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THE ENTROPY BASED ON PSEUDO–ARITHMETICAL OPERATIONS

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ABSTRACT. The entropy of the partitions of measurable spaces equipped with a $\perp$-decomposable or $\oplus$-decomposable measure ($\perp$ is a continuous Archimedean t-conorm, $\oplus$ is a pseudo-addition) is presented. The pseudo-arithmetical operations are used to build it up. Further the relationship between this kind of the entropy and the classical (Kolmogorov–Sinaj) entropy is shown.

1. Introduction

The entropy of partitions of the probability space introduced by Kolmogorov and Sinaj [2, 8] can be assumed to be a suitable tool for studying of the dynamics systems. One of successful attempts to generalize the notion of probability space has been made by Weber [9]. Weber replaced the probability measure by a $\perp$-decomposable measure, where $\perp$ is a continuous Archimedean t-conorm, and he used these spaces for the building up of a non-additive theory of integration. The similar access can be found in Sugeno and Murofushi [4], though they worked with a more general model using $\oplus$-decomposable measure, where $\oplus$ is the pseudo-addition. Now we will extend the entropy of partitions on these spaces using the pseudo-arithmetical operations introduced in paper [3]. Later we will prove that this type of entropy is a g-transformation (where g is a generator of pseudo-arithmetical operations and of t-conorm $\perp$ as well) of the entropy on a probability space corresponding to the given space. In the end we will introduce the conditional g-entropy and we will show that the same conclusions hold for it.

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2. Preliminaries

Let $\mathbf{X}$ be a nonempty set, $(\mathbf{X}, \mathcal{S})$ be a measurable space and the function g be a generator of the consistent system of pseudo-arithmetical operations $\{\oplus, \odot, \ominus, \oslash\}$ (see [3]). Thus $g : [-\infty, +\infty] \to [-\infty, +\infty]$ is such a continuous, strictly increasing and odd function that $g(0) = 0$, $g(1) = 1$, $g(+\infty) = +\infty$ and

$$x \oplus y = g^{-1}\big(g(x) + g(y)\big), \quad x \odot y = g^{-1}\big(g(x) \cdot g(y)\big),$$
$$x \ominus y = g^{-1}\big(g(x) - g(y)\big), \quad x \oslash y = g^{-1}\Big(\frac{g(x)}{g(y)}\Big), \tag{1}$$

for every $x, y \in [-\infty, +\infty]$ where expressions $g(x) - g(y)$ and $\frac{g(x)}{g(y)}$ make sense. If we take the restriction of this function g on the interval $[0,1]$ then, by Schweizer and Sklar [7], the binary operation $\perp\colon [0,1] \to [0,1]$ given by

$$a \perp b = g^{(-1)}\big(g(a) + g(b)\big) \tag{2}$$

is a continuous Archimedean t-conorm where $g^{(-1)}$ is the pseudo-inverse of g, i.e., $g^{(-1)}(x) = g^{-1}(\min\{x, 1\})$, $\forall x \in [0, +\infty)$. Moreover, it is nonstrict, therefore, it is nilpotent.

On the other hand, if $\perp$ is a continuous Archimedean t-conorm, then there exists such a continuous, strictly increasing function $h\colon [0,1] \to [0, +\infty]$, $h(0) = 0$; that the t-conorm $\perp$ is generated by the formula

$$a \perp b = h^{(-1)}\big(h(a) + h(b)\big).$$

Let the conorm $\perp$ be nonstrict (it is nilpotent); then the function h is bounded, so we can take a normalized generator g $(g(1) = 1)$ and extend it on the interval $[0, +\infty]$ so that it generates the pseudo-arithmetical operations on $[0, +\infty]$ and further on $[-\infty, +\infty]$ (Remark 3.12, [3]).

Now let the function $m\colon \mathcal{S} \to [0,1]$ have the following properties:

(M1) $m(\emptyset) = 0$, $m(\mathbf{X}) = 1$,
(M2) $\mathbf{A}, \mathbf{B} \in \mathcal{S}$, $\mathbf{A} \cap \mathbf{B} = \emptyset \implies m(\mathbf{A} \cup \mathbf{B}) = m(\mathbf{A}) \perp m(\mathbf{B})$,
(M3) $\{\mathbf{A}_n\} \subset \mathcal{S}$, $\mathbf{A}_n \nearrow \mathbf{A} \implies m(\mathbf{A}_n) \nearrow m(\mathbf{A})$.

