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THE DIFFERENCE POSET OF MONOTONE FUNCTIONS

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ABSTRACT. The finite chains are the simplest ordered structures which can be organized into a difference poset (D–poset). But even in easy cases the set of all monotone functions from one chain into another endowed with the pointwise order will not carry any difference structure. If one sharpens the pointwise order to the monotone difference order, the monotone functions from one D–poset into another D–poset again form a D–poset.

1. Ideals and cartesian products of D–posets

Kôpka and Chovanec [4] have introduced the notion of a difference poset (D–poset). Soon it became clear that this influenced a fruitful investigation (e.g., [1], [2], [5], [6]).
We assume a D–poset to be a mathematical object of the following kind

$$\mathbb{D} = (D, \leq, 0, -):$$

1. $(D, \leq, 0)$ is a partial ordered set (poset) with the smallest element 0.

2. $-$ is a partial defined binary operation on D such that the following conditions are satisfied:

   (i) $b - a$ is defined iff $a \leq b$.

   (ii) $a - 0 = a$ for all $a \in D$.

   (iii) For all $a \leq b$ it always holds $b - a \leq b$ and $b - (b - a) = a$.

   (iv) For $a \leq b \leq c$ it always holds $b - a = (c - a) - (c - b)$.

Remark. We do not assume that there must be a greatest element — a so-called order unit $\mathbb{1}$. (For such generalized difference posets see [3] or the RI–posets of Kalmbach and Riečanová).
$(\mathbb{N}, \leq, 0, -)$ — the chain of natural numbers with respect to the natural order and natural difference structure is a D–poset.

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The following facts can be easily seen.

Proposition 1. *Let $\boldsymbol{D} = (D, \leq, 0, -)$ be a D–poset and I an order ideal of $(D, \leq)$, i.e., I is a non-empty inductive subset of D:*

$$x \leq y \in I \implies x \in I\,.$$

Then $\boldsymbol{I} = (I, \leq, 0, -)$ with respect to the induced order and the difference is a D–poset.

Proposition 2. *Let $(\boldsymbol{D}_\alpha)_{\alpha\in A}$ be any family of D–posets. Then the cartesian product $\prod_{\alpha\in A} D_\alpha$ with respect to the coordinatewise order and coordinatewise operation is a D–poset, too. This will be called the cartesian product $\boldsymbol{D} = \prod_{\alpha\in A} \boldsymbol{D}_\alpha$ of the family.*

Examples.

1.1. Each finite chain is order-isomorphic to an ideal of the D–poset $\boldsymbol{N}$. Thus it is a D–poset with a unit. (For the unicity of the difference structure see [6]).

1.2. The non-isomorphic order ideals of the cartesian product of a two-element chain and a 3-element chain with itself are listed in Figure 1.

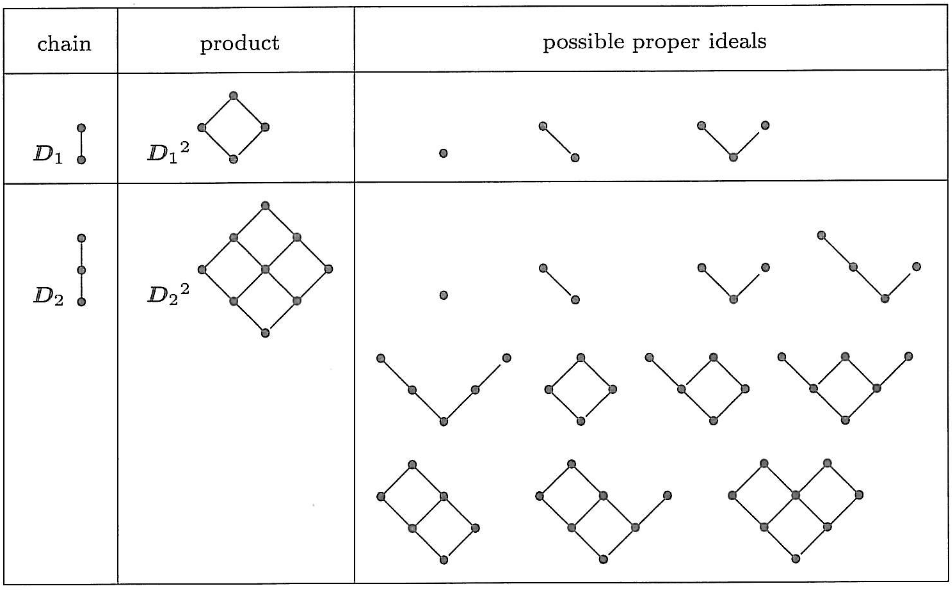


Figure 1. Table for examples 1.2.

