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COMPARISON OF FUZZY NUMBERS IN DECISION MAKING

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ABSTRACT. The problem of ordering trapezoidal fuzzy numbers is studied. The family of distances d_p determined on the space of all trapezoidal fuzzy numbers is introduced and investigated. These distances induce the family of linear orders; we will try to compare the alternatives in decision making problems under fuzzy environment. The optimistic, pessimistic and mixed approach in decision making are studied.

1. Introduction

The problem of ordering fuzzy numbers plays an important role in decision making problems under fuzzy environment. Many methods for ranking fuzzy numbers have been suggested in literature. Some methods are surveyed in paper [1]. Generally we can distinguish two ways. The first way is based on ranking function and the second uses the fuzzy preference relations.

In this paper we propose the method of ordering trapezoidal fuzzy numbers. This method is based on the distance between such fuzzy numbers. We introduce and investigate the family of distances d_p indexed by parameter p, determined on the space of all trapezoidal fuzzy numbers T treated as elements of four-dimensional space. These distances generate the class of linear orders under a fixed upper horizon H. From this family of orders we prefer one based on distance with parameter $p = 1$. Such order does not depend on the horizon and it is univocaly determined by an expected value of trapezoidal fuzzy numbers.

Finally we study the family of orders under a fixed lower and upper horizon. These orders combine the pessimistic and optimistic approach in decision making problems under fuzzy environment.

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2. Fuzzy numbers

Now we recall some notions connected with fuzzy numbers. The *fuzzy number* is a normal fuzzy subset of the real line with the upper semi-continuous membership function. This definition implies that every r-level of such a fuzzy set A is a closed interval $A_r = [a_r, b_r]$. We also assume that its support, i.e., the set $\operatorname{supp} A = \{x : \mu_A(x) > 0\}^{\mathrm{cl}}$, is the bounded interval [4], [5]. The membership function of the fuzzy number A has the following form:

$$\mu_A(x) = \begin{cases} 0 & \text{for } x < a_1, \\ f(x) & \text{for } a_1 \le x < a_2, \\ 1 & \text{for } a_2 \le x \le a_3, \\ g(x) & \text{for } a_3 < x \le a_4, \\ 0 & \text{for } a_4 < x, \end{cases}$$

where f is an increasing and g is a decreasing function. We call the functions f and g the left and the right side, a_1, a_4 the outer borders and a_2, a_3 the inner borders of fuzzy number A, respectively.

We call the fuzzy numbers with linear sides the *trapezoidal fuzzy numbers* (t.f.n.). We denote the trapezoidal fuzzy number as $T(a_1, a_2, a_3, a_4)$ and the family of all t.f.n. as T. When $a_2 = a_3$ we obtain a triangular fuzzy number. The *expected interval* and the *expected value* of fuzzy number A are equal

$$EI(A) = [E_{s_1}, E_{s_2}],$$
$$EV(A) = (E_{s_1} + E_{s_2})/2,$$

where $Es_1 = a_2 - \int_{a_1}^{a_2} f(x)dx$ and $Es_2 = a_3 + \int_{a_3}^{a_4} g(x)dx$ [6].

In the case of the trapezoidal fuzzy numbers we obtain:

$$EI(A) = \left[(a_1 + a_2)/2,\ (a_3 + a_4)/2\right],$$
$$EV(A) = (a_1 + a_2 + a_3 + a_4)/4.$$

The borders of trapezoidal fuzzy number univocaly determine this fuzzy number. In such situation we can treat every t.f.n. as point in the four-dimensional space and class T as subset $R^4 \subset \mathbb{R}^4$:

$$R^4 = \left\{(x_1, x_2, x_3, x_4,) \colon x_1 \le x_2 \le x_3 \le x_4\right\}.$$

The above interpretation of trapezoidal fuzzy numbers is very useful for us; now let us define the distance between t.f.n.. We construct the family of distances indexed by parameter p (see also [2], [3]) using the metrics in space $\mathbb{R}^4$.

DEFINITION 1. Let $A = T(a_1, a_2, a_3, a_4)$ and $B = T(b_1, b_2, b_3, b_4)$, the distance between t.f.n. A and B is the following real function determined on Cartesian product $T \times T$:

$$d_p(A,B) = \begin{cases} \sqrt[p]{0.25 \sum_{i=1}^{4} |a_i - b_i|^p}, & \text{for } 1 \le p < \infty, \\ \max_i (|a_i - b_i|), & \text{for } p = \infty. \end{cases}$$

This function is a metric in the space T and the space (T, d_p) is a complete metric space [2].

3. Order determined by distance between trapezoidal fuzzy numbers

The problem of ordering trapezoidal fuzzy numbers plays an important role in the decision making problem under uncertainty, when the environment of such problems is described by means of fuzzy sets. In this case we have some alternatives and the final values of them by payoff or utility function are the trapezoidal fuzzy number. If we want to choose the best one we ought to have a method of ordering those fuzzy numbers.

