BOOLEAN UNIVERSE VERSUS FUZZY SETS

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ABSTRACT. We construct the universe of fuzzy sets in analogy with the Boolean valued universe of set theory (with urelements, respectively) and we discuss their connection for measure algebras. We define elements of ℓ^∞ to be fuzzy reals as counterpart of reals in the Boolean universe and investigate some notions and operations. We note that both Boolean and fuzzy two valued sets and notions coincide with those for classical sharp sets.

In many areas of mathematics and its applications multivalued logic and multivalued objects have appeared. Besides fuzzy sets ([Z]) and fuzzy logic recall at least Boolean valued universe of set-theoretical forcing constructions ([C, Sc, So, Vo]), model theory and the use of multivalued logic in expert systems and artificial intelligence (approximate reasoning and/or uncertain reasoning).

In this paper, we give first a brief review of the construction of Boolean valued universe in set theory and we develop similar transfinite construction for real-valued — i.e., fuzzy-sets. For Boolean algebras with measure there is a natural translation of Boolean sets to fuzzy sets. Note that there are connections of our presentation to the one which in different setting and motivation appeared in [DMS] and [St] already. Moreover note, that our approach is not the one of nonstandard analysis. Our standard reference source for set theory is [J] and for fuzzy sets [M].

Construction of the Boolean valued universe of set theory. Assume B to be a Boolean algebra. By U we denote the (possibly empty) set of urelements. We think of Boolean valued universe as of a generalization of sharp sets represented by $0, 1$-valued characteristic functions. Here we take characteristic functions taking values in a Boolean algebra B (with $0, 1$ represented as that of B). Moreover in a typical case (mainly in applications) fuzziness appears

just from a certain level on (say, e.g., by reals or sets of reals). That is why we construct our universe from urelements (which are untouched by fuzziness, e.g., natural numbers or reals).

DEFINITION. For a Boolean algebra B and a set of urelements U we define the *Boolean valued universe* $V^B(U)$ by transfinite induction:

(1) $V_0^B(U) = U$,

(2) $V_{\alpha+1}^B(U) = \{f \colon \operatorname{dom}(f) \subseteq V_\alpha^B(U) \quad \text{and} \quad \operatorname{rng}(f) \subseteq B\}$,

(3) $V_\lambda^B(U) = \bigcup_{\alpha<\lambda} V_\alpha^B(U)$ for λ limit,

(4) $V^B(U) = \bigcup_{\alpha\in On} V_\alpha^B(U)$.

Note that for main applications we do not work with whole $V^B(U)$ defined along all ordinals but it is enough to work with $V_{\aleph_0}^B(U)$ or sometimes even just with $V_1^B(U)$.

Recall from [J] the definition of Boolean truth value $\|\psi\|_B$, ψ being a closed formula or a formula with parameters from $V^B(U)$. Up to this point note that $V^B(U)$ is a well-founded universe, i.e., to each $f \in V^B(U)$ we can assign its rank α, where α is the first step where f in transfinite construction occurred. Note that $h \in \operatorname{dom}(f)$ implies $\operatorname{rank}(h) < \operatorname{rank}(f)$. Now assuming that values for smaller ranks (in the lexicographical order of pairs of ordinals) are already defined (for elements of U take classical $0,1$ values) we define Boolean truth values first for atomic and then for more complex formulas.

DEFINITION.
$$\|f \in g\|_B = \sum_{h\in\operatorname{dom}(g)} \left(\|f = h\|_B \wedge g(h)\right),$$

$$\|f \subseteq g\|_B = \prod_{h\in\operatorname{dom}(f)} \left(f(h) \Rightarrow \|h \in g\|_B\right),$$

for further logical connectives and quantifiers just develop the process using appropriate Boolean operations, see [J].

OBSERVATION. *For $A \subseteq U$ the intersection of two $f, g \colon A \longrightarrow B$ is defined by*
$$(f \cap g)(u) = f(u) \wedge g(u),$$
and the union by
$$(f \cup g)(u) = f(u) \vee g(u).$$

P r o o f . Notice that $f \cap g$ fulfills the formula being an intersection of f and g with Boolean value 1_B. $\qquad\square$

Construction of the fuzzy universe. Let $[0,1]$ denote the unit interval.

