Tatra Mountains Math. Publ. **5** (1995), 101–105

TATRA MOUNTAINS
Mathematical Publications

ON CATEGORIES OF STRONGLY REGULAR RELATIONAL SYSTEMS

JOSEF ŠLAPAL

Dedicated to Professor J. Jakubík on the occasion of his 70th birthday

ABSTRACT. For any ordinal $\alpha > 1$ we define a category of monorelational systems of type α and show that each of these categories is an exponential supercategory of the category of preordered sets. We also give an application of the results obtained into topology.

The categories in which there exist well-behaved function spaces play important roles in many branches of mathematics and nowadays they have been intensively used in computer science. It is therefore worthwhile to deal with such categories and the presented note is a contribution to this area.

Given an ordinal $\alpha > 1$, by a *relation R of type α on a set X* we understand a subset $R \subseteq X^\alpha$. The pair (X, R) is then called a *relational system of type α*.

A relational system of type α is said to be *strongly regular* if the following two conditions are satisfied:

(i) $(x_i \mid i < \alpha) \in R$ whenever there is $x \in X$ such that $x_i = x$ for all ordinals $i < \alpha$,

(ii) if $(x_i \mid i < \alpha) \in X^\alpha$ has the property that for any ordinal i_0, $0 < i_0 < \alpha$, there are $(y_j \mid j < \alpha) \in R$ and an ordinal j_0, $0 < j_0 < \alpha$, such that $x_{i_0} = y_{j_0}$ and $\{y_j : j < j_0\} \subseteq \{x_i : i < i_0\}$, then $(x_i \mid i < \alpha) \in R$.

All categories in this note are supposed to be concrete categories of structured sets and structure-compatible maps. We denote by Rel_α the category of strongly regular relational systems of type α with relational homomorphisms (i.e., maps $f\colon (X,R) \to (Y,S)$ fulfilling $(x_i \mid i < \alpha) \in R \Rightarrow \big(f(x_i) \mid i < \alpha\big) \in S\big)$ as morphisms. For any pair of objects $G, H \in \mathrm{Rel}_\alpha$ we denote by $\mathrm{Hom}(G,H)$ the set of all homomorphisms of G into H.

AMS Subject Classification (1991): 18B15.
Key words: relational system, strongly regular relational system, diagonal relational system, an exponential supercategory of a category.

EXAMPLE. Let Top be the category of Čech's topological spaces [1] with continuous maps as morphisms, i.e., the category whose objects are pairs (X, u) where $u\colon \exp X \to \exp X$ is a map with $u\emptyset = \emptyset$, $A \subseteq X \Rightarrow A \subseteq uA$ and $A \subseteq B \subseteq X \Rightarrow uA \subseteq uB$, and whose morphisms are maps $f\colon (X,u) \to (Y,v)$ fulfilling $f(uA) \subseteq vf(A)$ whenever $A \subseteq X$. Let $\alpha > 1$ be an ordinal and for any space $(X,u) \in$ Top let $R_u \subseteq X^\alpha$ be the relation given by $(x_i \mid i < \alpha) \in R_u \Leftrightarrow x_{i_0} \in u\{x_i\colon i < i_0\}$ for all ordinals i_0, $0 < i_0 < \alpha$. Then (X, R_u) is a strongly regular relational system and the assignation $(X,u) \mapsto (X, R_u)$ defines a concrete functor from Top into Rel_α.

Let $\widetilde{\mathrm{Rel}}_\alpha$ denote the full subcategory of Rel_α given by the objects $(X,R) \in \mathrm{Rel}_\alpha$ that are *diagonal*, i.e., that fulfil the following condition:

> if $\big((x_{ij} \mid j < \alpha) \mid i < \alpha\big) \in R^\alpha$ has the property that $(x_{ij} \mid i < \alpha) \in R$ for each $j < \alpha$, then $(x_{ii} \mid i < \alpha) \in R$.

Now, in accordance with [6], we define

DEFINITION. If $\mathcal{K}$ is a category with finite products and $\mathcal{L}$ its full isomorphism-closed subcategory, then $\mathcal{K}$ is called an *exponential supercategory* of $\mathcal{L}$ provided that for any pair of objects $G \in \mathcal{L}$, $H \in \mathcal{K}$ there exists an object $G^H \in \mathcal{L}$ with $\mid G^H \mid = \mathrm{Mor}_{\mathcal{K}}(H,G)$ such that the pair (G^H, e), where e is the evaluation map for G^H (i.e., the map $e\colon H \times G^H \to G$ given by $e(y,f) = f(y)$), is a co-universal map for G with respect to the functor $H \times - : \mathcal{K} \to \mathcal{K}$.

