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FUZZY DISCRETE DYNAMIC SYSTEMS — EFFICIENT ALGORITHMS USING DIGRAPHS

KATARÍNA CECHLÁROVÁ

ABSTRACT. Several problems connected with fuzzy discrete dynamic systems are stated and their formulation in the language of matrices over fuzzy algebra is demonstrated. To design efficient algorithms for solving these problems we use graph-theoretical methods.

Introduction

The aim of this paper is to study some questions related to *fuzzy discrete dynamic systems* (FDDS for short). A system $\mathcal{S}$, having n qualitative characteristics, is observed in discrete time points $t = 0, 1, \dots$ Its complete description in time t is given by means of the *state vector* $\mathbf{x}^t = (x_1^t, \dots, x_n^t)$, where $x_i^t \in \langle 0, 1\rangle$ for each time point t and each index i. The value x_i^t is the *intensity of presence* of the ith characteristic in time t. The evolution of system $\mathcal{S}$ over time is described by a square matrix $\mathbf{A} = (a_{ij})$, called the *transition matrix*. Its entries are again taken from the interval $\langle 0, 1\rangle$ and a_{ij} is the intensity with which the jth characteristic contributes to the development of the ith characteristic in the following time point. In the language of fuzzy sets, $\mathbf{A}$ represents a *fuzzy relation* on the set characteristics. We can write

$$x_i^{t+1} = \max\{\min\{a_{i1}, x_1^t\}, \dots, \min\{a_{in}, x_n^t\}\}.$$

If the operations max and min are denoted by $\oplus$ and $\otimes$, respectively, and used formally in the the same way as the normal addition and multiplication are used in the classical linear algebra, then the state vector in time $t+1$ can be expressed in the form

$$\mathbf{x}^{t+1} = \mathbf{A} \otimes \mathbf{x}^t. \tag{1}$$

In this paper we present a unified approach to several problems connected with FDDS, based on the language of linear algebra and digraphs. These problems have

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been originally formulated and solved in the classical systems theory. Recently, however, discrete dynamic systems gave rise to the study of new topics concerning *synchronization* and *linearity*, cf. [1], and several works have appeared using *maxalgebra* for their description, e.g., [7], [8], [14]. Similar questions are also interesting for fuzzy discrete dynamic systems. We will show how digraphs associated with the transition matrix can help solve them.

An extensive study of relations between various extremal (i.e., using max or min instead of usual addition or multiplication) algebraic structures and digraphs can be found in [16]. Other works where these connections were studied are [2–6], [11]. To evaluate the efficiency of algorithms proposed, we will use the notions of computational complexity as defined in [9].

Definitions and notations

DEFINITION 1. The *fuzzy algebra* is the triple $\mathscr{F} = (\langle 0,1\rangle, \oplus, \otimes)$ where $\oplus =$ max and $\otimes =$ min.

Clearly, the fuzzy algebra is a *semiring* with some additional properties, i.e., both operations are associative, commutative, idempotent and distributive with respect to each other and the result of any operation is always equal to one of its operands. In what follows, $\mathscr{F}_n$ will denote the set of all n-vectors over $\mathscr{F}$ and $\mathscr{F}_{nn}$ the set of all square matrices of order n over $\mathscr{F}$.

A *digraph* is a pair $\mathcal{G} = (V, H)$ where V is the set of *nodes* and H, a set of ordered pairs of nodes, is called the *arcs set.* A *path* in $\mathcal{G}$ is a sequence of nodes $p = (v_0, v_1, \dots, v_k)$ such that $(v_i, v_{i+1}) \in H$ for all $i = 0, 1, \dots, k-1$. The *length* $\ell(p)$ of a path p is the number of arcs on it. If the arcs (v_i, v_{i+1}) are assigned capacities c_i, then the *capacity* of a path p is defined by

