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ON COMPLETE SYSTEMS OF ALMOST CONTINUOUS FUNCTIONS

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Dedicated to Professor J. Jakubík on the occasion of his 70th birthday

ABSTRACT. In this note we consider the conditions under which it is possible to extend a given family of almost continuous functions to a complete system of almost continuous functions.

In 1959 J. Stallings ([10]) introduced the notion of an almost continuous function: A function $f\colon X \to Y$ is *almost continuous* (in the sense of Stallings) if, for each open set $U \subset X \times Y$ containing the graph of f, U contains the graph of some continuous function $g\colon X \to Y$. This property was introduced in order to generalize the Brouwer fixed point theorem.

Lately, there have appeared many papers concerning operations performed on almost continuous functions (e.g., [4, 9, 8, 6, 7]). The interest of scientists in these problems follows mainly from the fact that, even after performing simple operations on some almost continuous functions, one can obtain a function which does not possess this property (in our considerations, we shall confine ourselves to the examination of real functions defined on the unit interval, although these problems are also extremely interesting when transformations defined on other spaces are concerned — many facts concerning these problems were included in [8]). So, it seems interesting to ask the question about the possibility of distinguishing as wide a class of almost continuous functions, being closed under the fundamental operations, as possible. This problem leads to the consideration of the notion of a complete system, introduced in [1] (cf. also [5]):

Let Ξ be a class of real functions. The class Ξ is a *complete system* if

1°. $f, g \in \Xi$ imply $\max(f,g) \in \Xi$, $\min(f,g) \in \Xi$;
2°. Ξ contains all constants;
3°. $f, g \in \Xi$ imply $f+g \in \Xi$, $fg \in \Xi$ and $f/g \in \Xi$ for g such that $\{x\colon g(x)=0\} = \emptyset$;
4°. the uniform limit of a sequence $\{f_n\} \subset \Xi$ belongs to Ξ.

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Our considerations will be carried out in order to obtain the answer to the following question: *what conditions must a class $\mathcal{F}$ of almost continuous functions satisfy in order that the fundamental operations performed on functions from this class lead to almost continuous functions?*. The precise statement of this question and its treatment in terms of mathematical considerations can take the form:

> *What conditions must a family $\mathcal{F}$ of almost continuous real functions defined on the unit interval $\mathcal{I}$ satisfy in order that it may be extended to a complete system of almost continuous functions?*

The above problem constitutes the fundamental question in this paper.

Throughout the paper, we use the commonly known symbols and notations (such as, for instance, in [3]). In particular, we denote by the symbol $\mathcal{I}$ the unit interval with the natural topology.

Let $\mathcal{A}$ be a fixed family of subsets of a space $X \subset \mathbb{R}$ (where $\mathbb{R}$ denotes the set of real numbers). Then, for $B \subset X$, $\mathcal{A}|_B = \{A \cap B\colon\ A \in \mathcal{A}\}$. The symbol $\bigcup \mathcal{A}$ stands for $\bigcup_{A \in \mathcal{A}} A$. We also say that $\mathcal{A}$ is dense if $\bigcup \mathcal{A}$ is a dense set.

Let A be some subset of $\mathcal{I}$. Then by A^* we denote the class of all components of A.

Let $\mathcal{F}$ be a fixed family of sets. Then by the symbol $\mathcal{F}|_A$ we denote the family $\{f|_A\colon\ f \in \mathcal{F}\}$. If $*$ is a property that can be vested in functions, then $\mathcal{F}|_A$ is said to possess the property $*$ if $f|_A$ possesses it for $f \in \mathcal{F}$. We also adopt the following symbols: $D_{\mathcal{F}} = \bigcup_{f \in \mathcal{F}} D_f$ (where D_f stands for the set of all discontinuity points of f); moreover, for functions defined on $\mathcal{I}$, $C_{\mathcal{F}} = \mathcal{I} \setminus D_{\mathcal{F}}$.

By the symbol const_y we denote a constant function equal to y.