Then m is said to be a $\perp$-decomposable measure on $\mathcal{S}$ (see [9]). If we replace the property (M2) by

(M2′) $\mathbf{A}, \mathbf{B} \in \mathcal{S}$, $\mathbf{A} \cap \mathbf{B} = \emptyset \implies m(\mathbf{A} \cup \mathbf{B}) = m(\mathbf{A}) \oplus m(\mathbf{B})$,

then m will be called a $\oplus$-decomposable measure on $\mathcal{S}$ (see [4]).

It is obvious that this measure m is σ-$\perp$ $(\oplus)$-decomposable too, i.e., (M4) $\mathbf{A}_n \in \mathcal{S}$, $n = 1, 2, \dots$, $\mathbf{A}_i \cap \mathbf{A}_j = \emptyset$, $i \neq j$

$$\Longrightarrow m\Big(\bigcup_{n\in\mathbb{N}} \mathbf{A}_n\Big) = \mathop{\perp}_{n\in\mathbb{N}}\Big(\bigoplus_{n\in\mathbb{N}}\Big) m(\mathbf{A}_n).$$

Using some results by Klement and Weber [1] we can divide the $\perp$-decomposable measures $m\colon \mathcal{S} \to [0,1]$ into two types ($\perp$ is an Archimedean t-conorm with the additive generator g):

— m is of the type (A) iff $g \circ m$ is a σ-additive measure on $\mathcal{S}$, i.e., $(h \circ m)\Big(\bigcup\limits_{n\in\mathbb{N}} \mathbf{A}_n\Big) = \sum\limits_{n\in\mathbb{N}} (h \circ m)(\mathbf{A}_n)$ for every sequence $\{\mathbf{A}_n\}_{n\in\mathbb{N}}$ of disjoint sets in $\mathcal{S}$,

— m is of the type (P) iff $g \circ m$ is a pseudo-σ-additive measure on $\mathcal{S}$, i.e., there exists such a sequence $\{\mathbf{A}_n\}_{n\in\mathbb{N}}$ of disjoint sets in $\mathcal{S}$ that $(h \circ m)\Big(\bigcup\limits_{n\in\mathbb{N}} \mathbf{A}_n\Big) < \sum\limits_{n\in\mathbb{N}} (h \circ m)(\mathbf{A}_n)$.

Considering that the t-conorm $\perp$ is nonstrict only, according to the classification by Weber [9], the measure m of the type (A) is always of the type (NSA), therefore, $g \circ m$ is a finite σ-additive measure. Similarly, the measure of the type (P) is identical to the type (NSP).

3. g-entropy

Let $(\mathbf{X}, \mathcal{S})$ be a measurable space and $\{\oplus, \odot, \ominus, \oslash\}$ be a consistent system of pseudo-arithmetical operations on $[-\infty, +\infty]$ generated by the function g. Further, let m be a $\perp$-decomposable measure on $\mathcal{S}$, where $\perp$ is a nilpotent t-conorm with the normalized additive generator g (the formula (2)). Note that this condition is not the restriction, since $g \circ m$ is an infinite σ-additive measure for strict Archimedean t-conorms and this fact excludes the possibility of defining the entropy.

DEFINITION 3.1. A finite collection $\mathcal{A} = \{\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n\} \subset \mathcal{S}$, is said to be a *measurable partition* of $\mathbf{X}$ iff it satisfies the following conditions:

(P1) $\mathbf{A}_i \cap \mathbf{A}_j = \emptyset$, $i \neq j$, $i, j = 1, 2, \dots, n$,

(P2) $\bigcup\limits_{i=1}^{n} \mathbf{A}_i = \mathbf{X}$.