Let $H = T(h_1, h_2, h_3, h_4)$ be some fixed trapezoidal fuzzy number called *upper horizon*. We can determine the subset

$$T_H = \{A \in T \colon a_4 \le h_1\}$$

of all trapezoidal fuzzy numbers $A = T(a_1, a_2, a_3, a_4)$ dominated by horizon H. It is the subset of space T.

Finally, we can define the following real function on subset T_H:

$$\rho_H(A) = d_p(A, H),$$

for the fixed upper horizon H and parameter p and $A \in T_H$. The function ρ_H projects subset T_H to the real line $\mathbb{R}$.

Now we define the order on a set T_H using a natural order on $\mathbb{R}$.

DEFINITION 2. Let $A, B \in T_H$, the *order under horizon* H is the following relation:

$$A <_H B \iff \rho_H(A) \ge \rho_H(B).$$

THEOREM 1. *The relation $<_H$ is the reflexive, transitive and complete, i.e., preference relation on space T_H.*

Proof. The proof is elementary. □

We can interpret the upper horizon H in decision making problem as the best, ideal alternative. This is an optimistic approach. In many cases, for instance when the horizon H is an abstract, non-real point, we want to choose the distance between t.f.n. from the family of orders indexed by parameter p, which does not depend on the upper horizon H.

THEOREM 2.

(i) *If $p = 1$, then the order $<_H$ does not depend on horizon H.*

(ii) *If $p = \infty$ and H is a crisp real number $(h_1 = h_4)$, then the order $<_H$ does not depend on horizon H.*

(iii) *In other cases this order depends on the horizon.*

Proof. We will use another, equivalent form of definition of order $<_H$:

$$A <_H B \iff \rho_H(A) - \rho_H(B) \geq 0 .$$

(i) Let $p = 1$ then

$$\begin{aligned} d_1(A,H) - d_1(B,H) &= \Big(\sum_{i=1}^{4}(h_i - a_i) - \sum_{i=1}^{4}(h_i - b_i)\Big)/4 = \\ &= \sum_{i=1}^{4}(b_i - a_i)/4 \end{aligned}$$

and this subtraction does not depend on the horizon H.

(ii) Let $p = \infty$, $H = T(h,h,h,h)$, $G = T(g,g,g,g)$, $g < h$ and $A, B \in T_H$. If $\max_i\{h - a_i\} = h - a_j$ then $\max_i\{g - a_i\} = g - a_j$ and we obtain

$$\begin{aligned} d_p(A,H) - d_p(B,H) &= \max_i\{h - a_i\} - \max_i\{h - b_i\} = \\ &= h - a_j - h + b_k = b_k - a_j \end{aligned}$$

and it does not depend on a horizon which is a crisp number.

(iii) We will show that in the case $1 < p < \infty$ the order depends on horizon H even if it is a crisp real number h. Let $H = T(h,h,h,h)$, $G = T(g,g,g,g)$, $dh = g - h > 0$, $r_H = d_p(A,H)$ and $r_G = d_p(A,G)$; then

$$d_p(H,G) = g - h = dh ,$$

for any $1 < p < \infty$. The function d_p is a metric in space T. Then

$$r_G = d_p(A,G) < d_p(A,H) + d_p(H,G) = r_H + dh , \tag{1}$$

for such p. Now we find two crisp real numbers b_H i b_G, such that

$$\begin{aligned} d_p(A,H) &= d_p(B_H,H) , \\ d_p(A,G) &= d_p(B_G,G) , \end{aligned}$$

where $B_H = T(b_H, b_H, b_H, b_H)$, $B_G = T(b_G, b_G, b_G, b_G)$. We obtain

$$b_H = h - r_H \,,$$
$$b_G = g - r_G \,,$$

and using (1) we have

$$b_G - b_H = g - r_G - h + r_H > g - r_H - dh - h + r_H = 0 \,.$$

Let $b = (b_H + b_G)/2$, $B = T(b, b, b, b)$ then $b_H < b < b_G$ and

$$d_p(A, H) = h - b_H > h - b = d_p(B, H) \,,$$
$$d_p(A, G) = g - b_G < g - b = d_p(B, G) \,.$$

Finally, we obtain

$$d_p(A, H) > d_p(B, H) \quad \text{and} \quad d_p(A, G) < d_p(B, G) \,,$$

i.e., the order $<_H$ depends on horizon H.

For $p = 1$ and $p = \infty$ we have equality in (1). □

EXAMPLE 1. Let $A = T(3, 5, 6, 7)$, $B = T(4, 5, 5, 6)$, $H_1 = T(8, 8, 8, 8)$, $H_2 = T(9, 9, 9, 9)$ and $p = 2$. Then $\rho_1(A) = \sqrt{39}$, $\rho_1(B) = \sqrt{38}$, $\rho_2(A) = \sqrt{65}$, $\rho_2(B) = \sqrt{66}$ and $A <_1 B$, $B <_2 A$.

EXAMPLE 2. Let $A = T(2, 3, 4, 5)$, $B = T(1, 3, 4, 6)$, $H_1 = T(8, 10, 11, 13)$, $H_2 = T(7, 8, 9, 10)$ and $p = \infty$. Then $\rho_1(A) = 8$, $\rho_1(B) = 7$, $\rho_2(A) = 5$, $\rho_2(B) = 6$ and $A <_1 B$, $B <_2 A$.