The symbol $\mathcal{G}_f$ will stand for the graph of the function f, and $f \nabla g$ — for combinations of the mappings f and g (i.e., if $f\colon X_1 \to Y$, $g\colon X_2 \to Y$ and $f|_{X_1 \cap X_2} = g|_{X_1 \cap X_2}$, then by $f \nabla g\colon X_1 \cup X_2 \to Y$ we denote a function such that $f \nabla g(x) = f(x)$ for $x \in X_1$ and $f \nabla g(x) = g(x)$ for $x \in X_2$).

Let now $\mathcal{P}$ be a fixed filter-base in the space X. Then, if $f\colon X \to Y$, the symbol $f(\mathcal{P})$ will denote the filter-base $\{f(P)\colon\ P \in \mathcal{P}\}$. So, consider a family of mappings $f_t\colon X \to \mathbb{R}$ ($t \in \mathrm{T}$). We say that a system of elements $\{\alpha_t\}_{t \in \mathrm{T}}$ ($\alpha_t \in \mathbb{R}$ for $t \in \mathrm{T}$) is the limit of a system of filter-bases $\{f_t(\mathcal{P})\}_{t \in \mathrm{T}}$, which is written down as $\{\alpha_t\}_{t \in \mathrm{T}} \in \lim_{t \in \mathrm{T}} f_t(\mathcal{P})$, if, for any $\varepsilon > 0$, there exists $P \in \mathcal{P}$ such that $f_t(P) \subset (\alpha_t - \varepsilon, \alpha_t + \varepsilon)$ for $t \in \mathrm{T}$. Of course, if $\mathrm{T} = \{t_0\}$, then this definition is identical with that of the ordinary limit of a filter-base ([3], p.77).

The answer to the question raised in the introduction will be much simpler after introducing a suitable notional apparatus connected with the term: *selection.*

So let $\mathcal{A}$ be a fixed family of subsets of a space $X \subset \mathcal{I}$. Then a set $S \subset \bigcup \mathcal{A}$ is called an *$\mathcal{A}$–selection (with respect to X)* if $A \cap S \neq \emptyset$ for any $A \in \mathcal{A}$. The $\mathcal{A}$–selection S is called a *perfect $\mathcal{A}$–selection (with respect to X)* if each point $x \in X \setminus \bigcup \mathcal{A}$ is a bilateral limit point of S (of course, if $x = 0$ or $x = 1$, then we only demand that x be a unilateral limit point of S), i.e., there exist sequences $\{x_n^-\}, \{x_n^+\} \subset S$ such that $x_n^- \nearrow x \swarrow x_n^+$. If $X = \mathcal{I}$, we shortly write: S is a (perfect) $\mathcal{A}$–selection, instead of a (perfect) $\mathcal{A}$–selection (with respect to $\mathcal{I}$).

Let $\mathcal{S}$ be a fixed perfect $\mathcal{A}$–selection and let $x \in \mathcal{I} \setminus \bigcup \mathcal{A}$. We say that the family $\mathcal{S}_x$ of subsets of $\mathcal{S}$ agrees with the pair $(x, \mathcal{S})$ if, for any $S \in \mathcal{S}_x$, there exists $\delta_S > 0$ such that S is a perfect $\mathcal{A}|_{(x-\delta_S, x+\delta_S)}$–selection (with respect to $(x - \delta_S,\ x + \delta_S)$).

Let now $\mathcal{F} = \{f_t\}$ be the family of transformations mapping $\mathcal{I}$ into $\mathbb{R}$ and let $\mathcal{A}$ be a dense family of sets, $\mathcal{S}$ — a perfect $\mathcal{A}$–selection, and $x_0 \in \mathcal{I} \setminus \bigcup \mathcal{A}$.

DEFINITION. We say that $\{\alpha_f\}_{f\in\mathcal{F}}$ is an *$\mathcal{S}$–cluster element of $\mathcal{F}$ at x_0* (which is written down as $\{\alpha_f\}_{f\in\mathcal{F}} \in L_{\mathcal{S}}(\mathcal{F}, x_0)$) if there exists a filter-base $\mathcal{S}_{x_0}$ agreement with $(x_0, \mathcal{S})$, such that $\{\alpha_f\}_{f\in\mathcal{F}} \in \lim_{f\in\mathcal{F}} (f(\mathcal{S}_{x_0}))$.