The objects G^H from the definition will be called *powers* of G and H.

The substantial property of the exponentiality defined is the validity of the first exponential law $(G^H)^K \cong G^{H \times K}$ for the powers. (Of course, the powers are unique whenever $\mathcal{K}$ is transportable.) If $\mathcal{K}$ is an exponential supercategory of itself, then $\mathcal{K}$ is cartesian closed in the sense of [3].

THEOREM 1. Rel_α *is an exponential supercategory of* $\widetilde{\mathrm{Rel}}_\alpha$ *for any ordinal* $\alpha > 1$.

P r o o f . It is obvious that Rel_α has finite products. Let $G = (X,R) \in \widetilde{\mathrm{Rel}}_\alpha$, $H = (Y,S) \in \mathrm{Rel}_\alpha$ be a pair of objects and put $G^H = \big(\mathrm{Hom}(H,G), T\big)$ where $T \subseteq \big(\mathrm{Hom}(H,G)\big)^\alpha$ is given by $(f_i \mid i < \alpha) \in T \Leftrightarrow \big(f_i(y) \mid i < \alpha\big) \in R$ for each $y \in Y$. Then evidently $G^H \in \widetilde{\mathrm{Rel}}_\alpha$. Let $e\colon H \times G^H \to G$ be the evaluation map for G^H. Denote $(Z,U) = H \times G^H$ and let $(g_i \mid i < \alpha) \in U$. Then there are $(y_i \mid i < \alpha) \in S$ and $(f_i \mid i < \alpha) \in T$ such that $g_i = (y_i, f_i)$ for each $i < \alpha$. Now we have $\big((f_i(y_j) \mid j < \alpha) \mid i < \alpha\big) \in R^\alpha$ (because $f_i \in \mathrm{Hom}(H,G)$ for each $i < \alpha$) and $\big(f_i(y_j) \mid i < \alpha\big) \in R$ for each $j < \alpha$. Hence $\big(f_i(y_i) \mid i < \alpha\big) \in R$. But $\big(f_i(y_i) \mid i < \alpha\big) = \big(e(y_i, f_i) \mid i < \alpha\big) = \big(e(g_i) \mid i < \alpha\big)$. We have shown that $e \in \mathrm{Hom}(H \times G^H, G)$. Let $K = (W,V) \in \mathrm{Rel}_\alpha$ and $h \in \mathrm{Hom}(H \times K, G)$.

Let $h^*\colon W \to \mathrm{Hom}(H,G)$ be the map given by $h^*(w)(y) = h(y,w)$. (Indeed, $h^*(w) \in \mathrm{Hom}(H,G)$ whenever $w \in W$ because for any $(y_i \mid i < \alpha) \in S$ we have $\big(h^*(w)(y_i) \mid i < \alpha\big) = \big(h(y_i,w) \mid i < \alpha\big) \in R$.) Let $(w_i \mid i < \alpha) \in V$. For any $y \in Y$ we have $\big(h^*(w_i)(y) \mid i < \alpha\big) = \big(h(y,w_i) \mid i < \alpha\big) \in R$, hence $\big(h^*(w_i) \mid i < \alpha\big) \in T$. Consequently, $h^* \in \mathrm{Hom}(K, G^H)$. As h^* is clearly the only map with $h = e \circ (\mathrm{id}_Y \times h^*)$, the pair (G^H, e) is a co-universal map for G w.r.t. the functor $H \times - : \mathrm{Rel}_\alpha \to \mathrm{Rel}_\alpha$. □

R e m a r k . For $\alpha = 2$ the Theorem is quite obvious because Rel_2 is the category of reflexive binary relational systems and $\widetilde{\mathrm{Rel}_2}$ is the category of preordered sets.

THEOREM 2. $\widetilde{\mathrm{Rel}_\alpha}$ *is concretely isomorphic to* $\widetilde{\mathrm{Rel}_2}$ *for each ordinal* $\alpha > 1$.