We see that for $p = 1$ the order $<_H$ does not depend on horizon H. Now we show that in this case $<_h$ is univocaly determined by an expected value of t.f.n.

THEOREM 3. *If $p = 1$ and $A \in T_H$, then*

$$d_p(A, H) = EV(H) - EV(A) \,.$$

P r o o f. From the definitions of d_p and T_H we obtain

$$d_p(A, H) = (h_1 - a_1 + h_2 - a_2 + h_3 - a_3 + h_4 - a_4)/4 = EV(H) - EV(A) \,.$$

□

Theorems 2(i) and 3 guarantee simple calculation of the distance d_1, and we prefer such distance in many applications.

In decision making problems we often want to order a sequence of n t.f.n. $A_i = T(a_1^i, a_2^i, a_3^i, a_4^i)$. Then we may choose an upper horizon H equal to a crisp real number

$$H = \max_i \{a_4^i\} \,.$$

We can also order such a sequence of trapezoidal fuzzy numbers considering a pessimistic approach using the lower horizon G. We may interpret such a horizon as the worst, non ideal alternative and choose for instance the crisp real number equal

$$G = \min_i \{a_1^i\} .$$

We can define the order under lower horizon G in the similar way as in the upper case. This order must be determined on the subset

$$T^G = \{A \in T \colon h_4 \le a_1\} .$$

Linking optimistic and pessimistic approaches we obtain another way of ranking fuzzy numbers. This method uses the geometrical interpretation of trapezoidal fuzzy number. We join the lower G and upper H horizons by line q. Next we project the points which represent the trapezoidal fuzzy numbers A_i on this line and obtain the sequence of points A_i'. We can order them using a natural linear order on this line.

In the case when the horizons are crisp real numbers, the line q contains all real numbers and the images by projection are the crisp real numbers, too. We assume that image $A' = T(a', a', a', a')$ of t.f.n. A satisfies the natural condition:

$$a_1 \le a' \le a_4 .$$

We can take the expected value of fuzzy number as the image of A_i by this projection, i.e.,

$$A_i' = EV(A_i) .$$

This method of projection is very simple, but it minimizes the distance, i.e., it satisfies the condition:

$$d_p(A_i, A_i') = \inf\{d_p(A_i, B) \colon B \in q\}$$

for $p = 2$ only.

THEOREM 4. *Let $A \in T$, the horizons be the crisp real numbers and A' be the image of A by projection on line q. The above images minimize the distance d_p:*

(i) *If $p = 2$:* $A' = EV(A)$,
(ii) *If $p = 1$:* $a_2 \le A' \le a_3$,
(iii) *If $p = \infty$:* $A' = (a_1 + a_4)/2$.

P r o o f.

(i) Let $EV(A) = (a_1 + a_2 + a_3 + a_4)/4$, $B = T(b, b, b, b) \in q$, $1 < p < \infty$, $f(b) = \big(d_p(A, B)\big)^p$ and we assume that $a_1 \le b \le a_2 \le a_3 \le a_4$. Then

$$f(b) = \big((b - a_1)^p + (a_2 - b)^p + (a_3 - b)^p + (a_4 - b)^p\big)/4 ,$$
$$f'(b) = p\big((b - a_1)^{p-1} - (a_2 - b)^{p-1} - (a_3 - b)^{p-1} - (a_4 - b)^{p-1}\big)/4 .$$

Now we assume that $p = 2$. In this case

$$f'(b) = (4b - a_1 - a_2 - a_3 - a_4)/2$$

and $f'(b) = 0$ for $b_o = (a_1 + a_2 + a_3 + a_4)/4 = EV(A)$. Of course, if we have $a_1 \leq EV(A) \leq a_2$. The case $a_2 \leq b \leq a_3$ and other are investigated in a similar way and we obtain the same result.

In case $1 < p < 2$, we can find a_1, a_2, a_3, a_4 satisfying inequality
$(b-a_1)^{p-1} - (a_2-b)^{p-1} - (a_3-b)^{p-1} - (a_4-b)^{p-1} > (4b-a_1-a_2-a_3-a_4)^{p-1}$, and for $2 < p < \infty$
$(b-a_1)^{p-1} - (a_2-b)^{p-1} - (a_3-b)^{p-1} - (a_4-b)^{p-1} < (4b-a_1-a_2-a_3-a_4)^{p-1}$.
In such situations $b_o \neq EV(A)$, where $f'(b_o) = 0$

(ii) Let $p = 1$ and $a_1 \leq b \leq a_2$, then

$$d_1(A, B) = (a_2 + a_3 + a_4 - a_1 - 2b)/4$$

and $b_o = a_2$.
Let $a_2 \leq b \leq a_3$, then

$$d_1(A, B) = (a_3 + a_4 - a_1 - a_2)/4$$

and $a_2 \leq b_o \leq a_3$.
Other cases are investigated in a similar way.

(iii) Let $p = \infty$, then

$$d_\infty(A, B) = \max\{b - a_1, a_4 - b\}$$

and $b_o = (a_1 + a_4)/2$. □

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