b \leq a \vee b \leq a \oplus b$, for every $a, b \in \mathcal{F}$.

[1.13] If $a \leq b$, then $a \oplus c \leq b \oplus c$ and $a \odot c \leq b \odot c$, for every $c \in \mathcal{F}$.

[1.14] The following three conditions are equivalent:
(i) $a \leq b$, (ii) $a^* \oplus b = 1$, (iii) $a \odot b^* = 0$.

[1.15] If $a \leq b$, then $b = a \oplus (b \odot a^*)$.

[1.16] $(a \vee b)^* = a^* \wedge b^*$ and $(a^* \vee b)^* = a^* \vee b^*$.

In [2] a binary operation $\ominus$ is defined on MV algebra by the formula: $b \ominus a := b \odot a^*$ for any $a, b \in \mathcal{F}$ and the next properties are proved for $a, b, c \in \mathcal{F}$:

[1.17] If $a \leq b$, then $b \ominus (b \ominus a) = a$ and $b = a \oplus (b \ominus a)$,

[1.18] If $a \leq b^*$, then $a = (a \oplus b) \ominus b$,

[1.19] If $a \leq b \leq c$, then
(i) $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$,
(ii) $b \ominus a \leq c \ominus a$ and $(c \ominus a) \ominus (b \ominus a) = c \ominus b$,

[1.20] $(a \vee b) \ominus a = b \ominus (a \wedge b)$,

[1.21] $(a \oplus b) \ominus a = b \ominus (a \odot b)$,

[1.22] $(a \oplus b) \ominus (a \vee b) = (a \wedge b) \ominus (a \odot b)$,

[1.23] let $a \leq b$ and $d \leq c$. If $b \ominus a = c \ominus d$, then $a \oplus c = b \oplus d$),

[1.24] $(a \vee b) \oplus (a \wedge b) = a \oplus b = (a \oplus b) \oplus (a \odot b)$,

[1.25] if $a \leq b \leq c$, then $(c \ominus b) \vee (b \ominus a) = c \ominus a$,

[1.26] $(a \wedge b) \oplus (b \ominus a) = b$,

[1.27] $c \ominus (a \wedge b) = (c \ominus a) \vee (c \ominus b)$,

[1.28] $c \ominus (a \vee b) = (c \ominus a) \wedge (c \ominus b)$.

An MV algebra $\mathcal{F}$ is said to be an *MV σ-algebra*, if each countable sequence of elements from $\mathcal{F}$ has the supremum in $\mathcal{F}$.

3. The measure extension theorem

In the measure extension theorem we will use a σ-continuous MVσ-algebra according to the next definition:

DEFINITION 1. We will say that $\mathcal{F}$ is a *σ-continuous MV σ-algebra* if it holds:

[2.1] if $x_n \nearrow x$ and $y_n \nearrow y$, then $x_n \wedge y_n \nearrow x \wedge y$,

[2.2] if $x_n \nearrow x$ and $y_n \searrow y$, then $y_n \ominus x_n \searrow y \ominus x$, $\quad x_n \ominus y_n \nearrow x \ominus y$,

where $(x_n)_n \subset \mathcal{F}$, $(y_n)_n \subset \mathcal{F}$, $x, y \in \mathcal{F}$.

It is not difficult to prove that σ-continuous MV σ-algebra is σ-complete and σ-continuous lattice such that:

[2.3] if $x_n \nearrow x$ and $y_n \nearrow y$, then $x_n \oplus y_n \nearrow x \oplus y$,

[2.4] if $x_n \nearrow x$ and $y_n \nearrow y$, then $x_n \odot y_n \nearrow x \odot y$,

where $x_n, y_n, x, y \in \mathcal{F}$ for all n.

Now let $\mathcal{A}$ be an MV subalgebra of MV σ-algebra $\mathcal{F}$, i.e., $\mathcal{A} \subseteq \mathcal{F}$ and $\mathcal{A}$ is an MV algebra.

There is given a measure $m\colon \mathcal{A} \to [0,1]$ satisfying the following conditions:

[3.1] $m(1) = 1$,

[3.2] if $a, b \in \mathcal{A}$, $a \leq b$, then $m(a) \leq m(b)$ and $m(b \ominus a) = m(b) - m(a)$,

[3.3] if $a_n \nearrow a$, $a_n, a \in \mathcal{A}$ for all n, then $m(a_n) \nearrow m(a)$.