Remark 3.2. If $\mathcal{A} = \{\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n\}$ is a measurable partition then

$$\mathop{\perp}_{i=1}^{n} m(\mathbf{A}_i) = 1 \quad \text{because} \quad 1 = m(\mathbf{X}) = m\Big(\bigcup_{i=1}^{n} A_i\Big) = \mathop{\perp}_{i=1}^{n} m(\mathbf{A}_i).$$

DEFINITION 3.3. Let $\mathcal{A} = \{\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n\}$ be a measurable partition of $\mathbf{X}$. Then its *g-entropy* is defined by

$$H_m(\mathcal{A}) = -\bigoplus_{i=1}^{n} \Phi\big(m(\mathbf{A}_i)\big)\,,$$

where

$$\Phi(x) = \begin{cases} 0\,, & \text{if } x = 0, \\ x \odot \log_g x\,, & \text{if } x > 0, \end{cases}$$

and $\log_g x = g^{-1}\big(\log g(x)\big)$ is the g-logarithmic function (see [6]).

The following theorem states the relationship between this entropy and the entropy introduced by Kolmogorov and Sinaj [2,8].

THEOREM 3.4. *Let a $\perp$-decomposable measure m on the measurable space $(\mathbf{X}, \mathcal{S})$ be of the type (NSA). Then there exists such a probability measure P on $\mathcal{S}$ that $m = g^{-1} \circ P$ where g is the normalized additive generator of t-conorm $\perp$ and*

$$H_m(\mathcal{A}) = g^{-1}\big(H_P(\mathcal{A})\big)\,,$$

for every measurable partition $\mathcal{A}$.

The quantity $H_P(\mathcal{A})$ is an entropy of the partition $\mathcal{A}$ on the probability space $(\mathbf{X}, \mathcal{S}, P)$, i.e.,

$$H_P(\mathcal{A}) = -\sum_{i=1}^{n} \varphi\big(P(\mathbf{A}_i)\big)\,,$$

where

$$\varphi(x) = \begin{cases} 0\,, & \text{if } x = 0\,, \\ x \cdot \log x\,, & \text{if } x > 0\,. \end{cases}$$

Proof. Let $m\colon \mathcal{S} \to [0,1]$ be a $\perp$-decomposable measure of the type (NSA), $\perp$ be a continuous Archimedean t-conorm given by the formula (2) and the function g be a generator of the consistent system of pseudo-arithmetical operations $\{\oplus, \odot, \ominus, \oslash\}$ (and also a normalized generator of the t-conorm $\perp$). By Weber [9] $g \circ m$ is a finite σ-additive measure on $\mathcal{S}$ and, moreover, $(g \circ m)(\mathbf{X}) = g\big(m(\mathbf{X})\big) = 1$, therefore $P = g \circ m$ is a p on $\mathcal{S}$. Hence $m = g^{-1} \circ P$. Further, let $\mathcal{A} = \{\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n\}$ be a measurable partition of $\mathbf{X}$ and $H_m(\mathcal{A}) = -\bigoplus_{i=1}^{n} \Phi\big(m(\mathbf{A}_i)\big)$ be its g-entropy.

Assume that $m(\mathbf{A}_i) > 0$, $i = 1, 2, \dots, n$ $\big(m(B) = 0 \implies P(B) = g\big(m(B)\big) = 0\big)$; using formula (1) and the definition of the function Φ, we can rewrite the previous formula

$$\begin{aligned} H_m(\mathcal{A}) &= -\bigoplus_{i=1}^{n} \Phi\big(m(\mathbf{A}_i)\big) = -\bigoplus_{i=1}^{n} \Phi\Big(g^{-1}\big(P(\mathbf{A}_i)\big)\Big) = \\ &= -\bigoplus_{i=1}^{n} \Big(g^{-1}\big(P(\mathbf{A}_i)\big) \odot \log_g g^{-1}\big(P(\mathbf{A}_i)\big)\Big) = \\ &= -\bigoplus_{i=1}^{n} \Big(g^{-1}\big(P(\mathbf{A}_i)\big) \odot g^{-1}\big(\log g\big(g^{-1}(P(\mathbf{A}_i))\big)\big)\Big) = \\ &= -\bigoplus_{i=1}^{n} \Big(g^{-1}\big(P(\mathbf{A}_i)\big) \cdot \log P(\mathbf{A}_i)\Big) = -g^{-1}\Big(\sum_{i=1}^{n} \big(P(\mathbf{A}_i) \cdot \log P(\mathbf{A}_i)\big)\Big) = \\ &= g^{-1}\Big(-\sum_{i=1}^{n} \big(P(\mathbf{A}_i) \cdot \log P(A_i)\big)\Big) = g^{-1}\big(H_P(\mathcal{A})\big), \end{aligned}$$