$$c(p) = \min\{c_0, \dots, c_{k-1}\},$$

or, in fuzzy algebra notation

$$c(p) = c_0 \otimes c_1 \otimes \dots \otimes c_{k-1}.$$

A path is a *cycle* if $v_0 = v_k$. A node u is *precyclic* in $\mathcal{G}$ if there is a path from u to a cycle in $\mathcal{G}$. A digraph $\mathcal{G}$ is *strongly connected* if for each pair u, v of its nodes there is a path from u to v and a path from v to u in $\mathcal{G}$. A digraph $\mathcal{G}' = (V', H')$ is a subgraph of $\mathcal{G} = (V, H)$, if $V' \subseteq V$ and $H' \subseteq H$. A *strongly connected component* (SCC for short) of $\mathcal{G}$ is its maximal strongly connected subgraph. A SCC is *nontrivial*, if it contains at least one cycle. A *period* of a digraph is the greatest common divisor (gcd for short) of its cycle lengths.

Definition 2. Let $\mathbf{A} \in \mathscr{F}_{nn}$ be a given fuzzy matrix. Then the *associated digraph* $\mathcal{G}(\mathbf{A})$ of $\mathbf{A}$ is the complete digraph on n nodes, where each arc (i,j) is assigned a capacity equal a_{ij}.

Definition 3. Let $\mathbf{A} \in \mathscr{F}_{nn}$ and $h \in \langle 0,1 \rangle$ be given. Then the *associated threshold digraph* $\mathcal{G}(\mathbf{A},h)$ of $\mathbf{A}$ at the threshold h is the digraph on n nodes such that the pair (i,j) is its arc if and only if $a_{ij} \geq h$.

In what follows, we will review several questions connected with FDDS, their solutions using digraphs and some open problems. In particular, for every problem we mention in this paper, it is still open, whether the known algorithm can be improved.

Problem 1: When does the system not change?

If the transition matrix $\mathbf{A}$ is given, what should the state vector look like, if the system is to be in exactly the same state in the following time point? In the fuzzy algebra notation, a matrix $\mathbf{A}$ is given and we are looking for a vector $\mathbf{x}$ such that

$$\mathbf{A} \otimes \mathbf{x} = \mathbf{x}\,. \tag{2}$$

Such a vector can be called the *eigenvector* of matrix $\mathbf{A}$. Clearly, $\mathbf{x} = (0,0,\dots,0)$ is always an eigenvector, but already in [15] Sanchez argued that every matrix has *exactly one greatest eigenvector* (GEV for short) and also gave some algorithms for its computation, however, without any formal statements or proofs. Eigenvectors in extremal structures were also studied in [11] and [16] in a broader algebraic context.

In [2] the following theorem was proved

Theorem 1. *Let $\mathbf{A} \in \mathscr{F}_{nn}$ be given. Then a vector $\mathbf{x}(\mathbf{A})$ defined by*

$$x_k(\mathbf{A}) = \max\left\{h \in \langle 0,1 \rangle\,;\ k \text{ is precyclic in } \mathcal{G}(\mathbf{A},h)\right\}$$

is the greatest eigenvector of $\mathbf{A}$.

Theorem 1 also gives a hint for a direct computation of $\mathbf{x}(\mathbf{A})$: threshold graphs for $\mathbf{A}$ and a sequence of thresholds, given by the entries of $\mathbf{A}$ ordered decreasingly, are constructed. At each threshold the nodes are sought, which became precyclic for the first time. Since there are at most n^2+1 different threshold graphs for a given matrix and searching a graph on n nodes takes $O(n^2)$ time, this algorithm has time complexity $O(n^4)$.

In [2] also the following was proved.

THEOREM 2. *For a given matrix* $\mathbf{A} \in \mathscr{F}_{nn}$, *the sequence*

$$x_i^{(1)}(\mathbf{A}) = \bigoplus_{j=1}^{n} a_{ij} \quad \text{for all} \quad i \in \mathbb{N},$$

$$\mathbf{x}^{(k+1)}(\mathbf{A}) = \mathbf{A} \otimes \mathbf{x}^{(k)} \quad \text{for all} \quad k = 1, 2, \dots$$

has the following two properties:

$$\mathbf{x}^{(n)} = \mathbf{x}^{(r)} \quad \text{for all } r \geq n, \tag{1}$$

$$\mathbf{x}^{(n)} = \mathbf{x}(\mathbf{A}). \tag{2}$$

Hence, the computational complexity of the iterative procedure based on Theorem 2 is $O(n^3)$. In [3], even a method with time complexity $O(n^2 \log n)$ was derived, based on ideas of Sanches.