DEFINITION. For a set of urelements U we define the *fuzzy universe* $F(U)$ by transfinite induction:

(1) $F_0(U) = U$,

(2) $F_{\alpha+1}(U) = \{f \colon \operatorname{dom}(f) \subseteq F_\alpha(U) \quad \text{and} \quad \operatorname{rng}(f) \subseteq [0,1]\}$,

(3) $F_\lambda(U) = \bigcup_{\alpha < \lambda} F_\alpha(U)$ for λ limit,

(4) $F(U) = \bigcup_{\alpha \in On} F_\alpha(U)$.

To analogize the definition of truth values we need to replace Boolean join and union by some t-norm and its conorm. Nevertheless there is a problem that in the definition of $\|\cdot\|_B$ some — possibly infinite — joins and meets appeared. We can try to overcome this for the countable case, where we define for $a \in \ell^\infty$, $|a|_\infty \leq 1$,

$$T(a) = \lim_{n \to \infty} T(a_0, a_1, \ldots, a_n), \quad \text{if it exists}.$$

For fuzzy sets taking uncountably many values, we can approximate them by functions taking at most countably many values. A limit according to the net of approximations, if it exists, could be used as an appropriate fuzzy truth value. We will do this at some other place, here we use Boolean values translated by some measure.

Boolean universe versus fuzzy sets. We interpret Boolean sets to fuzzy sets, roughly speaking, along the transfinite construction in such a way that whenever the value is $x \in B$, the fuzzy target will have the value $\mu(x)$.

DEFINITION. Assume $\mu \colon B \longrightarrow [0,1]$ to be a measure on a Boolean algebra, U to be the set of urelements. We define

$$i_\mu \colon V^B(U) \longrightarrow F(U)$$

by transfinite induction as follows

$$x \in U = V_0^B(U), \quad \text{then put} \quad i_\mu(x) = x$$

having defined $i_\mu | V_\alpha^B(U) \colon V_\alpha^B(U) \longrightarrow F_\alpha(U)$, then for $f \in V_{\alpha+1}^B(U)$ we define $i_\mu(f)$ as follows

and
$$\operatorname{dom}\big(i_\mu(f)\big) = \{i_\mu(h) \colon h \in \operatorname{dom}(f)\} \subseteq F_\alpha(U)$$

$$i_\mu(f)\big(i_\mu(h)\big) = \mu\big(f(h)\big).$$

Note that it would be probably better to define

$$i_\mu(f)\big(i_\mu(h)\big) = \mu\big(\|h \in f\|_B\big),$$

but in this paper it makes no difference.

NOTATION.

(1) From now on, elements of $V^B(U)$ will be denoted by $f^B, g^B, \ldots$ and elements of $F(U)$ by $f^F, g^F, \ldots$

(2) Moreover, fix B being the algebra of Borel subsets of unit interval factorized by the ideal of sets of measure zero.

OBSERVATION AND/OR EXAMPLE.

(1) *Assume* $f^F \in F_1(U)$ *and* $U \neq \emptyset$, *then there are at least* $2^{\aleph_0}$-*many different Boolean names* f^B *such that* $i_\mu(f^B) = f^F$.

(2) *As for any* $x, y \in B$

$$\max\bigl(0, \mu(x) + \mu(y) - 1\bigr) \leq \mu(x \wedge y) \leq \min\bigl(\mu(x), \mu(y)\bigr),$$

we get the same estimation for values of $i_\mu(f^B \cap g^B)$ *according to values* $i_\mu(f^B)$ *and* $i_\mu(f^B)$ *(which clearly points to extremal t-norms* T_0 *and* T_∞ *of Frank's family, see* [F] *and* [M]).

This gives us the possibility to define the set of all possible fuzzy truth values as a μ-interpretation of the boolean truth values of all possible Boolean sources of parameters involved.