The introduction of the above notional apparatus is not accidental since a profound analysis of the problem presented at the beginning with reference to a simple case has led to the following observation.

PROPOSITION. *Let $\mathcal{F}$ be a finite family consisting of almost continuous functions from $\mathcal{I}$ to $\mathbb{R}$ such that $D_{\mathcal{F}} = \{x_0\}$. Then, if an extension of $\mathcal{F}$ to some complete system Ξ of almost continuous functions is possible, there exists a perfect $C^*_{\mathcal{F}}$–selection S such that:*

$$\{f(x_0)\}_{f\in\mathcal{F}} \in \mathrm{L}_{\mathcal{S}}(\mathcal{F}, x_0).$$

Proof. Let $\mathcal{F} = \{f_1, f_2, \dots, f_n\}$. Denote $\hat{f}_i = f_i - \mathrm{const}_{f_i(x_0)} \in \Xi$ $(i = 1, 2, \dots, n)$. Moreover, let $f = \sum_{i=1}^{n} |\hat{f}_i| \in \Xi$. Of course, f fulfils the condition of Young ([11]); therefore, there exist two sequences $\{x_k^-\}_{k=1}^\infty$, $\{x_k^+\}_{k=1}^\infty$ such that:

$$x_k^- \nearrow x_0 \swarrow x_k^+ \quad \text{and} \quad \lim_{k\to\infty} f(x_k^-) = f(x_0) = \lim_{k\to\infty} f(x_k^+).$$

It is not difficult to see that:

$$\lim_{k\to\infty} f_i(x_k^-) = f_i(x_0) = \lim_{k\to\infty} f_i(x_k^+) \qquad (i = 1, 2, \dots, n). \tag{1}$$

So, let $\mathcal{S} = \{x_k^+ : k = 1, 2, \dots\} \cup \{x_k^- : k = 1, 2, \dots\}$. Thus $\mathcal{S}$ is a perfect $C^*_{\mathcal{F}}$–selection. It is easy to see that the family $\mathcal{S}_{x_0} = \{(x_0 - \frac{1}{m},\ x_0 + \frac{1}{m}) \cap \mathcal{S} : m = 1, 2, \dots\}$ agrees with the pair $(x_0, \mathcal{S})$ and $\mathcal{S}_{x_0}$ is a filter-base.

Let $\varepsilon > 0$. From (1) we may infer that:

$$\forall_{i=1,2,\dots,n} \exists_{l_i} \forall_{l \geq l_i} \ f_i(x_l^-),\ f_i(x_l^+) \in \big(f_i(x_0) - \varepsilon,\ f_i(x_0) + \varepsilon\big)\,. \tag{2}$$

Let us put $l_0 = \max\{l_i \colon\ i = 1,2,\dots,n\}$ and let m_0 be a positive integer such that $\big(x_0 - \frac{1}{m_0},\ x_0 + \frac{1}{m_0}\big) \cap \big(\{x_l^- \colon\ l < l_0\} \cup \{x_l^+ \colon\ l < l_0\}\big) = \emptyset$. Then $\big(x_0 - \frac{1}{m_0},\ x_0 + \frac{1}{m_0}\big) \cap \mathcal{S} \in \mathcal{S}_{x_0}$ and, according to (2),

$$f_i\Big(\Big(x_0 - \frac{1}{m_0},\ x_0 + \frac{1}{m_0}\Big) \cap \mathcal{S}\Big) \subset \big(f_i(x_0) - \varepsilon,\ f_i(x_0) + \varepsilon\big) \qquad (i = 1,2,\dots,n)\,,$$

which ends the proof. □

The observation made above has led to the search for solutions of the problem mentioned in the introduction. A consequence of this search is the following theorem.