where $H_P(\mathcal{A}) = -\sum_{i=1}^{n} \big(P(\mathbf{A}_i) \cdot \log P(\mathbf{A}_i)\big)$ is the entropy of the partition on the probability space $(\mathbf{X}, \mathcal{S}, P)$. □

By Lemma 2.2 [1] it follows that this theorem also holds conversely: If $(\mathbf{X}, \mathcal{S}, P)$ is a probability space and a function g is the generator of the consistent system of pseudo-arithmetical operations (and the normalized generator of the t-conorm $\perp$ given by the formula (2) as well) then $m = g^{-1} \circ P$ is a $\perp$-decomposable measure of the type (NSA) on $\mathcal{S}$.

Moreover,

$$H_P(\mathcal{A}) = g\big(H_m(\mathcal{A})\big)$$

for every measurable partition $\mathcal{A}$.

In consequence of Theorem 3.4 we can easily transform the questions connected with the g-entropy of the partitions on measurable spaces $(\mathbf{X}, \mathcal{S})$ equipped with a $\perp$-decomposable measure m into probability spaces $(\mathbf{X}, \mathcal{S}, P)$ where $P = g \circ m$. Thus we obtain directly the properties of g-entropy by the corresponding g-transformation, for instance:

Let $\mathcal{A} = \{\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n\}$ and $\mathcal{B} = \{\mathbf{B}_1, \mathbf{B}_2, \dots, \mathbf{B}_m\}$ be two measurable partitions of $\mathbf{X}$. Then

$$\mathcal{A} \vee \mathcal{B} = \{\mathbf{A}_i \cap \mathbf{B}_j\,; \quad \mathbf{A}_i \in \mathcal{A},\ \mathbf{B}_j \in \mathcal{B},\ i = 1, 2, \dots, n,\ j = 1, 2, \dots, m\}$$

is a measurable partition of $\mathbf{X}$, too. For the entropy of these partitions on a probability space $(\mathbf{X}, \mathcal{S}, P)$ it holds

$$H_P(\mathcal{A} \vee \mathcal{B}) \le H_P(\mathcal{A}) + H_P(\mathcal{B}).$$

Then

$$g^{-1}\big(H_P(\mathcal{A} \vee \mathcal{B})\big) \le g^{-1}\big(H_P(\mathcal{A}) + H_P(\mathcal{B})\big),$$

since

$$g^{-1}\big(H_P(\mathcal{A}\vee\mathcal{B})\big)\le g^{-1}\Big(g\big(g^{-1}(H_P(\mathcal{A}))\big)+g\big(g^{-1}(H_P(\mathcal{B}))\big)\Big),$$

and hence

$$g^{-1}\big(H_P(\mathcal{A}\vee\mathcal{B})\big)\le g^{-1}\big(H_P(\mathcal{A})\big)\oplus g^{-1}\big(H_P(\mathcal{B})\big).$$

Using Theorem 3.4. we have the following property for the g- entropy

$$H_m(\mathcal{A}\vee\mathcal{B})\le H_m(\mathcal{A})\oplus H_m(\mathcal{B}).$$

If a $\bot$-decomposable measure m on a measurable space $(\mathbf{X},\mathcal{S})$ is of the type (NSP), then the notion of the g-entropy is meaningless. The fact that the measure $g\circ m$ is only pseudo-additive often evokes the defect in the informative sense of the g-entropy (see the following example). □

EXAMPLE 3.5. Let $\mathbf{X}=[0,2)$, $\mathcal{S}=\mathcal{B}\big([0,2)\big)$ and $m\colon\mathcal{S}\to[0,1]$; $m(\mathbf{A})=\min\big\{1,\lambda(\mathbf{A})\big\}$, where λ is the Lebesgue measure on $\mathcal{S}$. Take the generator $g(x)=x$ that generates the system of common arithmetical operations and define the t-conorm $\bot$ by formula (2). Then

$$a\bot b=g^{(-1)}\big(g(a)+g(b)\big)=\min\{a+b,g(1)\}=\min\{a+b,1\},$$

so that t-conorm $\bot$ is identical with the Giles operation S_∞.