It has been proved in [2] that the measure has the following properties:

[3.4] $m(0) = 0$,

[3.5] $m(a^*) = 1 - m(a)$,

[3.6] $m(a \vee b) = m(a) + m(b \ominus a) = m(b) + m(a \ominus b)$,

[3.7] if $a \leq b$, then $m(b) = m(a) + m(b \ominus a)$,

[3.8] $m(a \oplus b) + m(a \odot b) = m(a) + m(b) = m(a \vee b) + m(a \wedge b)$,

[3.9] if $a \leq b^*$, then $m(a \oplus b) = m(a) + m(b)$.

In the next we will require that measure m has the following property:

[3.10] if $a_n \searrow a$ and $b_n \nearrow b$, a_n, $b_n \in \mathcal{A}$, $a \leq b$, then

$$\lim_{n\to\infty} m(b_n \ominus a_n) = \lim_{n\to\infty} m(b_n) - \lim_{n\to\infty} m(a_n) .$$

From the definition of m and [3.6] we can see that:

[3.11] if $a_n \nearrow a$ and $b_n \searrow b$, $a_n, b_n \in \mathcal{A}$, then

$$\lim_{n\to\infty} m(b_n) \leq \lim_{n\to\infty} m(a_n) + \lim_{n\to\infty} m(b_n \ominus a_n) .$$

THEOREM 1. *Let $\mathcal{F}$ be a σ-continuous MV σ-algebra, $\mathcal{A}$ be an MV subalgebra of $\mathcal{F}$ and $m\colon \mathcal{A} \to [0,1]$ a measure on $\mathcal{A}$ satisfying the property [3.10]. Let $\mathcal{S}(\mathcal{A})$ be an MV σ-subalgebra generated by $\mathcal{A}$. Then there exists exactly one measure $\mu\colon \mathcal{S}(\mathcal{A}) \to [0,1]$ which is an extension of m.*

The extension proces will be made by a standard way:

$$A^+ = \{a \in \mathcal{F};\ \exists (a_n)_n \subset \mathcal{A}\colon a_n \nearrow a\},\ m^+\colon \mathcal{A}^+ \to [0,1],\ m^+(a) = \lim_{n\to\infty} m(a_n)\,,$$

$$\mathcal{A}^- = \{b \in \mathcal{F};\ \exists (b_n)_n \subset \mathcal{A}\colon b_n \searrow b\},\ m^-\colon \mathcal{A}^- \to [0,1],\ m^-(b) = \lim_{n\to\infty} m(b_n)\,.$$

It is easy to prove that m^+ and m^- are defined correctly, i.e., $m^+(a)$ $(m^-(b))$ do not depend on the choise of $(a_n)_n$ $((b_n)_n)$, they are non-decreasing, valuations (i.e., [3.6] is satisfied) and semicontinuous.

We will define the maps $m^\circ\colon \mathcal{F} \to [0,1]$ and $m_\circ\colon \mathcal{F} \to [0,1]$ by the formulas:

$$m^\circ(x) = inf\{m^+(a),\ a \in \mathcal{A}^+,\ a \ge x\}\,,$$
$$m_\circ(x) = \sup\{m^-(b),\ b \in \mathcal{A}^-,\ b \le x\}\,.$$

The last step in our construction is the set $\mathcal{L} = \{x \in \mathcal{F};\ m^\circ(x) = m_\circ(x)\}$.

Later we will prove that $\mathcal{L} \supset \mathcal{S}(\mathcal{A})$ and $m^\circ/\mathcal{S}(\mathcal{A})$ is the proposed extension. Before the proof of the above mentioned theorem we will introduce some helpful lemmas.

LEMMA 1. *Let $a \in \mathcal{A}^+$, then $a^* \in \mathcal{A}^-$ and $m^+(a) + m^-(a^*) = 1$.*

Proof. Let $a \in \mathcal{A}^+$ and $(a_n)_n \subset \mathcal{A}$ such that $a_n \nearrow a$. Then $a^* \le a_n^*$, $a^* \le \bigwedge_n a_n^* = b$. Since $b \le a_n^*$, $a_n \le b^*$, we obtain $a = \bigvee_n a_n \le b^*$, $b \le a^*$. Then $a^* = \bigwedge_n a_n^*$ and $a^* \in \mathcal{A}^-$.