THEOREM. *Let $\mathcal{F}$ be a family consisting of functions from $\mathcal{I}$ to $\mathbb{R}$ such that there exists a set $A \subset \mathcal{I}$, open and dense in $\mathcal{I}$, satisfying the following conditions: $\mathcal{F}\big|_A$ is a family of approximately continuous functions and there exists a perfect A^* selection $\mathcal{S}$ such that $\{f(x)\}_{f \in \mathcal{F}} \in L_{\mathcal{S}}(\mathcal{F}, x)$ for any $x \notin A$. Then $\mathcal{F}$ is a family of almost continuous functions and $\mathcal{F}$ has an extension to some complete system Ξ of almost continuous functions, containing all continuous functions.*

Proof. To begin with, let us agree upon that, when speaking of the interval $\mathcal{J}$ contained in the domain of the transformations considered, we shall always mean the intersection $\mathcal{J} \cap \mathcal{I}$.

In the next considerations we omit the obvious case when $A = \mathcal{I}$.

Let $\mathcal{G}$ denote the density topology. For any $x \in A$, let B_x denote an arbitrary local base at x for the topological space $([0,1], \mathcal{G})$, consisting of sets included in the same component of A which contains x.

Let now $x \notin A$.

Let $\mathcal{S}_x$ be a filter-base agreeing with $(x, \mathcal{S})$, such that $\{f(x)\}_{f \in \mathcal{F}} \in \lim_{f \in \mathcal{F}} \big(f(\mathcal{S}_x)\big)$. This means that, for every positive integer n, there exists $\hat{S}_n^x \in \mathcal{S}_x$ such that $f(\hat{S}_n^x) \subset \big(f(x) - \frac{1}{n}, f(x) + \frac{1}{n}\big)$ (for $f \in \mathcal{F}$). Thus there exists $\delta_{\hat{S}_n^x} > 0$ such that $\hat{S}_n^x$ is a perfect $A^*\big|_{(x - \delta_{\hat{S}_n^x}, x + \delta_{\hat{S}_n^x})}$–selection (with respect to $(x - \delta_{\hat{S}_n^x}, x + \delta_{\hat{S}_n^x})$). Let us denote $S_n^x = \hat{S}_n^x \cap \big(x - \min(\delta_{\hat{S}_n^x}, \frac{1}{n}), x + \min(\delta_{\hat{S}_n^x}, \frac{1}{n})\big)$.

Then:

$$S_n^x \text{ is a perfect } A^*\big|_{(\inf S_n^x,\, \sup S_n^x)}\text{–selection with respect to } (\inf S_n^x,\ \sup S_n^x). \tag{3}$$

Assume

$$B_x = \Big\{\{x\} \cup \bigcup_{n=k}^{\infty} S_n^x \ : \ k = 1, 2, \dots\Big\}.$$

The collection $\{B_x\}_{x\in[0,1]}$ is a quasi-neighbourhood system. Let $\mathcal{T}$ be the topology generated by this system, i.e.,

$U \in \mathcal{T}$ if and only if, for any $x \in U$, there exists $V_x \in B_x$ such that $V_x \subset U$.

By the symbol Ξ we denote the family consisting of all continuous functions from $\big([0,1], \mathcal{T}\big)$ to $\mathbb{R}$. Of course, Ξ is a complete system. We shall show that Ξ is the required complete system.

Observe first that $\mathcal{T}$ is finer than the natural topology of $\mathcal{R}$. Thus Ξ contains the set of all continuous functions in the natural topology.

We shall now show that

$$\mathcal{F} \subset \Xi. \tag{4}$$

So, let $f \in \mathcal{F}$ and let $\alpha, \beta \in \mathbb{R}$. Put $V = f^{-1}\big((\alpha,\beta)\big)$. It suffices to show that $V \in \mathcal{T}$. Let $v \in V$. Let us consider the following cases:

1^o. $v \in A$. Hence by the approximate continuity of $f\big|_A$, there exists $Z_v \in B_v$ such that $f(Z_v) \subset (\alpha,\beta)$. Of course, $Z_v \in \mathcal{T}$.

2^o. $v \notin A$. Let n_0 be a positive integer such that $\big(f(v) - \frac{1}{n_0},\, f(v) + \frac{1}{n_0}\big) \subset (\alpha,\beta)$. Put $Z'_v = \{v\} \cup \bigcup\limits_{n=n_0}^{\infty} S_n^v \in B_v$. By the approximate continuity of $f\big|_A$, we infer that, for any $x \in \bigcup\limits_{n=n_0}^{\infty} S_v^n$, there exists $U_x \in B_x$ such that $f(U_x) \subset (\alpha,\beta)$. Then put $Z_v = \{v\} \cup \bigcup\limits_{x\in Z'_v\setminus\{v\}} U_x$. Thus Z_v is a $\mathcal{T}$–neighbourhood of v; moreover, $Z_v \subset V$. This ends the proof of (4).