1.3. The subset X (see Fig. 2) of the product $\boldsymbol{D_2}^2$ is not an ideal in $\boldsymbol{D_2}^2$, but it can be organized into a suitable D–poset having $b - a = a = c - a$, $b - b = c - c = a - a = 0$.

1.4. The product $\boldsymbol{D_1}^2$ of the two-element chain carries the D–structure of the Boolean algebra $\mathbf{2}^2$ but it can be organized also into another D–poset which is not Boolean, namely, we put $1 - a =: a$, $1 - b =: b$. Thus the 4-point lattice $\boldsymbol{D_1}^2$ is the smallest poset with two non-isomorphic D–poset structures!

1.5. The subset Y (see Fig. 2) of the product $\boldsymbol{D_1}^3$ is not an ideal in this product but it can also carry two non-isomorphic difference structures, namely, $1 - x = x$ for all three atoms a, b, c or only for one atom, e.g., $x = a$ and, furthermore, $1 - b = c$, $1 - c = b$.

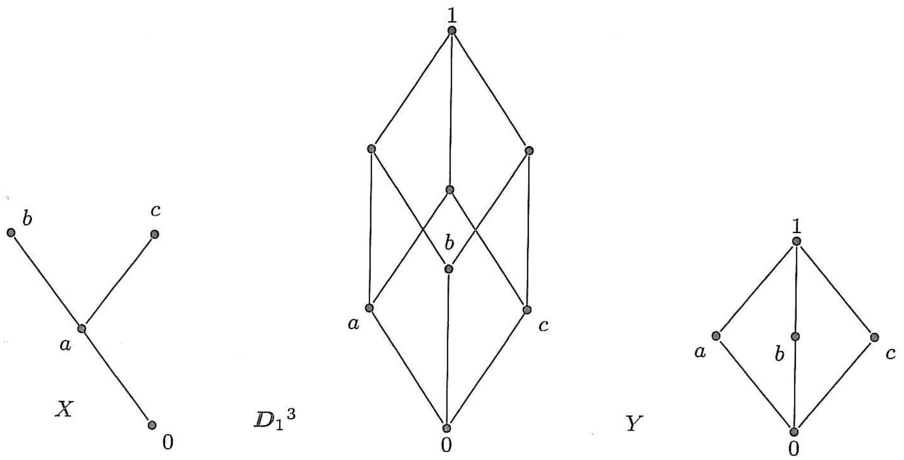


FIGURE 2. Concerning examples 1.3, 1.5.

2. Monotone functions on D–posets

Let $\boldsymbol{D_1}$ and $\boldsymbol{D_2}$ be two given D–posets. By $\mathrm{Mon}(\boldsymbol{D_1}, \boldsymbol{D_2})$ we understand the set of all monotone functions from $\boldsymbol{D_1}$ into $\boldsymbol{D_2}$, i.e., $f \in \mathrm{Mon}(\boldsymbol{D_1}, \boldsymbol{D_2}) \Longleftrightarrow f : D_1 \to D_2$ which leaves the order invariant, thus $f(x) \leq f(y)$ for all $x \leq y$ in D_1.

EXAMPLES 2.1. For the chains $\boldsymbol{D_1}$:, $\boldsymbol{D_2}$: of two resp. three elements the set $\mathrm{Mon}(\boldsymbol{D_1}, \boldsymbol{D_2})$ has a table of elements given in Fig. 3.

With respect to the pointwise order it gives the shown Hasse diagram (see Fig. 3). This poset cannot be equipped with any difference structure such that

	0	1
0	0	0
a	0	1
b	0	2
c	1	1
d	1	2
1	2	2

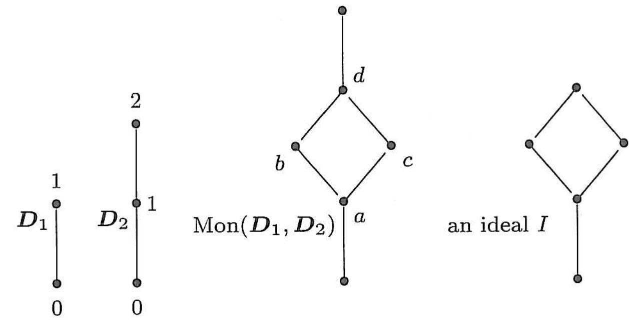


FIGURE 3. Lattice of monotone functions

it would be a D–poset. Namely, for $a < b < d$ and $a < c < d$ it should be $d-a > d-b$, $d-a > d-c$. If $d-a = b$ and $d-b = a$ there must be $d-c = a$ but for the tripes $0 < c < d$ it has to be $c - 0 = (d-0) - (d-c) = d - a = b$, which is a contradiction.

