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G–NORMALITY OF A SEQUENTIAL CONVERGENCE ON AN ℓ–GROUP

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Dedicated to Professor J. Jakubík on the occasion of his 70th birthday

ABSTRACT. We show that for an ℓ-group G a compatible convergence $\mathfrak{L}$ is determined by the set $\mathfrak{L}^+$ of all positive sequences which converge to the neutral element of G. A characterization of $\mathfrak{L}^+$ in terms of G–normality and an example of $\mathfrak{L}^+$ which is not a normal subset of the set of all positive sequences of G are given.

Sequential convergences on different types of algebras with and without an order were investigated by J. Novák [8], by J. Jakubík [5] and others (cf. [2]). Some concrete convergences on ℓ-groups were dealt with in [1, 6, 9]. For groups, abelian ℓ-groups and Boolean algebras a compatible convergence of sequences is determined by the set of all neutral sequences, i.e., sequences converging to the neutral element, and their characterization can be found in [3, 4] and [5], respectively.

We show that for an ℓ-group G a compatible convergence $\mathfrak{L}$ is determined by the set $\mathfrak{L}^+$ of all positive neutral sequences and give a characterization of $\mathfrak{L}^+$ in terms of G–normality. We construct an example such that $\mathfrak{L}^+$ fails to be a normal subset of the set $(G^+)^{\mathbb{N}}$ of all positive sequences.

1. Preliminaries

A triple $(G,+,\leq)$ is said to be an ℓ-group, if G is a group with respect to $+$, G is a lattice with respect to $\leq$ and $a \leq b$ implies $g_1 + a + g_2 \leq g_1 + b + g_2$ whenever $a, b, g_1, g_2 \in G$.

The following notation will be used:

$\mathbb{N}$ ($\mathbb{Z}$, $\mathbb{R}$) denotes the set of all positive integers (integers, real numbers);

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0 denotes the neutral element of the group $(G,+)$;
$|a| = -a \vee a$;
$A^{\mathbb{N}}$ denotes the set all of sequences with elements belonging to A;
G^+ denotes the set of all positive elements of G;
$\leq$ denotes the order on G or the pointwise order on $G^{\mathbb{N}}$;
Mon denotes the set of all monotone mappings of $\mathbb{N}$ into $\mathbb{N}$;
u, v, w denotes elements of Mon;
$S \circ u$ denotes a subsequence of the sequence S;
$S(n)$ denotes the n-th term of the sequence S;
$\mathrm{const}(g)$ denotes the constant sequence $\{g\}^{\mathbb{N}}$;
$R + S$ denotes the sequence whose n-th term is $R(n) + S(n)$;
$R \wedge S$, $R \vee S$, $-S$, $|S|$ are defined analogously;
$(x; y)$ denotes an ordered pair.
Instead of $S + \mathrm{const}(g)$ we write simply $S + g$, similarly $S \leq g$, $|S - g|$, etc.

1.1. Definition. Let $(G,+,\leq)$ be an ℓ-group. A set $\mathfrak{L} \subseteq G^{\mathbb{N}} \times G$ is said to be a *convergence* on $(G,+,\leq)$, if the following conditions are satisfied:

(i) $(S;s) \in \mathfrak{L}$ implies $(S \circ u; s) \in \mathfrak{L}$ for every $u \in \mathrm{Mon}$.
(ii) If $(S;s) \in G^{\mathbb{N}} \times G$ and for each $u \in \mathrm{Mon}$ there exists $v \in \mathrm{Mon}$ such that $(S \circ u \circ v; s) \in \mathfrak{L}$, then $(S;s) \in \mathfrak{L}$.
(iii) $(\mathrm{const}(s); s) \in \mathfrak{L}$ whenever $s \in G$.
(iv) $(S;a) \in \mathfrak{L}$ and $(S;b) \in \mathfrak{L}$ imply $a = b$.
(v) $(S;s) \in \mathfrak{L}$ implies $(-S;-s) \in \mathfrak{L}$.
(vi) $(R;r) \in \mathfrak{L}$ and $(S;s) \in \mathfrak{L}$ imply $(R+S; r+s) \in \mathfrak{L}$.
(vii) $(R;r) \in \mathfrak{L}$ and $(S;s) \in \mathfrak{L}$ imply $(R \wedge S; r \wedge s) \in \mathfrak{L}$.
(viii) $(R;r) \in \mathfrak{L}$ and $(S;s) \in \mathfrak{L}$ imply $(R \vee S; r \vee s) \in \mathfrak{L}$.
(ix) If $(R;g) \in \mathfrak{L}$ and $(T;g) \in \mathfrak{L}$ and $R \leq S \leq T$, then $(S;g) \in \mathfrak{L}$.