By [3.5] we have $m(a_n) + m(a_n^*) = 1$ for all $n \in \mathbb{N}$. Taking $n \to \infty$ we obtain $m^+(a) + m^-(a^*) = 1$. □

LEMMA 2. *The maps $m^\circ\colon \mathcal{F} \to [0,1]$ and $m_\circ\colon \mathcal{F} \to [0,1]$ have the following properties:*

[i] *they are extensions of m,*

[ii] *they are non-decreasing,*

[iii] $m^\circ(y \ominus x) \le m^\circ(y) - m_\circ(x)$, *for all $x, y \in \mathcal{F}$, $x \le y$,*

[iv] $m_\circ(y \ominus x) \ge m_\circ(y) - m^\circ(x)$, *for all $x, y \in \mathcal{F}$, $x \le y$,*

[v] $m_\circ(x) \le m^\circ(x)$, *for all $x \in \mathcal{F}$.*

Proof. The first and second assertions follow from the definition of m° and m°.

[iii] Let $x, y \in \mathcal{F}$, $b \ge y \ge x \ge a$, $a \in \mathcal{A}^-$, $b \in \mathcal{A}^+$.

Then $y \ominus x \le b \ominus a$ and [2.2] implies $b \ominus a \in \mathcal{A}^+$.

From the definition of m°, $m_\circ$ and [3.10] we obtain:

$$m^\circ(y \ominus x) \le m^+(b \ominus a) = m^+(b) - m^-(a)\,,$$
$$m^\circ(y \ominus x) \le m^\circ(y) - m^-(a)\,,$$
$$m^-(a) \le m^\circ(y) - m^\circ(y \ominus x)\,,$$
$$m_\circ(x) \le m^\circ(y) - m^\circ(y \ominus x) \quad \text{and hence}$$
$$m^\circ(y \ominus x) \le m^\circ(y) - m_\circ(x)\,.$$

[iv] Let $x, y \in \mathcal{F}$, $x \le y$, $x \le a$, $a \in \mathcal{A}^+$, $b \le y$, $b \in \mathcal{A}^-$.

Then $y \ominus x \ge b \ominus x \ge b \ominus a$ and [2.2] implies $b \ominus a \in \mathcal{A}^-$.

Analogously as in the proof of [iii], from definition of m°, $m_\circ$ and [3.11] we obtain:

$$m_\circ(y \ominus x) \ge m^-(b \ominus a) \ge m^-(b) - m^+(a)\,,$$
$$m_\circ(y \ominus x) \ge m_\circ(y) - m^+(a)\,,$$
$$m^+(a) \ge m_\circ(y) - m_\circ(y \ominus x)\,,$$
$$m^\circ(x) \ge m_\circ(y) - m_\circ(y \ominus x) \quad \text{and hence}$$
$$m_\circ(y \ominus x) \ge m_\circ(y) - m^\circ(x)\,.$$

[v] Due to [3.10] it is easy to see that if $a \in \mathcal{A}^-$, $b \in \mathcal{A}^+$, $a \le b$, then $m^-(a) \le m^+(b)$.

Let $x \in \mathcal{F}$ be such that $a \le x \le b$. Then $m_\circ(x) \le m^+(b)$. Since b is an arbitrary element from $\mathcal{A}^+$ such that $b \ge x$, we have $m_\circ(x) \le m^\circ(x)$. □

The following three results can be proved by the same ways as the corresponding proofs in [3]:

LEMMA 3. *Let $x \in \mathcal{L}$. Then $m^\circ(x) + m^\circ(x^*) = 1$.*

Proof. Take $a \in \mathcal{A}^+$, $a \ge x^*$. Then $a^* \in \mathcal{A}^-$, $a^* \le x$. From the inequality $m_\circ(x) \ge m^-(a^*) = 1 - m^+(a)$ and $m_\circ(x) = m^\circ(x)$ for all $x \in \mathcal{L}$ it follows $m^\circ(x) + m^+(a) \ge 1$ for every $a \in \mathcal{A}^+$, $a \ge x^*$ and then $m^\circ(x) + m^\circ(x^*) \ge 1$. And now let $b \in \mathcal{A}^-$, $b \le x$. Then $b^* \ge x^*$, $b^* \in \mathcal{A}^+$ and $m^+(b^*) \ge m^\circ(x^*)$. But then $1 \ge m^-(b) + m^\circ(x^*)$, hence $1 \ge m_\circ(x) + m^\circ(x^*) = m^\circ(x) + m^\circ(x^*)$. □

LEMMA 4. *Let $x \in \mathcal{L}$. Then $x^* \in \mathcal{L}$.*

Proof. Take $a \in \mathcal{A}^+$, $a \ge x$. Then $a^* \in \mathcal{A}^-$, $a^* \le x^*$ and then $m^-(a^*) \le m_\circ(x^*)$, i.e., $1 - m^+(a) \le m_\circ(x^*)$, $1 \le m^+(a) + m_\circ(x^*)$.

