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COMPATIBILITY IN D–POSETS OF FUZZY SETS

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ABSTRACT. In this paper the notion of compatibility in D–posets of fuzzy sets is introduced. Further the characterization of the finite compatible subsets of D–poset of fuzzy sets is given.

1. Introduction

In many structures of fuzzy sets, for example, in fuzzy measurable spaces introduced by Klement [8], soft σ-algebras introduced by Piasecki [15], F–quantum spaces introduced by Riečan [17], F–quantum posets introduced by Dvurečenskij and Chovanec [6], the original Zadeh [18] fuzzy connectives, which are induced by the partial ordering of fuzzy sets, are used.

There are a lot of systems of fuzzy sets, for example, fuzzy quantum logic introduced by Pykacz [16], fuzzy measurable spaces introduced by Butnariu [1] and Mesiar [13], T–measurable spaces introduced by Butnariu and Klement [2], h-fuzzy quantum logics and full fuzzy difference posets introduced by Mesiar [11], [12], in which the operations of fuzzy sets are defined by Gilles connectives [7], or they are introduced by means of t-conorms and t-norms.

Recently has appeared a new mathematical model, a D–poset of fuzzy sets or difference poset of fuzzy sets, introduced by author [9], in which a difference operation is a primary notion and, moreover, a difference of fuzzy sets has the same properties as the difference of crisp sets.

Some questions of the probability theory have been studied on D–posets of fuzzy sets (see [3], [4]). The aim of this paper is to study a compatibility in D–posets of fuzzy sets, which is a useful notion from the mathematical point of view and, on the other hand, it has its own physical meaning. Compatible pairs play an important role in the axiomatics of D–posets of fuzzy sets theories, since they represent simultaneously verifiable events.

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Though the compatibility of a finite set of elements in a quantum logic means that they belong to the same Boolean sublogic [14], in general, we cannot say anything similar about the existence of such a Boolean subalgebra in the case of D–posets of fuzzy sets, which enable a new look at the compatibility.

We note that there is a generalization of D–posets of fuzzy sets on any partially ordered set, a D–poset, introduced by the author and F. Chovanec [10]. D–posets generalize, for example, quantum logics, orthoalgebras, the set of efects, MV algebras.

2. D–posets of fuzzy sets

A *D–poset* of fuzzy sets [9] is a partially ordered set $\mathcal{F} \subseteq [0,1]^X$ with a partial ordering $\leq$, a greatest element $1(x) = 1$ for every $x \in X$ and with a partial binary operation $\searrow : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$, called a *difference*, such that, for $f, g \in \mathcal{F}$, $g \searrow f$ is defined if and only if $f \leq g$, and for $\searrow$ the following axioms hold for $f, g, h \in \mathcal{F}$:

(1) $g \searrow f \leq g$,

(2) $g \searrow (g \searrow f) = f$,

(3) if $f \leq g \leq h$, then $h \searrow g \leq h \searrow f$ and $(h \searrow f) \searrow (h \searrow g) = g \searrow f$.

The following statements have been proved in [10].

Proposition 1. *Let f, g, h, d be the elements of a D-poset of fuzzy sets $\mathcal{F}$. Then*

(i) $1 \searrow 1$ *is the least element of $\mathcal{F}$; denote it by* 0.

(ii) $f \searrow 0 = f$.

(iii) $f \searrow f = 0$.

(iv) *If $f \leq g$, then $g \searrow f = 0$ if and only if $g = f$.*

(v) *If $f \leq g$, then $g \searrow f = g$ if and only if $f = 0$.*

(vi) *If $f \leq g \leq h$, then $g \searrow f \leq h \searrow f$ and $(h \searrow f) \searrow (g \searrow f) = h \searrow g$.*

(vii) *If $g \leq h$ and $f \leq h \searrow g$, then $(h \searrow g) \searrow f = (h \searrow f) \searrow g$.*

(viii) *If $f \leq g \leq h$, then $f \leq h \searrow (g \searrow f)$ and $\big(h \searrow (g \searrow f)\big) \searrow f = h \searrow g$.*

(ix) *If $f \leq h$ and $g \leq h$, then $h \searrow f = h \searrow g$ if and only if $f = g$.*

(x) *If $d \leq f \leq h$, $d \leq g \leq h$, then $h \searrow f = g \searrow d$ if and only if $h \searrow g = f \searrow d$.*

□

3. $\bigoplus$-orthogonal systems of fuzzy sets

Let $\mathcal{F}$ be a D–poset of fuzzy sets. We put $f^{\perp} := 1 \setminus f$ for any $f \in \mathcal{F}$. Then the unary operation $\perp$ on $\mathcal{F}$ has the following properties: (i) $(f^{\perp})^{\perp} = f$; (ii) if $f \leq g$ then $g^{\perp} \leq f^{\perp}$.