It remains to show that

ξ is almost continuous for any $\xi \in \Xi$.

So, let $\xi \in \Xi$, and let $H \subset [0,1] \times \mathbb{R}$ be an arbitrary open set containing the graph of ξ.

Let us fix $x \notin A$. Then let n_x be a positive integer such that a closed cube with vertices $\big(x-\frac{1}{n_x},\, \xi(x)-\frac{1}{n_x}\big)$, $\big(x+\frac{1}{n_x},\, \xi(x)-\frac{1}{n_x}\big)$, $\big(x+\frac{1}{n_x},\, \xi(x)+\frac{1}{n_x}\big)$, $\big(x-\frac{1}{n_x},\, \xi(x)+\frac{1}{n_x}\big)$ is included in H. Let $k_x > n_x$ be a positive integer such that $\xi(S_{k_x}^x) \subset \big(\xi(x)-\frac{1}{n_x},\, \xi(x)+\frac{1}{n_x}\big)$.

By the symbols x' and x'' we denote elements of $S_{k_x}^x$ such that $x - \frac{1}{n_x} < x' < x < x'' < x+\frac{1}{n_x}$ (if $x-\frac{1}{n_x} \le 0$, then we assume $x' = 0$, and by the interval (x', x'') we understand $[0, x'')$; the similar remark refers to the number 1).

The above ascertainments have been made for any $x \notin A$. So, let $W = \{(x', x''):\ x \notin A\} \cup \{A_n:\ n = 1, 2, \dots\}$. Then W is an open (in the natural topology) cover of $[0,1]$. Let

$$W' = \{(x_1', x_1''), (x_2', x_2''), \dots, (x_t', x_t''),\ A_1, A_2, \dots, A_p\}$$

be a finite subcover of W. We may assume that $x_1 < x_2 < \dots < x_t$.

Let us now define a continuous function $h\colon [0,1] \to \mathbb{R}$ such that $\mathcal{G}_h \subset H$.

If $0 \in (x_1', x_1'')$, then we assume $h(z) = \xi(x_1)$ for $z \in [0, x_1]$.

Let us consider the case when $0 \notin (x_1', x_1'')$. Assume that A_1 is a component of A such that $0 \in A_1$. Moreover, let $\hat{x}_1 = \min\{x_i':\ i = 1, 2, \dots, t\}$. Of course, $\hat{x}_1 \in A_1$. Assume that $\hat{x}_1 = x_{i_0}'$. Then, by virtue of (3), there exists $x_1^* \in A_1 \cap S_{k_{x_{i_0}}}^{x_{i_0}}$ such that $x_1^* > \hat{x}_1$. Note that $\xi(x_1^*) \in \big(\xi(x_{i_0}) - \frac{1}{n_{x_{i_0}}},\ \xi(x_{i_0}) + \frac{1}{n_{x_{i_0}}}\big)$. Let $V_1 = \big(H \cap \{(p,q) \in [0,1] \times \mathbb{R}:\ p < x_1^*\}\big) \cup \big\{(p,q) \in [0,1] \times \mathbb{R}\colon p = x_1^* \wedge q \in (\xi(x_{i_0}) - \frac{1}{n_{x_{i_0}}},\ \xi(x_{i_0}) + \frac{1}{n_{x_{i_0}}})\big\}$. Then V_1 is an open neighbourhood of the graph of $\xi|_{[0,x_1^*]}$. Since $\xi|_{[0,x_1^*]}$ is approximately continuous, then it is a Darboux function of the first class of Baire. Thus ([8, 2]) there exists a continuous function $h_1\colon [0, x_1^*] \to \mathbb{R}$ such that $\mathcal{G}_{h_1} \subset V_1$. Assume $\alpha_1 = h_1(x_1^*) \in \big(\xi(x_{i_0}) - \frac{1}{n_{x_{i_0}}},\ \xi(x_{i_0}) + \frac{1}{n_{x_{i_0}}}\big)$.