DEFINITION. Assume $f_1^F, f_2^F, \ldots, f_n^F \in F(U)$ and ψ to be a statement. Then the set $\mathrm{TV}\bigl(\psi(f_1^F, f_2^F, \ldots, f_n^F)\bigr)$ of all *fuzzy-Boolean supported-truth values* is defined to equal the set of all reals of the form $\mu\bigl(\bigl\|\psi(g_1^B, g_2^B, \ldots, g_n^B)\bigr\|_B\bigr)$ where g_i^B range through all objects with $i_\mu(g_i^B = f_i^F)$

This gives us a possibility to define (Boolean supported) fuzzy truth values

DEFINITION. Assume $f_1^F, f_2^F, \ldots, f_n^F \in F(U)$ and ψ to be a statement. Then put

$$\bigl\|\psi(f_1^F, f_2^F, \ldots, f_n^F)\bigr\|_F^{\inf} = \inf\Bigl(\mathrm{TV}\bigl(\psi(f_1^F, f_2^F, \ldots, f_n^F)\bigr)\Bigr),$$

and

$$\bigl\|\psi(f_1^F, f_2^F, \ldots, f_n^F)\bigr\|_F^{\sup} = \sup\Bigl(\mathrm{TV}\bigl(\psi(f_1^F, f_2^F, \ldots, f_n^F)\bigr)\Bigr).$$

OBSERVATION. *Assume* $x \in U$ *and* $f \colon A \longrightarrow [0, 1]$ *for some* $A \subseteq U$, *then*

$$\bigl\|x \in f^F \cap g^F\bigr\|_F^{\inf} =$$
$$= \max\bigl(0, \, f^F(x) + g^F(x) - 1\bigr) = T_\infty\Bigl(\bigl\|x \in f^F\bigr\|_F^{\inf}, \, \bigl\|x \in g^F\bigr\|_F^{\inf}\Bigr),$$

and

$$\bigl\|x \in f^F \cap g^F\bigr\|_F^{\sup} = \min\bigl(f^F(x), \, g^F(x)\bigr) = T_0\Bigl(\bigl\|x \in f^F\bigr\|_F^{\sup}, \, \bigl\|x \in g^F\bigr\|_F^{\sup}\Bigr).$$

P r o o f. In this case, we use the fact that we confined ourselves to the case where μ takes all values in $[0, 1]$. $\qquad\square$

Note that according to [P], [ŠT], [ŠTV] and [U] we can discuss also other structures as values of our many-valued universe (e.g., relational valued or lattice valued sets) and then translate them to fuzzy sets according to some submeasure or real valued evaluation of the structure.

Fuzzy reals and operations on them. We look for fuzzy reals as fuzzy analogue (via i_μ) of Boolean reals and such which in the sharp case coincides with classical reals. As both Boolean and fuzzy approach generalizes sharp sets being $0-1$–valued characteristic functions (and all we did up to now was consistent with and coincided in classical, sharp case), we have to use a representation of reals along this lines — these are reals in binary expansion.

Indeed, a real (we restrict ourselves to the unit interval with operations modulo one) $x = 0.\varepsilon_0\varepsilon_1\ldots\varepsilon_n\ldots$ (where $\varepsilon_i \in \{0, 1\}$) can be identified with $f: \mathbb{N} \longrightarrow \{0, 1\}$, where $f(n) = \varepsilon_n$, that is with the characteristic functions of subsets of natural numbers. So it is natural to consider Boolean subsets of $\mathbb{N}$ and fuzzy subsets of $\mathbb{N}$.

DEFINITION. A function $f^B \in V_1^B(\mathbb{N})$ is said to be a *Boolean real*. A function $f^F \in F_1(\mathbb{N})$ is said to be a *fuzzy real*.