Conditions (i)–(vi) define a *convergence group* (cf. [8]) or a *FLUSH-convergence* on G (see [7]). The last three conditions concern the compatibility of a convergence with the order on G.

If $(S;s) \in \mathfrak{L}$ then we say that S converges to s and we denote by $\mathrm{Conv}\, G$ the set of all convergences on G.

1.2. Lemma. *If $\mathfrak{L} \in \mathrm{Conv}\, G$ then the following conditions are equivalent:*

$$(S;s) \in \mathfrak{L}\,; \tag{a}$$
$$(|S-s|;0) \in \mathfrak{L}\,; \tag{b}$$
$$\big(|-s+S|\,;0\big) \in \mathfrak{L}\,. \tag{c}$$

Proof. (a) implies (b): Let $(S;s) \in \mathfrak{L}$. By (ii) and (vi) we have $(S-s;0) \in \mathfrak{L}$. From (iii), (v) and (vi) we get $(s-S;0) \in \mathfrak{L}$. Applying (viii) we obtain (b). (b) implies (a): If $\big(|S-s|;0\big) \in \mathfrak{L}$ then by (v) also $\big(-|S-s|;0\big) \in \mathfrak{L}$. But $-|S-s| \le S-s \le |S-s|$ and so with respect to (ix) it is $(S-s;0) \in \mathfrak{L}$. Now, by (iii) and (vi) the condition (a) is fulfilled. The equivalence of (a) and (c) can be shown analogously. □

The previous lemma indicates that a convergence can be given by the set of positive sequences which converge to the neutral element of G. This fact will be described precisely in Theorem 2.3 below.

2. Main results

Let us denote the set $\{S \in (G^+)^{\mathbb{N}}\colon (S;0) \in \mathfrak{L}\}$ by $\mathfrak{L}^+$.

2.1. Definition. The subset P of $(G^+)^{\mathbb{N}}$ is *G–normal* if $g+S-g \in P$ whenever $S \in P$ and $g \in G$.

2.2. Lemma. *Let $\mathfrak{L} \in \operatorname{Conv} G$. Then $\mathfrak{L}^+$ is a convex G–normal subsemigroup of the semigroup $(G^+)^{\mathbb{N}}$ with the following properties :*

- (α) *$\mathfrak{L}^+$ is closed under taking subsequences.*
- (β) *Let $S \in (G^+)^{\mathbb{N}}$. If for every $u \in \mathrm{Mon}$ there exists $v \in \mathrm{Mon}$ such that $S \circ u \circ v \in \mathfrak{L}^+$, then $S \in \mathfrak{L}^+$.*
- (γ) *$\operatorname{const}(s) \in \mathfrak{L}^+$ if and only if $s = 0$.*

The proof is straightforward and it is omitted. □

The next theorem shows that also the converse assertion is true.

2.3. Theorem. *Let P be a convex G–normal subsemigroup of the semigroup $(G^+)^{\mathbb{N}}$ and let (α)–(γ) be fulfilled when $\mathfrak{L}^+$ is replaced by P. Then there exists $\mathfrak{L} \in \operatorname{Conv} G$ such that $\mathfrak{L}^+ = P$. Moreover, if $\mathcal{K} \in \operatorname{Conv} G$ and $\mathcal{K}^+ = P$, then $\mathcal{K} = \mathfrak{L}$.*

The proof of Theorem 2.3 is based on the next three lemmas.

2.4. Lemma. *Let P be a convex subsemigroup of the semigroup $(G^+)^{\mathbb{N}}$ containing $\operatorname{const}(0)$ and let $S \in G^{\mathbb{N}}$, $s \in G$. Then the following conditions are equivalent :*

- (a) *$|S-s| \in P$.*
- (b) *There exist $A, B \in P$ such that $S = A - B + s$.*
- (c) *There exist $A, B \in P$ such that $S = -B + A + s$.*

P r o o f . (a) implies (b) and (c):

Let $|S-s| \in P$. Let us denote $A=(S-s)\vee 0$ and $B=(s-S)\vee 0$. Since $S-s+B=A$, we have $S=A-B+s$. Analogously $A+s-S=B$ gives $S=-B+A+s$. Since $0\leq A=(S-s)\vee 0\leq |S-s|$ and P is a convex set, $A\in P$. Similarly, $0\leq B\leq |s-S| = |S-s|$ gives $B\in P$.