In the case when $1 \in (x_1', x_1'')$ we have the required function h by putting $h(z) = h_1(z)$ for $z \in [0, x_1^*]$ and $h(z) = \alpha_1$ for $z \in [x_1^*, 1]$.

So, let $1 \notin (x_1', x_1'')$. Let us consider the following possibilities:

1) There exists $i \in \{1, \dots, t\}$ such that $x_i' \le x_{i_0}''$ and $x_i'' > x_{i_0}''$. Let i_1 be an element of $\{1, \dots, t\}$ such that $x_{i_1}'' = \max\{x_i'';\ 1 \le i \le t \wedge x_i' \le x_{i_0}'' \wedge x_i'' > x_{i_0}''\}$. Let A^2 be a component of A containing x_{i_0}''. By virtue of (3), there exist points

- $x_2^* \in A^2 \cap (x_1^*, x_{i_0}'') \cap S_{k_{x_{i_0}}}^{x_{i_0}}$, of course $\xi(x_2^*) \in \big(\xi(x_{i_0}) - \frac{1}{n_{x_{i_0}}},\ \xi(x_{i_0}) + \frac{1}{n_{x_{i_0}}}\big)$;
- $x_3^* \in A^2 \cap (x_{i_0}'', x_{i_1}'') \cap S_{k_{x_{i_1}}}^{x_{i_1}}$ such that there exists $\hat{x}_3^* \in A^2 \cap (x_3^*, x_{i_1}'') \cap S_{k_{x_{i_1}}}^{x_{i_1}}$, of course $\xi(x_3^*) \in \big(\xi(x_{i_1}) - \frac{1}{n_{x_{i_1}}},\ \xi(x_{i_1}) + \frac{1}{n_{x_{i_1}}}\big)$.

Let $V_2 = \big(H \cap \{(p,q) \in [0,1] \times \mathbb{R}:\ x_2^* < p < x_3^*\}\big) \cup \big\{(p,q) \in [0,1] \times \mathbb{R}:\ p = x_2^* \wedge q \in \big(\xi(x_{i_0}) - \frac{1}{n_{x_{i_0}}},\ \xi(x_{i_0}) + \frac{1}{n_{x_{i_0}}}\big)\big\} \cup \big\{(p,q) \in [0,1] \times \mathcal{R}:\ p = x_3^* \wedge q \in \big(\xi(x_{i_1}) - \frac{1}{n_{x_{i_1}}},\ \xi(x_{i_1}) + \frac{1}{n_{x_{i_1}}}\big)\big\}$. Thus V_2 is an open neighbourhood of the graph of $\xi|_{[x_2^*, x_3^*]}$. Similarly as above, there exists a continuous function $h_2'\colon [x_2^*, x_3^*] \to \mathbb{R}$ such that $\mathcal{G}_{h_2'} \subset V_2$. Let us assume

$$\alpha_2 = h_2'(x_2^*) \in \Big(\xi(x_{i_0}) - \frac{1}{n_{x_{i_0}}},\ \xi(x_{i_0}) + \frac{1}{n_{x_{i_0}}}\Big),$$

$$\alpha_3 = h_2'(x_3^*) \in \Big(\xi(x_{i_1}) - \frac{1}{n_{x_{i_1}}},\ \xi(x_{i_1}) + \frac{1}{n_{x_{i_1}}}\Big).$$

Then we define a function $h_2\colon [x_1^*, x_3^*] \to \mathbb{R}$ in the following way: h_2 is linear in the interval $[x_1^*, x_2^*]$, and $h_2(x_1^*) = \alpha_1$, $h_2(x_2^*) = \alpha_2$; moreover $h_2(x) = h_2'(x)$ for $x \in [x_2^*, x_3^*]$.

It is easy to see that $\mathcal{G}_{h_1 \nabla h_2} \subset H$.