To define sum (modulo 1) of Boolean and/or fuzzy reals we represent the sum of two classical reals $x = 0.\varepsilon_0\varepsilon_1\ldots\varepsilon_n\ldots$ and $y = 0.\delta_0\delta_1\ldots\delta_n\ldots$ as a limit

$$\lim_{i\to\infty} \left(0.\varepsilon_0\ldots\varepsilon_i0^- + 0.\delta_0\ldots\delta_i0^-\right)$$

and recall the rule when digit 1 transfers to the left. For $f^B \in V_1^B(\mathbb{N})$ we think of $f^B(n)$ as of Boolean truth value of the statement that on the nth position there is the digit 1, so here we have to transfer to the left the Boolean value of the fact that two (or three, with the one already transfered from before) 1's appeared. Note that following algorithm coincides in the sharp case with the classical one.

Let $\triangle$ denote the symmetric difference and $n = \{0, 1, \ldots, n-1\}$.

ALGORITHM B. Let $n \in \mathbb{N}$ and f^B, $g^B \in V_1^B(\mathbb{N})$. By induction through $i = 0$ to $n-1$ we construct $r(i) \in B$ and $(f^B + g^B)(n-i) \in B$ as follows:

step $i = 0$: $(f^B + g^B)(n-1) = f^B(n-1) \triangle g^B(n-1)$;

$\qquad r(0) = f^B(n-1) \wedge g^B(n-1)$,

step $i+1$: put $j = n - (i+1)$ and $f^B(j) = a$, $g^B(j) = b$ and $r(i) = c$, then

$$(f^B + g^B)(j) = \big(a - (b \vee c)\big) \vee \big(b - (a \vee c)\big) \vee \big(c - (a \vee b)\big) \vee (a \wedge b \wedge c),$$

these are the elements of B carrying information of one or three 1's which should be transferred to the left, and

$$r(i+1) = (a \wedge b) \vee (b \wedge c) \vee (a \wedge c),$$

this are elements of B carrying information of two or three digits 1.

OBSERVATION. *For any* f^B, $g^B \in V_1^B(\mathbb{N})$ *the limit* $\lim_{n \to \infty} (f^B|n + g^B|n)$ *exists (up to the fact that we work with one-to-one representation of reals, i.e., we do not handle reals taken from a point on all values equal 1).*

P r o o f. $V^B(\mathbb{N})$ is a Boolean model of ZFC, see [J]. $\qquad \square$

For the fuzzy case, fix some t-norm T, corresponding conorm S and symmetric difference $\triangle_T$.

ALGORITHM F. Let $n \in \mathbb{N}$ and f^F, $g^F \in F_1(\mathbb{N})$ and T be a fixed t-norm. By induction through $i = 0$ to $n - 1$ we construct $r(i) \in [0,1]$ and $(f^F \oplus_T g^F)(n - i) \in [0,1]$ as follows:

step $i = 0$: $(f^F \oplus_T g^F)(n-1) = f^F(n-1)\triangle_T g^F(n-1)$, and

$$r(0) = T\big(f^F(n-1), g^F(n-1)\big).$$

step $i + 1$: Put $j = n - (i+1)$ and $f^F(j) = a$, $g^F(j) = b$ and $r(i) = c$, then

$$(f^F \oplus_T g^F)(j) = S(x, y, u, v),$$

where $x = T\big(a,\, 1 - S(b,c)\big)$, $y = T\big(b,\, 1 - S(a,c)\big)$, $u = T\big(c,\, 1 - S(a,b)\big)$ and $v = T(a,b,c)$, and

$$r(i+1) \doteq S\big(T(a,b), T(b,c), T(a,c)\big).$$

Recall that c_0 are sequences of reals with the limit equal to 0.