By this argumentation we have seen that the poset I (see Fig. 3) cannot carry any difference structure. This and it dual are the smallest possible examples of this kind (an example with 7 points has been given in [6]).

2.2. The D–posets D_1, D_2 of Fig. 4 have the poset Mon(D_1, D_2) with the following table of monotone functions and the given Hasse diagram with respect to its pointwise order.

0	a	b
0	0	0
0	0	1
0	1	0
0	1	1
1	1	1

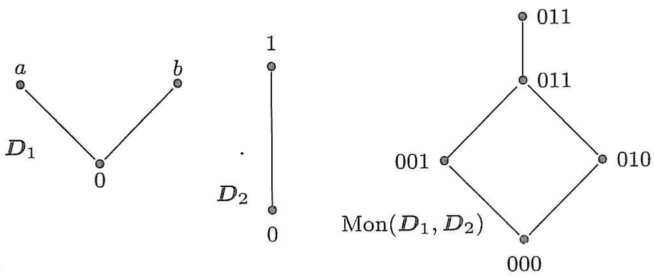


FIGURE 4. Another lattice of monotone functions.

But this ordered set cannot carry any difference structure, either.

Now we consider some special monotone functions for D–posets which preserve all of the D–structure. Thus a function $f\colon D_1 \to D_2$ from D_1 into D_2 has to be called a D–morphism, i.e., f is a *D–morphism* $\Longrightarrow$ 1. f is monotone, 2. $f(0) = 0$, 3. f is subtractive:

$$f(x - y) = f(x) - f(y) \quad \text{for all} \quad x \leq y \in D_1 .$$

EXAMPLES. 2.3. A finite chain $\boldsymbol{D}$ with at least 3 elements has only the trivial 2-valued D–morphism $f \equiv 0$. Namely, let $\boldsymbol{D} = \{0, 1, ..., n\}$, $n \geq 2$. Then, for $2 \in \boldsymbol{D}$, $f(2) - f(1) = f(2-1) = f(1)$. Thus $f(1) = 0$ and therefore $f(2) = 0$. By induction one gets for every $x \in \boldsymbol{D}$ the value $f(x) = 0$.

2.4. A chain D–poset $\boldsymbol{D}$ (abbreviated by D–chain) may have a non-trivial 2-valued D–morphism. It is not clear yet which chains can be made into a D–chain. The existence of a non-trivial 2–valued D–morphism on a D–chain is equivalent to the following property: There exists a decomposition $D = (I_0, I_1)$ into $I_0(\neq \emptyset)$ — the set of "small points", and $I_1(\neq \emptyset)$ — the set of "large points", $I_1 = D \setminus I_0$, for which

(1) I_0 is an order ideal (a inductive set: $I_0 \ni x > y \implies y \in I_0$).

(2) I_1 is an anti-order ideal ($I_1 \ni y < x \implies x \in I_1$).

(3) $x \in I_1$ and $y \in I_0 \implies x - y \in I_1$ (the difference of a large point and a small point is big).

(4) $x \in I_1$ and $y \in I_1$ with $x > y \implies x - y \in I_0$ (the difference of large points is always small).

For the wanted D–morphism one has only to take I_0 as the kernel. The decomposition property occurs for example in the following chain.

$$C = I_0 \cup I_1 , \quad \text{where} \quad I_0 = \{(0,n);\ n \in \mathbb{N}\}, \quad I_1 = \{(1,n);\ n \in \mathbb{N}\},$$

with the order

$$\begin{aligned} &(0,n) \leq (0,m) \iff n \leq m, \\ &(1,n) \leq (1,m) \iff n \geq m, \\ &(0,n) \leq (1,m) \qquad \text{for every} \quad n, m \in \mathbb{N}. \end{aligned}$$

The difference is as follows

$$\begin{aligned} &(0,m) - (0,n) = (0, m-n) &&\text{for} \quad m \geq n, \\ &(1,m) - (0,k) = (1, m+k) &&\text{for arbitrary} \quad m, k, \\ &(1,m) - (1,n) = (0, n-m) &&\text{for} \quad n \geq m. \end{aligned}$$

Hint: The construction of $I_0 \cup I_1$ is a specialization of an embedding procedure of a general D–poset into a D–poset with unit given by J. Hedlíková and S. Pulmannová [3].

Remark. On a D–poset $\boldsymbol{D}$ each pair x, y with $x - y = y$ belongs to the kernel of each 2-valued D–morphism f, i.e., $f(x) = f(y) = 0$.