P r o o f . For any object $(X,R) \in \widetilde{\mathrm{Rel}_\alpha}$ let $r_R \subseteq X^2$ be given by $(x,y) \in r_R \Leftrightarrow$ there is $(x_i \mid i < \alpha) \in R$ such that $x_0 = x$ and $x_{i_0} = y$ for each ordinal i_0, $0 < i_0 < \alpha$. In [6] it is shown that the assignation $(X,R) \mapsto (X, r_R)$ defines a concrete isomorphism of $\widetilde{\mathrm{Rel}_\alpha}$ onto $\widetilde{\mathrm{Rel}_2}$. □

Let $\alpha, \beta > 1$ be ordinals. For any object $(X,R) \in Rel_\alpha$ put $F_{\alpha,\beta}(X,R) = (X,S)$ where $S \subseteq X^\beta$ is given by $(y_j \mid j < \beta) \in S$ iff for each ordinal j_0, $0 < j_0 < \beta$, there are $(x_i \mid i < \alpha) \in R$ and an ordinal i_0, $0 < i_0 < \alpha$, such that $y_{j_0} = x_{i_0}$ and $\{x_i : i < i_0\} \subseteq \{y_j : j < j_0\}$.

The following statement is obvious

THEOREM 3. $F_{\alpha,\beta}$ *is a concrete functor from* Rel_α *into* Rel_β *for any ordinals* $\alpha, \beta > 1$.

THEOREM 4. *Let* α, β *be ordinals,* $1 < \alpha \leqslant \beta$. *Then* $F_{\alpha,\beta}$ *is a full concrete embedding of* Rel_α *into* Rel_β.

P r o o f . In virtue of Theorem 3, $F_{\alpha,\beta}$ is a concrete functor from Rel_α into Rel_β. Let $(X,R_1),(X,R_2) \in \mathrm{Rel}_\alpha$ and suppose that there is $(X,S) \in Rel_\beta$ with $F_{\alpha,\beta}(X,R_1) = F_{\alpha,\beta}(X,R_2) = (X,S)$. Let $(x_i \mid i < \alpha) \in R_1$ and let $(y_i \mid i < \beta) \in X^\beta$ be the sequence given by $y_i = x_i$ for all ordinals $i < \alpha$ and $y_i = x_0$ for all ordinals i, $\alpha \leqslant i < \beta$. Then $(y_i \mid i < \beta) \in S$. Thus, for each ordinal i_0, $0 < i_0 < \beta$, there are $(z_j \mid j < \alpha) \in R_2$ and an ordinal j_0, $0 < j_0 < \alpha$, such that $y_{i_0} = z_{j_0}$ and $\{z_j : j < j_0\} \subseteq \{y_i : i < i_0\}$. This yields $(y_i \mid i < \alpha) = (x_i \mid i < \alpha) \in R_2$. Hence $R_1 \subseteq R_2$. As the inverse inclusion can be proved in an analogous way, $F_{\alpha,\beta}$ is injective on objects. To prove that $F_{\alpha,\beta}$ is full, let $(X,R_1),(Y,R_2) \in \mathrm{Rel}_\alpha$ and $(X,S_1) = F_{\alpha,\beta}(X,R_1)$, $(Y,S_2) = F_{\alpha,\beta}(Y,R_2)$. Let $f \in \mathrm{Hom}\big((X,S_1),(Y,S_2)\big)$ and $(x_i \mid i < \alpha) \in R_1$. Let $(y_i \mid i < \beta) \in X^\beta$ be the sequence given in the same way as in the first part of the proof, i.e., $y_i = x_i$ for all ordinals $i < \alpha$ and $y_i = x_0$ for all ordinals i, $\alpha \leqslant i <$

β. Then $(y_i \mid i < \beta) \in S_1$ and hence $\big(f(y_i) \mid i < \beta\big) \in S_2$. Thus, for each ordinal i_0, $0 < i_0 < \beta$, there are $(z_j \mid j < \alpha) \in R_2$ and an ordinal j_0, $0 < j_0 < \alpha$, such that $f(y_{i_0}) = z_{j_0}$ and $\{z_j : j < j_0\} \subseteq \{f(y_i) : i < i_0\}$. This yields $\big(f(y_i) \mid i < \alpha\big) = \big(f(x_i) \mid i < \alpha\big) \in R_2$. Consequently, $f \in \mathrm{Hom}\big((X, R_1), (Y, R_2)\big)$, i.e., $F_{\alpha,\beta}$ is full. □

In the Example we have shown that there is a certain connectedness between the category Top and the categories Rel_α. Now we are going to present a result that originates in this connectedness.

Let Top_T be the full subcategory of Top given by the objects $(X, u) \in \mathrm{Top}$ having the property that for any subset $A \subseteq X$ and any point $x \in uA$ there exist an ordinal $i_0 > 0$ and a sequence $(x_i \mid i < i_0) \in A^{i_0}$ such that $x_j \in u\{x_i : i < j\}$ for each ordinal j, $0 < j < i_0$, and $x \in u\{x_i : i < i_0\}$.

It can easily be shown that Top_T is a coreflective modification of Top (i.e., a coreflective subcategory of Top with identity maps acting as coreflectors).