Problem 2: What is the long-term behaviour of the system?

Let us suppose that the system starts in time 0 in a state $\mathbf{x}^0 = \mathbf{b}$. Then its states in the subsequent time points are

$$\mathbf{x}^1 = \mathbf{A} \otimes \mathbf{x}^0, \quad \mathbf{x}^2 = \mathbf{A} \otimes \mathbf{x}^1, \quad \dots \quad \mathbf{x}^{t+1} = \mathbf{A} \otimes \mathbf{x}^t, \ \dots \tag{3}$$

or, equivalently,

$$\mathbf{x}^1 = \mathbf{A} \otimes \mathbf{b}, \quad \mathbf{x}^2 = \mathbf{A}^2 \otimes \mathbf{b}, \quad \dots \quad \mathbf{x}^{t+1} = \mathbf{A}^{t+1} \otimes \mathbf{b}, \ \dots \tag{4}$$

Hence, the long-run behaviour of the system is completely described by its starting state vector $\mathbf{b}$ and the power sequence of matrix $\mathbf{A}$. It is clear that the powers of $\mathbf{A}$ must eventually stabilize or oscillate with a finite period. Li Jian-Xin gave some bounds for the period of a fuzzy matrix and proposed an algorithm for its computation in [12] and [13]. He used purely algebraic methods, and the algorithm he gave involved computing very high powers of $\mathbf{A}$, so it was not efficient.

In [4] it was proved

THEOREM 3. *The powers of a fuzzy matrix* $\mathbf{A}$ *stabilize if and only if the period of every nontrivial* SCC *in every threshold graph of* $\mathbf{A}$ *is* 1.

This theorem was generalized by Gavalec in [10] to an arbitrary period d and an algorithm with computational complexity $O(n^3)$ for computing the period was derived.

In case the powers of $\mathbf{A}$ stabilize and hence the system stabilizes for arbitrary starting vector, we can also ask what the limiting state will be. The answer is given by the following theorem.

THEOREM 4. *Let $\mathbf{A}$ be a fuzzy matrix with stabilizing powers and $\lim_{k\to\infty} \mathbf{A}^k = \mathbf{A}^* = (a^*_{ij})$. Then*

$$a^*_{ij} = \max\{h, \text{ there is a path of length at least } n \text{ from } i \text{ to } j \text{ in } \mathcal{G}(\mathbf{A}, h)\}. \quad (5)$$

P r o o f . Let us note that formula (5) is equivalent to the following formulation

$$a^*_{ij} = \max\{h, \text{ there is a path from } i \text{ to } j \text{ in } \mathcal{G}(\mathbf{A}, h)\text{, containing a cycle}\}.$$

A path of length $r \geq n$ from i to j in $\mathcal{G}(\mathbf{A}, h)$ means that a^r_{ij}, the entry in $\mathbf{A}^r$, is greater than or equal to h. Such a path must contain a cycle, which is in a nontrivial SCC of $\mathcal{G}(\mathbf{A}, h)$, say $\mathcal{G}'$. $\mathbf{A}$ has stabilizing powers, hence the period of $\mathcal{G}'$ is equal to 1. This means that there are cycles $C_1, C_2, \dots, C_k$ in $\mathcal{G}'$ such that the gcd of their lengths $c_1, c_2, \dots, c_k$ is equal to 1. A well-known fact from the number theory then says that there is a positive integer Q such that for all integers $\ell \geq Q$ there exist nonnegative integers $d_1, d_2, \dots, d_k$ such that $\ell = d_1c_1 + d_2c_2 + \dots + d_kc_k$. This means that for every $\ell \geq Q$ there is a path of length ℓ from i to j in $\mathcal{G}(\mathbf{A}, h)$ —it suffices to add the cycles $C_1, C_2, \dots, C_k$ to the original path as many times as suggested by the previous expression, hence $a^*_{ij} \geq h$.