3. The monotone difference order

By examples 2.1. and 2.2. it becomes clear that we have to look for another suitable order in $\mathrm{Mon}(\boldsymbol{D}_1, \boldsymbol{D}_2)$ for D–posets $\boldsymbol{D}_1, \boldsymbol{D}_2$. The following sharpening of the pointwise order will be referred to as the *monotone difference order* in $\mathrm{Mon}(\boldsymbol{D}_1, \boldsymbol{D}_2)$:

$$f \leq_m g : \iff \begin{cases} 1. \quad f \leq g & \text{pointwise and} \\ 2. \quad g - f & \text{is monotone}. \end{cases}$$

Hence we get the

Theorem.

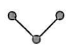
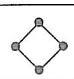
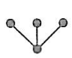
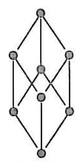
1. *Let $\boldsymbol{D}_1, \boldsymbol{D}_2$ be two D–posets. The monotone difference order in the set $\mathrm{Mon}(\boldsymbol{D}_1, \boldsymbol{D}_2)$ of all monotone functions from $\boldsymbol{D}_1$ into $\boldsymbol{D}_2$ makes it into a D–poset with the zero-element $f \equiv 0$ and the pointwise difference.*

2. *The D–poset $\mathrm{Mon}(\boldsymbol{D}_1, \boldsymbol{D}_2)$ has a greatest element iff ($\boldsymbol{D}_1$ has only one element and $\boldsymbol{D}_2$ has a unit element) or ($\boldsymbol{D}_2$ has only one element).*

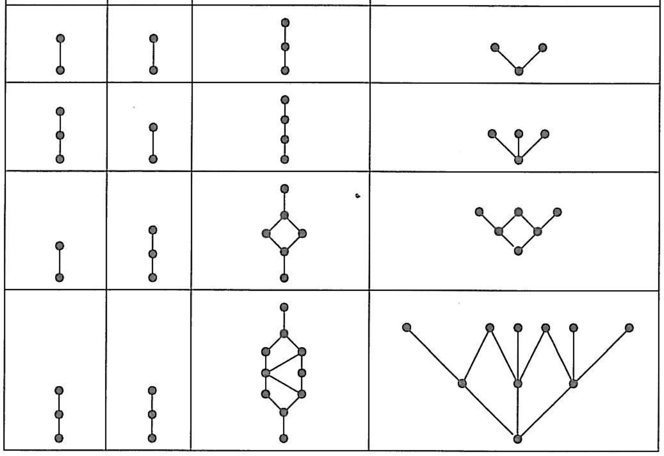
P r o o f . 1. For two given functions f, g from D_1 into D_2 the pointwise difference $g - f$ is defined only in the case of $f \leq_p g$ and $g - f$ has to be monotone to ensure $g - f \in \mathrm{Mon}(\boldsymbol{D}_1, \boldsymbol{D}_2)$.

2. If there is a unit in $\mathrm{Mon}(\boldsymbol{D}_1, \boldsymbol{D}_2)$, then this function is greater than any constant function. This concludes that $\boldsymbol{D}_2$ has a unit $\mathbf{1}$ and $f \equiv \mathbf{1}$ has to be the function unit. If card $\boldsymbol{D}_1 \geq 2$, then the function g with $g(0) = 0$, $g(x) = \mathbf{1}$ for all $x \neq 0$ is different from f. But $f - g$ is not monotone. Therefore card $\boldsymbol{D}_1 = 1$.

Now we show some examples of $\mathrm{Mon}(\boldsymbol{D}_1, \boldsymbol{D}_2)$ with monotone difference order and also $\mathrm{Hom}(\boldsymbol{D}_1, \boldsymbol{D}_2)$, whereby the latter means the set of all D–morphisms from $\boldsymbol{D}_1$ into $\boldsymbol{D}_2$.

Domain	Range	$\mathrm{Mon}(X, Y)$
X	Y	pointwise order / monotone difference order
		
		

Domain	Range	Mon(X,Y)	
X	Y	pointwise order	monotone difference order



REFERENCES

[1] DVUREČENSKIJ, A.: *Tensor product of difference posets*, Trans. Amer. Math. Soc. **347** (1995), 1043–1057.

[2] FOULIS, D. J.—BENNET, N. K.: *Effect algebras and unsharp quantum logics*, Found. Phys. **24** (1994).

[3] HEDLÍKOVÁ, J.—PULMANNOVÁ, S.: *Generalized difference posets and orthoalgebras*, preprint, MÚ SAV Bratislava.

[4] KÔPKA, F.—CHOVANEC, F.: *D-posets*, Math. Slovaca **44** (1994), 21–34.

[5] NAVARA, M.—PTÁK, P.: *Difference posets and orthoalgebras*, preprint, ČVUT Prague.

[6] RIEČANOVÁ, Z.—BRŠEL, D.: *Counterexamples in difference posets and orthoalgebras*, Internat. J. Theoret. Phys. **33** (1994), 133–141.

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