(b) implies (a) (and in the same way also (c) implies (a)):

Let $A,B\in P$ and let $S=A-B+s$. Then $0 \leq |S-s| \leq |A-B| \leq A+B+A$. Again, P is convex and thus $|S-s| \in P$. □

2.5. Lemma. *Let P be a convex subsemigroup of the semigroup $(G^+)^{\mathrm{N}}$ containing* const(0) *and let $S\in G^{\mathrm{N}}$, $s\in G$. Then the following conditions are equivalent:*

(a) $|-s+S| \in P$.
(b) *There exist $A,B\in P$ such that $S=s+A-B$.*
(c) *There exist $A,B\in P$ such that $S=s-B+A$.*

P r o o f . Put $A=(-s+S)\vee 0$ and $B=(-S+s)\vee 0$ and continue as in the previous proof. □

2.6. Lemma. *Let P be a G–normal subset of the set $(G^+)^{\mathrm{N}}$ and let $S\in G^{\mathrm{N}}$, $s\in G$. Then $|S-s| \in P$ if and only if $|-s+S| \in P$.*

P r o o f . The assertion follows easily from the fact that $s+|-s+S| = |S-s|+s$. □

T h e p r o o f o f T h e o r e m 2.3.

Let $\mathfrak{L}=\{(S;s)\in G^{\mathrm{N}}\times G\colon |S-s| \in P\}$. First we prove that $\mathfrak{L}\in \operatorname{Conv} G$. Conditions (i) – (iii) are trivial.

(iv): If $(S;a)\in\mathfrak{L}$ and $(S;b)\in\mathfrak{L}$, then $|S-a| \in P$ and $|S-b| \in P$. Moreover, $0\leq |a-b| = |a-S+S-b| \leq |a-S|+|S-b|+|a-S|$ and so $\operatorname{const}(|a-b|)\in P$. Thus $|a-b| =0$ and $a=b$.

(v): Let $(S;s)\in\mathfrak{L}$. Then $|S-s|\in P$ and by Lemma 2.6 also $|-s+S| \in P$. Therefore $(-S;-s)\in\mathfrak{L}$.

(ix): Let $(R;g)\in\mathfrak{L}$, $(T;g)\in\mathfrak{L}$, $S\in G^{\mathrm{N}}$ and $R\leq S\leq T$. According to Lemma 2.4 there are elements A_R, B_R, A_T, B_T in P such that $R=g+A_R-B_R$ and $T=g-B_T+A_T$. If we denote by $A_S=-g+S+B_R$ and $B_S=B_R$, then $S=g+A_S-B_S$. Since $A_R=-g+R+B_R\leq -g+S+B_R = A_S\leq -g+T+B_R = -B_T+A_T+B_R\leq A_T+B_R$, we obtain $A_S\in P$. By applying Lemmas 2.5 and 2.6 we have $|S-g| \in P$ and thus $(S;g)\in\mathfrak{L}$.

(vi): Let $(R;r)\in\mathfrak{L}$ and $(S;s)\in\mathfrak{L}$. There are A, B, C, D in P such that $R=A-B+r$, $S=s-C+D$. Let us denote by E and F the sequences $(r+s)+C-(r+s)$ and $(r+s)+D-(r+s)$, respectively. Since P is a convex semigroup and $0\leq |R-S-(r+s)| = |A-B-E+F| \leq |A-B|+$

$|-E+F|+|A-B| \le A+B+A \quad + \quad E+F+E \quad + \quad A+B+A$, we have $(R+S;r+s) \in \mathfrak{L}$.

(vii): Let $(R;r) \in \mathfrak{L}$ and $(S;s) \in \mathfrak{L}$. If we denote by $R_1 = R-r$ and $S_1 = S-s$, then $|R_1| \in P$ and $|S_1| \in P$. Since $0 \le |R_1 \vee S_1| \le |R_1|+|S_1|$ and $0 \le |R_1 \wedge S_1| \le |R_1|+|S_1|$, then $|R_1 \vee S_1| \in P$ and $|R_1 \wedge S_1| \in P$. Hence $(R_1 \vee S_1;0) \in \mathfrak{L}$ and $(R_1 \wedge S_1;0) \in \mathfrak{L}$. Now, $R_1 \wedge S_1 = (R-r)\wedge(S-s) \le \big(R+(-r\vee-s)\big)\wedge\big(S+(-r\vee-s)\big) = (R\wedge S)-(r\wedge s) = \big((R\wedge S-r)\vee(R\wedge S)-s\big) \le (R-r)\vee(S-s) = R_1 \vee S_1$ and by the properties (ix), (iii) and (vi) which we have proved, $(R\wedge S; r\wedge s) \in \mathfrak{L}$.

(viii): Let $(R;r) \in \mathfrak{L}$ and $(S;s) \in \mathfrak{L}$. Since $R \vee S = R-(R\wedge S)+S$, by applying (vii), (v) and (vi) we obtain $(R\vee S; r\vee s) \in \mathfrak{L}$.