Since h was maximum with this property, it follows that for every $h' > h$ we have $a^r_{ij} < h'$ for every $r \geq n$, therefore $a^*_{ij} = h$. □

The metric matrix of $\mathbf{A}$

$$\mathbf{M} = \mathbf{A} \oplus \mathbf{A}^2 \oplus \dots \oplus \mathbf{A}^n$$

can be used to find $\mathbf{A}^*$. $\mathbf{M}$ can be computed by Floyd–Warshall algorithm in $O(n^3)$ time and

$$m_{ij} = \max\{h; \text{ there is a path from } i \text{ to } j \text{ in } \mathcal{G}(\mathbf{A}, h)\}$$

and

$$m_{kk} = \max\{h; \text{ there is a cycle containing } k \text{ in } \mathcal{G}(\mathbf{A}, h)\}$$

Therefore,

$$a^*_{ij} = \max\{\min\{m_{ik}, m_{kk}, m_{kj}\} \text{ for } k = 1, 2, \dots, n\}$$

and this again can be computed in $O(n^3)$ time.

Problem 3: What is the most efficient description of the system?

Suppose that we know that the system evolves according to formula (3), but what we only observe is its scalar output in the form

$$g_t = \max\{\min\{c_1, x_1^t\}, \dots, \min\{c_n, x_n^t\}\},$$

or, denoting by $\mathbf{c}^T$ the *observation vector* $(c_1, \dots, c_n)$,

$$g_t = \mathbf{c}^T \otimes \mathbf{x}_t . \tag{6}$$

The sequence $\{g_t\}_{t=0}^{\infty}$ is called the *sequence of Markov parameters* (or Markov sequence, MS for short) of the system. Our task is to find a matrix $\mathbf{A}$ and vectors $\mathbf{b}$, $\mathbf{c}$ such that the description of the system by means of $\mathbf{A}, \mathbf{b}, \mathbf{c}$ will be consistent with the observed MS, i.e.,

$$g_t = \mathbf{c}^T \otimes \mathbf{A}^t \otimes \mathbf{b} \qquad \text{for all } t = 0, 1, \dots$$

To achieve the most efficient description of the system, the minimum possible dimension of $\mathbf{A}$ is desirable, so we are looking for what is called the *minimal dimension* $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ *realization.*

This is a problem known from the classical systems theory and its formulation for different algebraic structures was studied in [14], [7], [8]. Paper [5] is probably the first one addressing this question in fuzzy algebra.

First, it is clear that a MS can have an $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ realization only if it is ultimately periodic, i.e., there exist integers r and p such that $g_{t+p} = g_t$ for every $t > r$. p is called the *period*, $g_0, g_1, \dots, g_r$ the *transient part* and the number $r + p + 1$ is the *characteristic length* of the MS.

As was argued in [5] using the language of paths in digraphs, every ultimately periodic MS with characteristic length d has a trivial realization of dimension d. Moreover, a method for constructing the so called *up-one-place* realizations was proposed, where an up-one-place realization of order n is defined by the following three properties:

(1) $\mathbf{c}^T = (1, 0, \dots, 0)$,
(2) $\mathbf{b}^T = (g_0, g_1, \dots, g_{n-1})$ and
(3) the matrix $\mathbf{A}$ is such that for each t we have
$\mathbf{A}^t \otimes \mathbf{b} = (g_t, g_{t+1}, \dots, g_{t+n-1})$.

THEOREM 5. ([5]) *The order of any up-one-place realization of a* MS *with period p over the fuzzy algebra is greater than or equal to p.*

OPEN PROBLEM. Is it possible to prove a similar bound for the order of any $(\mathbf{A}, \mathbf{b}, \mathbf{c})$ realization over the fuzzy algebra?

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Department of Geometry and Algebra
P. J. Šafárik University
Jesenná 5
SK–041 54 Košice
SLOVAKIA

E-mail: cechlarova@duro.upjs.sk