We say that two fuzzy sets f and g are *orthogonal*, and write $f \perp g$, if $f \leq g^{\perp}$ (or equivalently $g \leq f^{\perp}$). For orthogonal fuzzy sets f and g we define their sum as follows:

$$f \oplus g := (g^{\perp} \setminus f)^{\perp}.$$

The partial binary operation $\oplus$ on $\mathcal{F}$ is commutative and associative (see [5]). It is not difficult to verify that the sum of orthogonal fuzzy sets has the following properties:

PROPOSITION 2. *Let $\mathcal{F}$ be a D–poset of fuzzy sets.*

(1) *If $f \leq g^{\perp}$ then $f \leq f \oplus g$ and $g \leq f \oplus g$;*

(2) *If $f \leq g^{\perp}$ then $(f \oplus g) \setminus f = g$ and $(f \oplus g) \setminus g = f$;*

(3) *If $f, g, h \in \mathcal{F}$, $f \leq g \leq h$, then there exists $(h \setminus g) \oplus (g \setminus f)$ in $\mathcal{F}$ and $(h \setminus g) \oplus (g \setminus f) = h \setminus f$.*

□

Let $\mathrm{F} = \{f_1, \dots, f_n\}$ be a finite sequence of $\mathcal{F}$. According to [5], recursively we define for $n \geq 3$

$$f_1 \oplus \cdots \oplus f_n := (f_1 \oplus \cdots \oplus f_{n-1}) \oplus f_n\,,$$

supposing that $f_1 \oplus \cdots \oplus f_{n-1}$ and $(f_1 \oplus \cdots \oplus f_{n-1}) \oplus f_n$ exist in $\mathcal{F}$. Definitorically we put $f_1 \oplus \cdots \oplus f_n := f_1$ if $n = 1$, and $f_1 \oplus \cdots \oplus f_n := 0$ if $n = 0$. Then for any permutation $(i_1, \dots, i_n)$ of $(1, \dots, n)$ and any k with $1 \leq k \leq n$ we have

$$f_1 \oplus \cdots \oplus f_n = f_{i_1} \oplus \cdots \oplus f_{i_n}\,,$$
$$f_1 \oplus \cdots \oplus f_n = (f_1 \oplus \cdots \oplus f_k) \oplus (f_{k+1} \oplus \cdots \oplus f_n)\,.$$

Let $\mathcal{F}$ be a D–poset of fuzzy sets. We say that a finite system $F = \{f_1, \dots, f_n\}$ of $\mathcal{F}$ is $\oplus$-orthogonal iff $f_1 \oplus \cdots \oplus f_n$ exists in $\mathcal{F}$ and write $f_1 \oplus \cdots \oplus f_n = \bigoplus_{i=1}^{n} f_i$. An arbitrary system G of $\mathcal{F}$ is $\oplus$-orthogonal if every finite subsystem F of G is $\oplus$-orthogonal.

4. Compatibility in D–posets of fuzzy sets

In this section we give the definition of compatibility of two fuzzy sets and the definition of a compatible subset of D–poset of fuzzy sets.

DEFINITION 1. Let $\mathcal{F}$ be a D–poset of fuzzy sets. We say that two fuzzy sets $f, g \in \mathcal{F}$ are *compatible* (in $\mathcal{F}$) if there exist such fuzzy sets $u, v \in \mathcal{F}$ that $v \leq f \leq u$, $v \leq g \leq u$ and $u \searrow f = g \searrow v$. We write $f \leftrightarrow g$.

The definition of compatibility of two elements is correct by the Proposition 1, statement (x).

PROPOSITION 3. *Let $\mathcal{F}$ be a D–poset of fuzzy sets.*

(i) *If $f \leq g$, then $f \leftrightarrow g$.*

(ii) *If f, g, h are elements of $\mathcal{F}$, $f \leq h$, $g \leq h \searrow f$ then $f \leftrightarrow g$.*

Proof.

(i) It suffices to put $u = g$ and $v = f$. Then $u \searrow f = g \searrow f = g \searrow v$.