Because a is an arbitrary element from $\mathcal{A}^+$, using [v] in Lemma 2 we have $1 \le m^\circ(x) + m_\circ(x^*) \le m^\circ(x) + m^\circ(x^*) = 1$.

Therefore $m_\circ(x^*) = m^\circ(x^*)$, i.e., $x^* \in \mathcal{L}$. □

Lemma 5. *Let $x_n \in \mathcal{L}$, $x_n \nearrow x$ (or $x_n \searrow x$), $x \in \mathcal{F}$. Then $x \in \mathcal{L}$ and $m^\circ(x) = \lim\limits_{n\to\infty} m^\circ(x_n)$.*

Proof. If $x_n \in \mathcal{L}$, then for every $\varepsilon > 0$ there exists $b_n \in \mathcal{A}^+$, $b_n \geq x_n$, such that for all $n \in \mathbb{N}$, $m^\circ(x_n) + \frac{\varepsilon}{2^n} > m^+(b_n)$. Take $u_n = \bigvee\limits_{i=1}^{n} b_i$. Then $u_n \in \mathcal{A}^+$, $u_n \nearrow \bigvee\limits_{n=1}^{\infty} u_n \geq \bigvee\limits_{n=1}^{\infty} x_n = x$.

Since m^+ is a valuation, it implies

$$m^+(u_2) = m^+(b_1 \vee b_2) = m^+(b_1) + m^+(b_2) - m^+(b_1 \wedge b_2) \leq$$
$$\leq m^+(b_1) + m^+(b_2) - m^\circ(x_1) < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + m^\circ(x_2).$$

By induction we prove $m^+(u_n) < m^\circ(x_n) + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \cdots + \frac{\varepsilon}{2^n}$ and hence $m^\circ(x) \leq m^+\Big(\bigvee\limits_{n=1}^{\infty} u_n\Big) = \lim\limits_{n\to\infty} m^+(u_n) \leq \lim\limits_{n\to\infty} m^\circ(x_n) + \sum\limits_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \lim\limits_{n\to\infty} m^\circ(x_n) + \varepsilon$. Therefore $m^\circ(x) \leq \lim\limits_{n\to\infty} m^\circ(x_n)$. The opposite inequality follows, since m^* is non-decreasing.

Further $m_\circ(x) \leq m^\circ(x) = \lim\limits_{n\to\infty} m^\circ(x_n) = \lim\limits_{n\to\infty} m_\circ(x_n) \leq m_\circ(x)$, hence $x \in \mathcal{L}$.

And now we will prove the second part of Lemma 5.

Let $x_n \searrow x$, $x_n \in \mathcal{L}$, $n \in \mathbb{N}$. Then $x_n^* \nearrow x^*$. As has been shown above $m^\circ(x^*) = \lim\limits_{n\to\infty} m^\circ(x_n^*)$ and $x^* \in \mathcal{L}$. But then $x \in \mathcal{L}$ too and we can write:

$$m^\circ(x) = 1 - m^\circ(x^*) = 1 - \lim_{n\to\infty} m^\circ(x_n^*) = 1 - \lim_{n\to\infty}\big(1 - m^\circ(x_n)\big) = \lim_{n\to\infty} m^\circ(x_n).$$

□

Lemma 6. *Let $\mathcal{A}$ be an MV subalgebra of a σ-continuous MV σ-algebra $\mathcal{F}$. Let $\mathcal{S}(\mathcal{A})$ be the MV σ-algebra generated by $\mathcal{A}$ and $M(\mathcal{A})$ be the least monotone set over $\mathcal{A}$, i.e., $\mathcal{M}(\mathcal{A})$ be the least set over $\mathcal{A}$ closed under the monotone sequences of $\mathcal{A}$. Then $\mathcal{S}(\mathcal{A}) = \mathcal{M}(\mathcal{A})$.*