2) $\{i\colon\ x_i'' > x_{i_0}''\} = \emptyset$. Then let A_2 be a component of A such that $1 \in A_2$. Thus $x_{i_0}'' \in A_2$ and, by virtue of (3), there exists $x_2^* \in A_2 \cap (x_{i_0}, x_{i_0}'') \cap S_{k_{x_{i_0}}}^{x_{i_0}}$. Analogously as in the case considered, if $0 \notin (x_1', x_1'')$, then there exists a continuous function $h_2\colon [x_2^*, 1] \to \mathbb{R}$ such that $\mathcal{G}_{h_2} \subset H$ and $h_2(x_2^*) \in \left(\xi(x_{i_0}) - \frac{1}{n_{x_{i_0}}}, \xi(x_{i_0}) + \frac{1}{n_{x_{i_0}}}\right)$. Thus a function h, defined by putting $h(x) = h_1(x)$ for $x \le x_1^*$, $h(x) = h_2(x)$ for $x \in [x_2^*, 1]$ and h linear in $[x_1^*, x_2^*]$ such that $h(x_1^*) = h_1(x_1^*)$ and $h(x_2^*) = h_2(x_2^*)$, is the required function.

3) $\{i\colon\ x_i'' > x_{i_0}''\} \ne \emptyset$ and if $x_i'' > x_{i_0}''$, then $x_i' > x_{i_0}'$ $(i \in \{1, \dots t\})$. So, let i_1 be an element of $\{1, \dots, t\}$ such that $x_{i_1}' = \min\{x_i'\colon\ x_i' > x_{i_0}''\}$. Since W' is a cover of $[0,1]$, there exists $k \in \{1, \dots, p\}$ such that $A_k \cap [0, x_{i_0}'') \ne \emptyset \ne A_k \cap (x_{i_1}', 1]$. From (3) it follows that there exist points

- $x_2^* \in A_k \cap (x_{i_0}, x_{i_0}'') \cap S_{k_{x_{i_0}}}^{x_{i_0}}$, of course $\xi(x_2^*) \in \left(\xi(x_{i_0}) - \frac{1}{n_{x_{i_0}}}, \xi(x_{i_0}) + \frac{1}{n_{x_{i_0}}}\right)$,
- $x_3^* \in A_k \cap (x_{i_1}', x_{i_1}) \cap S_{k_{x_{i_1}}}^{x_{i_1}}$, then $\xi(x_3^*) \in \left(\xi(x_{i_1}) - \frac{1}{n_{x_{i_1}}}, \xi(x_{i_1}) + \frac{1}{n_{x_{i_1}}}\right)$.

Let us now define (analogously as in 1)) a continuous function $h_2\colon [x_1^*, x_3^*] \to \mathbb{R}$ such that $\mathcal{G}_{h_2} \subset H$ and $h_2(x_3^*) \in \left(\xi(x_{i_1}) - \frac{1}{n_{x_{i_1}}}, \xi(x_{i_1}) + \frac{1}{n_{x_{i_1}}}\right)$.

It is easy to see that $\mathcal{G}_{h_1 \nabla h_2} \subset H$.

If case 2) has taken place in the construction under consideration, the proof is completed. Let us consider cases 1) and 3) of the above construction. In both of the cases, if $1 \in (x_{i_1}', x_{i_1}'')$, then it suffices to put $h = h_1 \nabla h_2 \nabla h_3$ where $h_3\colon [x_3^*, 1] \to \mathbb{R}$ is constant and equal to $h_2(x_3^*)$. Thus h is the required function. So, let us consider the situation when $1 \notin (x_{i_1}', x_{i_1}'')$. The following cases take place:

1') There exists $i \in \{1, \dots, t\}$ such that $x_i' \le x_{i_1}''$ and $x_i'' > x_{i_1}$. Then let i_2 be an element of $\{1, \dots, t\}$ such that $x_{i_2}'' = \max\{x_i''\colon\ 1 \le i \le t \wedge x_i' \le x_{i_1}'' \wedge x_i'' > x_{i_1}''\}$.