(ii) From $f \leq h$ and $g \leq h \searrow f$ we get
$f = h \searrow (h \searrow f) \leq h \searrow \big((h \searrow f) \searrow g\big)$,
$g = h \searrow (h \searrow g) \leq h \searrow (h \searrow g) \searrow f = h \searrow \big((h \searrow f) \searrow g\big)$.

We put $u = h \searrow \big((h \searrow f) \searrow g\big)$ and $v = 0$. Then

$$u \searrow f = \big[h \searrow \big((h \searrow f) \searrow g\big)\big] \searrow f = (h \searrow f) \searrow \big((h \searrow f) \searrow g\big) = g = g \searrow 0.$$

□

THEOREM 1. *Let $\mathcal{F}$ be a D–poset of fuzzy sets and let $f, g \in \mathcal{F}$. Then the following four assertions are equivalent.*

1) *There exist $u, v \in \mathcal{F}$, $v \leq f \leq u$, $v \leq g \leq u$ such that $u \searrow f = g \searrow v$.*

2) *There exists $u \in \mathcal{F}$, $f \leq u$, $g \leq u$ such that $u \searrow f \leq g$, $u \searrow g \leq f$, respectively.*

3) *There exists $v \in \mathcal{F}$, $v \leq f$, $v \leq g$ such that $g \searrow v \leq 1 \searrow f$, $f \searrow v \leq 1 \searrow g$, respectively.*

4) *There exists $\oplus$-orthogonal triplet $\{f_1, g_1, h_1\}$ of fuzzy sets from $\mathcal{F}$, such that $f = f_1 \oplus h_1$, $g = g_1 \oplus h_1$.*

Proof. Let the assertion 1) hold, then the assertion 2) is evident. Let us suppose that the assertion 2) is true, i.e., there exists $u \in \mathcal{F}$, $f \leq u$, $g \leq u$ and $u \searrow g \leq f$. Then

$$f \searrow (u \searrow g) = \big(u \searrow (u \searrow f)\big) \searrow (u \searrow g) = \big(u \searrow (u \searrow g)\big) \searrow (u \searrow f) = g \searrow (u \searrow f).$$

Put $v = f \searrow (u \searrow g) = g \searrow (u \searrow f)$. Then $v \leq f$, $v \leq g$ and $g \searrow v = g \searrow \big(g \searrow (u \searrow f)\big) = u \searrow f \leq 1 \searrow f$. The assertion 3) is proved.

Let there exist $v \in \mathcal{F}$, $v \leq f$, $v \leq g$ such that $g \searrow v \leq 1 \searrow f$, and $f \searrow v \leq 1 \searrow g$, respectively. Then

$$\begin{aligned}(1 \searrow g) \searrow (f \searrow v) &= (1 \searrow g) \searrow \big[(1 \searrow v) \searrow (1 \searrow f)\big] \\ &= (1 \searrow g) \searrow \big[((1 \searrow v) \searrow (g \searrow v)) \searrow ((1 \searrow f) \searrow (g \searrow v))\big] \\ &= (1 \searrow g) \searrow \big[(1 \searrow g) \searrow ((1 \searrow f) \searrow (g \searrow v))\big] \\ &= (1 \searrow f) \searrow (g \searrow v)\,.\end{aligned}$$

We put $u = 1 \searrow \big[(1 \searrow f) \searrow (g \searrow v)\big] = 1 \searrow \big[(1 \searrow g) \searrow (f \searrow v)\big]$. Then $1 \searrow f \geq (1 \searrow f) \searrow (g \searrow v)$ and so $f \leq 1 \searrow \big[(1 \searrow f) \searrow (g \searrow v)\big] = u$. The proof of the inequality $g \leq u$ is similar.
Then we have

$$u \searrow f = \big[1 \searrow ((1 \searrow f) \searrow (g \searrow v))\big] \searrow f = (1 \searrow f) \searrow \big[(1 \searrow f) \searrow (g \searrow v)\big] = g \searrow v\,,$$

and

$$u \searrow g = \big[1 \searrow ((1 \searrow g) \searrow (f \searrow g))\big] \searrow g = (1 \searrow g) \searrow \big[(1 \searrow g) \searrow (f \searrow g)\big] = f \searrow g\,,$$

i.e., 1) is true. We have just shown the equivalency of assertions 1)–3).