It is easy to see that m is the $\bot$-decomposable measure on $\mathcal{S}$ of the type (NSP).

Further consider the measurable partition $\mathcal{A}=\{\mathbf{A}_1,\mathbf{A}_2\}$, where $\mathbf{A}_1=[0,1)$ and $\mathbf{A}_2=[1,2)$. Obviously $m(\mathbf{A}_1)=m(\mathbf{A}_2)=1$. Then the g-entropy of this partition is

$$\begin{aligned}H_m(\mathcal{A})&=-\bigoplus_{i=1}^{2}\Phi\big(m(\mathbf{A}_i)\big)=-\Big(\big(m(\mathbf{A}_1)\odot\log_g m(A_1)\big)\oplus\big(m(A_2)\odot\log_g m(A_2)\big)\Big)\\&=-\Big(\big(1\odot g^{-1}(\log g(1))\big)\oplus\big(1\odot g^{-1}(\log g(1))\big)\Big)\\&=-\big((1\odot 0)\oplus(1\odot 0)\big)=0.\end{aligned}$$

Thus the g-entropy is equal to zero for the non-trivial partition.

Now we describe the relationship between the partition of a measurable space and the $\bot$-decomposable measures of the type (NSP) on this space.

LEMMA 3.6. *A $\bot$-decomposable measure m is of the type (NSP) iff there exists such a partition $\mathcal{A}=\{\mathbf{A}_1,\mathbf{A}_2,\dots,\mathbf{A}_n\}$, $n\in\mathbb{N}$ (either finite or infinite) of the measurable space $(\mathbf{X},\mathcal{S})$ that $\sum\limits_{i=1}^{n}g\big(m(\mathbf{A}_i)\big)>1$.*

P r o o f . Let a $\perp$-decomposable measure m on $\mathcal{S}$ be of the type (NSP), i.e., $(g\circ m)\Big(\bigcup_{k\in\mathbb{N}}\mathbf{B}_k\Big) < \sum_{k\in\mathbb{N}}(g\circ m)\,(\mathbf{B}_k)$ for the same disjoint system $\{\mathbf{B}_k\}_{k\in\mathbb{N}}\subset\mathcal{S}$. From formula (2) and the properties (M2) and (M3) it follows

$$m\Big(\bigcup_{k\in\mathbb{N}}\mathbf{B}_k\Big) = g^{(-1)}\Big(\sum_{k\in\mathbb{N}}(g\circ m)(\mathbf{B}_k)\Big).$$

Hence if $m\Big(\bigcup_{k\in\mathbb{N}}\mathbf{B}_k\Big) < 1$, then $(g\circ m)\Big(\bigcup_{k\in\mathbb{N}}\mathbf{B}_k\Big) = \sum_{k\in\mathbb{N}}(g\circ m)(\mathbf{B}_k)$.

This means that it holds $m\Big(\bigcup_{k\in\mathbb{N}}\mathbf{B}_k\Big) = 1$ and subsequently $\sum_{k\in\mathbb{N}}(g\circ m)(\mathbf{B}_k) > 1$ (the equality is excluded by the introductory condition).

Let us consider the system $\mathcal{A}=\{\mathbf{A}_n\}_{n\in\mathbb{N}}$, where $\mathbf{A}_1 = \Big(\bigcup_{k\in\mathbb{N}}\mathbf{B}_k\Big)^c$ and $\mathbf{A}_{k+1} = \mathbf{B}_k$, $k=1,2,\dots$. It is evident that this system is the partition of a measurable space and

$$\sum_{n\in\mathbb{N}}(g\circ m)(\mathbf{A}_n) \ge \sum_{k\in\mathbb{N}}(g\circ m)(\mathbf{B}_k) > 1\,.$$

The opposite implication will be obvious. □

R e m a r k 3.7. Let a measure m on a measurable space $(\mathbf{X},\mathcal{S})$ be $\oplus$-decomposable where $\oplus$ is a pseudo-addition (see [4], [3]). Then the notion of g-entropy is significative only if $\oplus=\perp$, where $\perp$ is an Archimedean t-conorm. But this case has been studied above.