THEOREM. *Assume that* f^F, $g^F \in F_1(\mathbb{N}) \cap c_0$ *and* T_∞ *is the smallest Archimedean t-norm. Then*

$$\lim_{n \to \infty} \big(f^F|n \oplus_{T_\infty} g^F|n\big)$$

exists (and we can put it equal to $f^F \oplus_{T_\infty} g^F$*).*

P r o o f. As $\lim_{n \to \infty} f^F(n) = \lim_{n \to \infty} g^F(n) = 0$ the new information that occurs at in the last digit position, and which possibly transfers to the left, is (from a point on) smaller than $\varepsilon = \frac{1}{4} > 0$. But then $r(0) = T\big(f^F(n),\, g^F(n)\big) = 0$ and so the digits established in previous steps do not change anymore and $(f^F \oplus_{T_\infty} g^F)(n) = f^F(n) + g^F(n)$ (the usual sum of reals). $\qquad \square$

OBSERVATION. *Assume that $f^B \in V_1^B(\mathbb{N})$ is such that there is a $P \subseteq B$ which is a partition of unity in B which refines the matrix $\{\{f^B(n), -f^B(n) : n \in \mathbb{N}\}\}$ consisting of elements of B. Then there are sharp reals $\{x_A : A \in P\}$ such that for every $A \in P$, $\left\| f^B = x_A \right\|_B = A$.*

P r o o f . For every $A \in P$ define $\varepsilon_n^A = 1$ if $A \leq f^B(n)$ and $\varepsilon_n^A = 0$ if $A \leq -f^B(n)$. Then $\left\| f^B = 0.\varepsilon_0^A \ldots \varepsilon_n^A \ldots \right\|_B = A$. $\qquad\square$

An analogue of this in fuzzy case is the following

OBSERVATION. *Assume $f^F \in F_1(\mathbb{N})$ and $x = 0.\varepsilon_0 \ldots \varepsilon_n \ldots$ to be a classical sharp real number. Denote $f^F(n)^0 = 1 - f^F(n)$ and $f^F(n)^1 = f^F(n)$. Then*

$$\left\| f^F = x \right\|_F^{\inf} \geq 1 - \sum_{n=0}^{\infty} \left(1 - f^F(n)^{\varepsilon_n} \right),$$

and

$$\left\| f^F = x \right\|_F^{\sup} \geq \inf \left\{ f^F(n)^{\varepsilon_n} : n \in \mathbb{N} \right\}.$$

We will finish our paper with a discussion on rational and/or irrational numbers. There are several possibilities to define a fuzzy real to be irrational.

Using previous theorem in Boolean case we have

$$\left\| f^B \text{ is irrational} \right\|_B = \bigvee \{ A : x_A \text{ is irrational} \}$$

so we could try to do the same in the fuzzy case, but we should be careful because of the following example.

EXAMPLE. Let $f^F(n) = \frac{1}{2}$ for all $n \in \mathbb{N}$. Then there is no sharp real x with $\left\| f^F = x \right\|_F^{\inf} > 0$. On the other hand, for every sharp real x, $\left\| f^F = x \right\|_F^{\sup} = \frac{1}{2}$.

For sharp reals in binary expansion to be rational also means to have just finitely many 1's — i.e., the sum of digits is $< \infty$. So to generalize this to fuzzy case we can define a fuzzy real f^F to be rational provided $\sum_{n=0}^{\infty} f^F(n) < \infty$, otherwise it is said to be irrational. So for an $f^F \in c_0$ we take the ideal

$$\mathcal{I}_{f^F} = \left\{ X \subseteq \mathbb{N} : \sum_{n \in X} f^F(n) < 1 \right\}.$$

Then

$$\left\| f^F \text{ has on } X \text{ digits } 0 \right\|_F^{\inf} \geq 1 - \sum_{n \in X} f^F(n).$$

So in a sense for X infinite, with $\mathbb{N} \setminus X$ being infinite too, this says f^F is irrational. These ideals are closely related to the structures in the following theorem.

THEOREM. ([V]) *If* $\mathfrak{p} = cf(2^{\aleph_0})$, *then the Boolean algebras* $\mathrm{RO}(\ell^\infty \setminus \ell^1, \leq^*)$ *and* $\mathrm{RO}\big(P(\mathbb{N})/_{fin}, \subseteq^*\big)$ *are isomorphic.*

In this paper, we tried to give some motivation, how using Boolean concepts one can define corresponding (boolean supported) fuzzy concepts. We did not formulate any problem, because there are many problems concerning relationships and properties of all these new objects.

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