Finally we prove the equivalence of the assertion 1) with the assertion 4). Let the assertion 4) hold. We put $u = f_1 \oplus g_1 \oplus h_1$ and $v = h_1$. Then

$$u \geq f, g, \quad v = h_1 = g \searrow g_1 \leq g\,, \quad v = h_1 = f \searrow f_1 \leq f\,,$$

and

$$u \searrow f = u \searrow (f_1 \oplus h_1) = g_1 = g \searrow h_1 = g \searrow v\,.$$

Now let there exist $v, u \in \mathcal{F}$, $v \leq f$, $g \leq u$ such that $u \searrow g = f \searrow v$ and $u \searrow f = g \searrow v$, respectively. Put

$$f_1 = f \searrow v = u \searrow g\,, \quad g_1 = g \searrow v = u \searrow f\,, \quad h_1 = v\,.$$

Then

$$f = (f \searrow v) \oplus v = f_1 \oplus h_1 \quad \text{and} \quad g = (g \searrow v) \oplus v = g_1 \oplus h_1\,.$$

The $\oplus$-orthogonality of triplet $\{f_1, g_1, h_1\}$ is evident. □

Definition 2. Let $\mathcal{F}$ be a D–poset of fuzzy sets. We say that the finite subset $F = \{f_1, f_2, \dots, f_n\} \subseteq \mathcal{F}$ is *compatible* (in $\mathcal{F}$) if there exists a $\oplus$-orthogonal system G of elements of $\mathcal{F}$, $G = \{g_t,\ t \in T\}$, such that $f_i = \oplus\{g_t;\ t \in T_i\}$, where T_i is a finite subset of T, for every $i = 1, \dots, n$.

An *arbitrary subset* $E \subseteq \mathcal{F}$ *is compatible* (in $\mathcal{F}$) if every finite subset of E is compatible (in $\mathcal{F}$).

If we consider pairs in $\mathcal{F}$, our "general" definition, Definition 2, agrees with Definition 1.

A *D-poset* ([10]) is a partially ordered set P with a partial ordering $\leq$, maximal element 1, and with a partial binary operation $\setminus \colon P \times P \to P$, called a difference, such that, for $a, b \in P$, $b \setminus a$ is defined if and only if $a \leq b$, where the following axioms hold for $a, b, c \in P$: (i) $b \setminus a \leq b$; (ii) $b \setminus (b \setminus a) = a$; (iii) $a \leq b \leq c \Rightarrow c \setminus b \leq c \setminus a$ and $(c \setminus a) \setminus (c \setminus b) = b \setminus a$.

If $\mathcal{B}$ is a Boolean algebra and $a, b \in \mathcal{B}$, $a \leq b$, then $b \setminus a := b \wedge a^{\perp}$ is a difference on $\mathcal{B}$, hence $\mathcal{B}$ is a D-poset.

Let P_1 and P_2 be two D-posets. According to [10] we say that a mapping $w \colon P_1 \to P_2$ is a *morphism* if $w(1_{P_1}) = 1_{P_2}$, $a, b \in P_1$, $a \leq b$, implies $w(a) \leq w(b)$ and $w(b \setminus a) = w(b) \setminus w(a)$.

LEMMA 1. *Let $\mathcal{F}_1$, $\mathcal{F}_2$ be two D-posets of fuzzy sets. Let $w : \mathcal{F}_1 \to \mathcal{F}_2$ be a morphism. Let F be a $\oplus$-orthogonal system of $\mathcal{F}_1$. Then $w(F)$ is an $\oplus$-orthogonal system of $\mathcal{F}_2$.*

Proof. By the definition of the operation $\oplus$ we have

$$\begin{aligned} w(f \oplus g) &= w\big[1 \setminus \big((1 \setminus f) \setminus g\big)\big] = w(1) \setminus \big[\big(w(1) \setminus w(f)\big) \setminus w(g)\big] = \\ &= 1 \setminus \big[\big(1 \setminus w(f)\big) \setminus w(g)\big] = w(f) \oplus w(g)\,. \end{aligned}$$

Let $G = \{w(f_1),\ \dots\ w(f_n),\ f_1, \dots f_n \in F\}$ be an arbitrary finite system of $w(F)$. Then, by previous, there exists $w(f_1 \oplus \dots \oplus f_n) = w(f_1) \oplus \dots w(f_n)$ in $\mathcal{F}_2$. Hence the system $w(F)$ is $\oplus$-orthogonal. □

The previous Lemma implies the following theorem.