Now we will introduce the conditional g-entropy on measurable spaces $(\mathbf{X},\mathcal{S})$ with a $\perp$-decomposable measure m. Note that g is a generator of the consistent system of pseudo-arithmetical operations $\{\oplus,\odot,\ominus,\oslash\}$.

DEFINITION 3.8. Let $(\mathbf{X},\mathcal{S})$ be a measurable space, m be a $\perp$-decomposable measure on $\mathcal{S}$ and $\mathcal{A}=\{\mathbf{A}_1,\mathbf{A}_2,\dots,\mathbf{A}_n\}$ be a finite measurable partition of $\mathbf{X}$. If $\mathbf{D}\in\mathcal{S}$, then the conditional g-entropy of the partition $\mathcal{A}$ given by $\mathbf{D}$ is defined via

$$H_m(\mathcal{A}/\mathbf{D}) = -\bigoplus_{i=1}^{n}\Phi\big(m(\mathbf{A}_i/\mathbf{D})\big)\,,$$

where

$$m(\mathbf{A}_i/\mathbf{D}) = \begin{cases} 0\,, & \text{if } m(\mathbf{D})=0\,,\\ m(\mathbf{A}_i\cap\mathbf{D})\,, & \text{if } m(\mathbf{D})>0\,.\end{cases}$$

If $\mathcal{B}=\{\mathbf{B}_1,\mathbf{B}_2,\dots,\mathbf{B}_m\}$ is a finite measurable partition of $\mathbf{X}$, then the conditional g-entropy of $\mathcal{A}$ given by $\mathcal{B}$ is defined via

$$H_m(\mathcal{A}/\mathcal{B}) = \bigoplus_{j=1}^{m}\big(m(\mathbf{B}j)\odot H_m(\mathcal{A}/\mathbf{B}_j)\,.$$

It is easy to see that if $\mathcal{A} = \{\mathbf{A}_1, \dots, \mathbf{A}_n\}$ and $\mathcal{B} = \{\mathbf{B}_1, \dots, \mathbf{B}_m\}$ are measurable partitions on $(\mathbf{X}, \mathcal{S})$, then

$$H_m(\mathcal{A}/\mathcal{B}) = -\bigoplus_{i\in I}\bigoplus_{j\in J} m(\mathbf{A}_i \cap \mathbf{B}_j) \odot \log_g m(\mathbf{A}_i/\mathbf{B}_j)\,,$$

where

$$J = \{j \in \{1,2,\dots,m\}\,;\ m(\mathbf{B}_j) > 0\} \qquad \text{and}$$
$$I = \{i \in \{1,2,\dots,n\}\,;\ m(\mathbf{A}_i/\mathbf{B}_j) > 0\,, \quad j \in J\}\,.$$

R e m a r k 3.9. The same conclusions hold for the conditional g-entropy as it has been shown for the g-entropy in Theorem 3.4. Thus, if the measure m is of the type (NSA) then $H_m(\mathcal{A}/\mathcal{B}) = g^{-1}\big(H_P(\mathcal{A}/\mathcal{B})\big)$, where $H_P(\mathcal{A}/\mathcal{B})$ is the conditional entropy on the probability space $(\mathbf{X}, \mathcal{S}, P)$, $P = g \circ m$. Therefore the properties of the conditional g-entropy can be obtained by the corresponding properties of the conditional entropy on this probability space, for example: from

$$H_P(\mathcal{A} \vee \mathcal{B}) = H_P(\mathcal{A}/\mathcal{B}) + H_P(\mathcal{B})$$

we obtain

$$H_m(\mathcal{A} \vee \mathcal{B}) = H_m(\mathcal{A}/\mathcal{B}) \oplus H_m(\mathcal{B})\,.$$

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