Proof. Since $\mathcal{S}(\mathcal{A})$ is a monotone set, it is $\mathcal{M}(\mathcal{A}) \subset \mathcal{S}(\mathcal{A})$. It is necessary to prove that $\mathcal{M}(\mathcal{A})$ is an MV σ-subalgebra of $\mathcal{F}$. Let $x \in \mathcal{A}$. Put $\mathcal{G} = \{y \in \mathcal{M}(\mathcal{A}) : x \oplus y \in \mathcal{M}(\mathcal{A})\}$. Evidently $\mathcal{A} \subset \mathcal{G}$. $\mathcal{G}$ is a monotone set, because if $y_n \in \mathcal{G}$ and $y_n \nearrow y$, then due to [2.3] $y_n \oplus x \nearrow y \oplus x$ which implies $y \oplus x \in \mathcal{M}(\mathcal{A})$ and so $y \in \mathcal{G}$. Therefore $\mathcal{M}(\mathcal{A}) \subset \mathcal{G}$. So, $x \oplus y \in \mathcal{M}(\mathcal{A})$ for all $y \in \mathcal{M}(\mathcal{A})$ and $x \in \mathcal{A}$. And now take $y \in \mathcal{M}(\mathcal{A})$ and put $\mathcal{K} = \{x \in \mathcal{M}(\mathcal{A}) : x \oplus y \in \mathcal{M}(\mathcal{A})\}$. $\mathcal{A} \in \mathcal{K}$. Let $x_n \in \mathcal{K}$ and $x_n \nearrow x$. Due to [2.3] $x_n \oplus y \nearrow x \oplus y$ and therefore $x \oplus y \in \mathcal{M}(\mathcal{A})$ which implies $x \in \mathcal{K}$. From this it follows that $\mathcal{K}$ is a monotone set, $\mathcal{M}(\mathcal{A}) \subset \mathcal{K}$ and $x \oplus y \in \mathcal{M}(\mathcal{A})$ for every $x, y \in \mathcal{M}(\mathcal{A})$.

Similary we can prove that if $x \in \mathcal{M}(\mathcal{A})$ then $x^* \in \mathcal{M}(\mathcal{A})$ too. As has been shown above, $\mathcal{M}(\mathcal{A})$ is closed with respect to $\oplus$ and $*$ for any $x, y \in \mathcal{M}(\mathcal{A})$. Since $x \odot y = (x^* \oplus y^*)^*$, $\mathcal{M}(\mathcal{A})$ is closed with respect to $\odot$ too. We proved that $\mathcal{M}(\mathcal{A})$ is an MV σ-subalgebra of $\mathcal{F}$. Therefore $\mathcal{M}(\mathcal{A}) \supset \mathcal{S}(\mathcal{A})$. □

And now we can prove Theorem 1.

1. Existence. As shown above, $\mathcal{S}(\mathcal{A}) = \mathcal{M}(\mathcal{A}) \subset \mathcal{L}$. Put $\mu = m^\circ/\mathcal{S}(\mathcal{A})$. It is easy to see that μ has the property [3.1] and μ is non decreasing. According to Lemma 6 $y \ominus x \in \mathcal{S}(\mathcal{A}) \subset \mathcal{L}$ for any $x, y \in \mathcal{S}(\mathcal{A})$ and $m^\circ(x) = m_\circ(x)$ holds on $\mathcal{L}$. By Lemma 2 for $x \le y$ it holds:

$$m^\circ(y \ominus x) \le m^\circ(y) - m_\circ(x) = m_\circ(y) - m^\circ(x) \le m_\circ(y \ominus x).$$

Therefore $\mu(y \ominus x) = \mu(y) - \mu(x)$ for any $x, y \in \mathcal{S}(\mathcal{A})$, $x \le y$. According to Lemma 5 μ is upper continuous.

Now we have to prove [3.10] for μ.

Let for any $n \in \mathbb{N}$ x_n, $y_n \in \mathcal{S}(\mathcal{A})$ and $x_n \searrow x$ and $y_n \nearrow y$. By [2.2] $y_n \ominus x_n \nearrow y \ominus x$. Let $x \le y$. Using the properties of μ which were mentioned above we can write:

$$\lim_{n\to\infty} \mu(y_n \ominus x_n) = \mu(y \ominus x) = \mu(y) - \mu(x) = \lim_{n\to\infty} \mu(y_n) - \lim_{n\to\infty} \mu(x_n).$$

2. Uniqueness. Let $\nu\colon \mathcal{S}(\mathcal{A}) \to [0,1]$ be a measure such that $\nu/\mathcal{A} = m$. Put $\mathcal{G} = \{x \in \mathcal{S}(\mathcal{A})\colon \mu(x) = \nu(x)\}$. Evidently $\mathcal{A} \subset \mathcal{G}$. But $\mathcal{G} \supset \mathcal{M}(\mathcal{A}) = \mathcal{S}(\mathcal{A})$ because $\mathcal{G}$ is closed under limits of monotone sequences.

Therefore $\nu(x) = \mu(x)$ for any $x \in \mathcal{S}(\mathcal{A})$. □

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