We denote by Qcat the quasicategory of all categories (with functors as morphisms).

THEOREM 5. *In* Qcat, Top_T *is a direct limit of the diagram* $\{\mathrm{Rel}_\alpha : \alpha > 1$ *an ordinal*$\}$ *with embeddings* $F_{\alpha,\beta}$, $1 < \alpha \leqslant \beta$, *as morphisms.*

Proof. For any ordinal $\alpha > 1$ and any object $(X, R) \in Rel_\alpha$ put $G_\alpha(X, R) = (X, u)$ where $u : \exp X \to \exp X$ is given by $uA = \{x \in X :$ there are $(x_i \mid i < \alpha) \in R$ and an ordinal i_0, $0 < i_0 < \alpha$, such that $x = x_{i_0}$ and $x_i \in A$ for all $i < i_0\}$. It is evident that G_α is a full concrete embedding of Rel_α into Top_T and for arbitrary object $(X, u) \in \mathrm{Top}_T$ we have $(X, u) \in G_\alpha(\mathrm{Rel}_\alpha)$ iff for any subset $A \subseteq X$ and any point $x \in uA$ there exist an ordinal i_0, $0 < i_0 < \alpha$, and a sequence $(x_i \mid i < i_0) \in A^{i_0}$ such that $x_j \in u\{x_i : i < j\}$ for each ordinal j, $0 < j < i_0$, and $x \in u\{x_i : i < i_0\}$. As $G_\alpha = G_\beta \circ F_{\alpha,\beta}$ clearly holds whenever $1 < \alpha \leqslant \beta$, the system $G_\alpha : \mathrm{Rel}_\alpha \to \mathrm{Top}_T$, $\alpha > 1$ ordinals, is a natural sink in Qcat. We are to show that, in Qcat, for any natural sink $H_\alpha : \mathrm{Rel}_\alpha \to \mathcal{K}$, $\alpha > 1$ ordinals, there is a unique functor $H : \mathrm{Top}_T \to \mathcal{K}$ such that $H_\alpha = H \circ G_\alpha$ for all ordinals $\alpha > 1$. Obviously, the functor H can be obtained as follows: For any object $(X, u) \in \mathrm{Top}_T$ we put $H(X, u) = H_\alpha(X, R)$ where $\alpha > 1$ is the least ordinal with $(X, u) \in G_\alpha(\mathrm{Rel}_\alpha)$ and $(X, R) \in \mathrm{Rel}_\alpha$ is the object with $G_\alpha(X, R) = (X, u)$; for any morphism $f : (X, u) \to (Y, v)$ we put $Hf = H_\alpha f$ where $\alpha > 1$ is the least ordinal with both $(X, u) \in G_\alpha(\mathrm{Rel}_\alpha)$ and $(Y, v) \in G_\alpha(\mathrm{Rel}_\alpha)$. This completes the proof. □

Denote by Top_S the full subcategory of Top given by those objects $(X, u) \in \mathrm{Top}$ that are *finitely generated topological spaces*, i.e., that fulfil $uuA = uA = \bigcup_{x \in A} u\{x\}$ whenever $\emptyset \neq A \subseteq X$. Clearly, Top_S is a (full) subcategory of Top_T.

As it is well known that Top_S is concretely isomorphic to $\widetilde{\mathrm{Rel}_2}$, the previous results yield

COROLLARY. Top_T *is an exponential supercategory of* Top_S.

EXAMPLES.

1. Let $X = \{a, b, c\}$ and define $u\colon \exp X \to \exp X$ as follows: $u\emptyset = \emptyset$, $u\{a\} = \{a, b\}$, $u\{b\} = \{b\}$, $u\{c\} = \{c\}$, $u\{a, b\} = u\{a, c\} = uX = X$, $u\{b, c\} = \{b, c\}$. Then $(X, u) \in \mathrm{Top}_T$ (but $(X, u) \notin Top_S$).

2. Let $(\omega + 1, u)$ be the topological space defined by $u\emptyset = \emptyset$, $uA = \omega$ whenever $A \subseteq \omega$ and $0 < \operatorname{card} A < \omega$, $uA = \omega + 1$ otherwise. Then $(\omega+1, u) \in \mathrm{Top}_T$, $(\omega+1, u) \notin \mathrm{Top}_S$, and the topology u is additive, i.e., $u(A \cup B) = uA \cup uB$ whenever $A, B \subseteq \omega + 1$.

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Received February 9, 1994

Department of Mathematics
Technical University of Brno
Technická 2
CZ–616 69 Brno
CZECH REPUBLIC