If case 3) has taken place in the previous step of this construction, then reasoning analogously as in case 1), we find a continuous function $h_3\colon [x_3^*, x_5^*] \to \mathbb{R}$ such that $\mathcal{G}_{h_1 \nabla h_2 \nabla h_3} \subset H$ and $x_5^* \in (x_{i_1}'', x_{i_2}'') \cap S_{k_{x_{i_2}}}^{x_{i_2}}$, $\xi(x_5^*) \in \left(\xi(x_{i_2}) - \frac{1}{n_{x_{i_2}}}, \xi(x_{i_2}) + \frac{1}{n_{x_{i_2}}}\right)$ and there exists $\hat{x}_5^* \in (x_{i_1}'', x_{i_2}'') \cap S_{k_{x_{i_2}}}^{x_{i_2}}$.

If case 1) has taken place, we consider a component A^3 of A such that $x''_{i_1} \in A^3$. If $\hat{x}^*_3 \in A^3$, then we put $x^*_4 = \hat{x}^*_3$. Conversely, if $\hat{x}^*_3 \notin A^3$ (according to (3)), let

- $x^*_4 \in A^3 \cap (\hat{x}^*_3, x''_{i_1}) \cap S^{x_{i_1}}_{k_{x_{i_1}}}$, of course $\xi(x^*_4) \in \left(\xi(x_{i_1}) - \frac{1}{n_{x_{i_1}}}, \xi(x_{i_1}) + \frac{1}{n_{x_{i_1}}}\right)$,
- $x^*_5 \in A^3 \cap (x''_{i_1}, x''_{i_2}) \cap S^{x_{i_2}}_{k_{x_{i_2}}}$ be a point such that there exists $\hat{x}^*_5 \in A^3 \cap (x^*_5, x''_{i_2}) \cap S^{x_{i_2}}_{k_{x_{i_2}}}$; then $\xi(x^*_5) \in \left(\xi(x_{i_2}) - \frac{1}{n_{x_{i_2}}}, \xi(x_{i_2}) + \frac{1}{n_{x_{i_2}}}\right)$.

Thus $h'_3\colon [x^*_4, x^*_5] \to \mathbb{R}$ and $h_3\colon [x^*_3, x^*_5] \to \mathbb{R}$ is defined analogously as in 1).

2') $\{i\colon\ x''_i > x''_{i_1}\} = \emptyset$.

If case 3) has taken place in the previous step of this construction, then, similarly as in 2), the proof is completed.

So, let us consider the situation when case 1) has taken place. Then let A_3 be a component of A such that $1 \in A_3$. Thus, if $\hat{x}^*_3 \in A_3$, then let $x^*_4 = \hat{x}^*_3$; in the opposite case, let $x^*_4 \in A_3 \cap (\hat{x}^*_3, x''_{i_1}) \cap S^{x_{i_1}}_{k_{x_{i_1}}}$ and, analogously as in 2), the proof is finished.

3') $\{i\colon\ x''_i > x''_{i_1}\} \neq \emptyset$ and if $x''_i > x''_{i_1}$, then $x'_i > x''_{i_1}$ $(i \in \{1, \dots, t\})$.

Then the reasoning is similar to that in case 3) with the employment of the possible modification signalled in 1') and 2') (i.e., $x^*_4 = \hat{x}^*_3$ or we choose x^*_4 according to (3)).

The method presented above allows one to construct a function h defined in $[0, 1]$ such that $\mathcal{G}_h \subset H$ (the finiteness of the cover W' guarantees the ending of this construction after finitely many steps).

The proof of the theorem is completed. □

The facts presented in this note incline us to formulate the following problems:

Problem 1. Proposition and Theorem allow one to build a necessary and sufficient condition for the existence of an extension to a complete system of almost continuous functions in the case of a finite family $\mathcal{F}$ with the property that $D_f = \{x_0\}$. The result giving a characterization for an infinite family $\mathcal{F}$ possessing a *large* set of points of discontinuity would be interesting.

Problem 2. Does there exist a topology $\mathcal{D}$ such that the set $\{x\colon\ B_{\mathcal{D}}(x) \neq B_{\mathcal{G}}(x)\}$ is dense in $\mathcal{I}$, where $B_{\mathcal{D}}(x)$ $(B_{\mathcal{G}}(x))$ denotes a local base of the space $([0, 1], \mathcal{D})$ $(([0, 1], \mathcal{G}))$ at x, such that the family of all real functions continuous in the topology $\mathcal{D}$ consists of almost continuous functions?

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