THEOREM 2. *Let $\mathcal{F}_1$, $\mathcal{F}_2$ be two D-posets of fuzzy sets. Let $w : \mathcal{F}_1 \to \mathcal{F}_2$ be a morphism. Let $F \subseteq \mathcal{F}_1$ be a compatible subset of fuzzy sets of $\mathcal{F}_1$. Then $w(F)$ is a compatible subset of fuzzy sets of $\mathcal{F}_2$.* □

The following theorem deals with the characterization of finite compatible subsets of D-poset of fuzzy sets.

THEOREM 3. *Let $\mathcal{F}$ be a D-poset of fuzzy sets. The finite subset $F = \{a_1, \dots, a_n\} \subseteq \mathcal{F}$ is compatible if and only if there exist a Boolean algebra $\mathcal{B}$ and morphism $w \colon \mathcal{B} \to \mathcal{F}$ such that $F \subseteq w(\mathcal{B})$.*

Proof. Let $\mathcal{B}$ be a Boolean algebra, $w \colon \mathcal{B} \to \mathcal{F}$ is a morphism and $F \subseteq w(\mathcal{B})$. Let $C = \{b_1, \dots, b_n\} \subseteq \mathcal{B}$ be a subset of elements of $\mathcal{B}$ such that $w(b_i) = a_i$ for every $i = 1, \dots, n$. Denote $\bigvee B = b_{B_1} \vee \dots \vee b_{B_k}$, and $\bigwedge B = b_{B_1} \wedge \dots \wedge b_{B_k}$ for every $B = \{b_{B_1}, \dots, b_{B_k}\} \subseteq C$, $2 \leq k \leq n$.

If $k = 1$ then $\bigvee B = \bigwedge B = b_{B_1}$, if $k = 0$ then $\bigvee B = 0_{\mathcal{B}}$ and $\bigwedge B = 1_{\mathcal{B}}$. Then in the system G_C of $\mathcal{B}$

$$G_C = \big\{d_B = (\textstyle\bigwedge B) \wedge \big[\bigvee (C \setminus B)\big]^{\perp},\ B \subseteq C\big\}$$

the following conditions are satisfied:

(i) $d_B \le b_i$ for every $b_i \in B$;

(ii) $d_B \perp b_j$ for every $b_j \in C$, $b_j \notin B$;

(iii) $b_i = \bigvee\{d_B,\ B \subseteq C,\ b_i \in B\}$, $i = 1, \dots, n$.

Therefore the system G_C is the system of mutually orthogonal elements. Since the operation $\oplus$ in $\mathcal{B}$ is identical with the operation $\vee$, G_C is $\oplus$-orthogonal system. The morphism properties imply $\oplus$-orthogonality of $G = w(G_C)$, and further

$$a_i = w(b_i) = w\big(\bigvee\{d_B,\ B \subseteq C,\ b_i \in B\}\big) = \oplus\big\{w(d_B),\ B \subseteq C,\ b_i \in B\big\},$$

$i = 1, \dots, n$.

Conversely, suppose that $F \subseteq \mathcal{F}$ is a compatible set, i.e., there exists a $\oplus$-orthogonal system G of elements of $\mathcal{F}$, $G = \{g_t;\ t \in T\}$, such that $a_i = \oplus\{g_t;\ t \in T_i\}$, where T_i is a finite subset of T, $i = 1, \dots, n$.
Denote $S = \bigcup_{i=1}^{n} T_i$ and $h = \oplus\{g_t;\ t \in S\}$. Let $\mathcal{B}$ be the algebra of all subsets of the set $\{g_t;\ t \in S\} \cup \{1 \setminus h\}$. The map $w:\ \mathcal{B} \to \mathcal{F}$ such that

(i) $w\big(\{g_t\}\big) = g_t$, for every $t \in S$;

(ii) $w\big(\{1 \setminus h\}\big) = 1 \setminus h$;

(iii) $w(B) = \oplus\big\{w(\{b\});\ b \in B\big\}$ for every $\in \mathcal{B}$

is the morphism. Further it holds

$$w\big(\{g_t;\ t \in T_i\}\big) = \oplus\big\{w\big(\{g_t\}\big);\ t \in T_i\big\} = \oplus\{g_t;\ t \in T_i\} = a_i$$

for every $i = 1, \dots, n$. Hence $F \subseteq w(\mathcal{B})$. $\square$

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