We have verified that $\mathfrak{L} \in \operatorname{Conv} G$ and assuredly $\mathfrak{L}^+ = P$. To complete the proof let $\mathcal{K} \in \operatorname{Conv} G$ with $\mathcal{K}^+ = P$ and let $(S;s) \in \mathcal{K}$. By Lemma 2.4 $|S-s| \in \mathcal{K}^+$ and since $\mathcal{K}^+ = P = L^+$, $|S-s| \in \mathfrak{L}^+$ and thus $(S;s) \in \mathfrak{L}$. We have $\mathcal{K} \subseteq \mathfrak{L}$. Analogously, $\mathfrak{L} \subseteq \mathcal{K}$. □

3. Convergence envelope

In this section of the paper we construct the least convergence in which the given positive sequences will converge to the neutral element. For groups the construction has been described by F. Zanolin in [10].

Let $A \subseteq (G^+)^{\mathbb{N}}$. We will use the following notation :

$$\delta A = \big\{S \in (G^+)^{\mathbb{N}}:\ \text{there exist } R \in A \text{ and } u \in \operatorname{Mon} \text{ such that } S = R\circ u\big\};$$

$$\langle A\rangle = \big\{S \in (G^+)^{\mathbb{N}}:\ \text{there exist } S_1, S_2, \dots, S_n \in A \text{ and } g_1, g_2, \dots, g_n \in G \text{ such that } S = (g_1+S_1-g_1)+(g_2+S_2-g_2)+\dots+(g_n+S_n-g_n)\big\};$$

$$[A] = \big\{S \in (G^+)^{\mathbb{N}}:\ \text{there exists } T \in A \text{ such that } S \le T\big\};$$

$$A^* = \big\{S \in (G^+)^{\mathbb{N}}:\ \text{for every } u \in \operatorname{Mon} \text{ there is } v \in \operatorname{Mon} \text{ such that } S\circ u\circ v \in A\big\}.$$

3.1. Remark. $[A] = A$ means that A is a convex set, $\delta A = A$ means that A is closed with respect to subsequences and $\langle A\rangle = A$ means that A is G–normal semigroup.

3.2. Lemma. *Let $A \subseteq B \subseteq (G^+)^{\mathbb{N}}$. Then the following assertions are valid:*

(a) $A \subseteq \delta A$. (b) $\delta(\delta A) = \delta A$. (c) $\delta A \subseteq \delta B$.
(d) $A \subseteq \langle A\rangle$. (e) $\langle\langle A\rangle\rangle \subseteq \langle A\rangle$. (f) $\langle A\rangle \subseteq \langle B\rangle$.
(g) $A \subseteq [A]$. (h) $[[A]] = [A]$. (i) $[A] \subseteq [B]$.

(j) $\delta A \subseteq (\delta A)^*$. (k) $(A^*)^* = A^*$. (l) $A^* \subseteq B^*$.
(m) $\delta[A] \subseteq [\delta A]$. (n) $\delta\langle A\rangle \subseteq \langle \delta A\rangle$. (o) $\langle [A]\rangle \subseteq [\langle A\rangle]$.
(p) $\delta(A^*) = A^*$. (r) *If* $\langle A\rangle = \delta A = A$ *then* $\langle A^*\rangle = A^*$.
(s) *If* $[A] = A$ *then* $[A^*] = A^*$.
(t) $\mathrm{const}(s) \in A$ *if and only if* $\mathrm{const}(s) \in A^*$.

Proof. We verify only (e), (j), (k), (r) and (s). The proofs of the other assertions are easy and they are left out.

(e): Let $S \in \langle\langle A\rangle\rangle$. There are $S_1, S_2, \dots, S_n \in \langle A\rangle$ and $g_1, g_2, \dots, g_n \in G$ such that $S = \sum\limits_{j=1}^{n}(g_i + S_i - g_i)$. For each $i \in \{1, 2, \dots, n\}$ there are $S_{i1}, S_{i2}, \dots, S_{im(i)} \in A$ and $g_{i1}, g_{i2}, \dots, g_{im(i)} \in G$ such that $S_i = \sum\limits_{j=1}^{m(i)}(g_{ij} + S_{ij} - g_{ij})$. Therefore
$S = \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m(i)}\big((g_i + g_{ij}) + S_{ij} - (g_i + g_{ij})\big)$ and thus $S \in \langle A\rangle$. We obtain $\langle\langle A\rangle\rangle \subseteq \langle A\rangle$. (d) implies the converse inclusion.

(j): If $S \in \delta A$ then there are $R \in A$ and $u \in \mathrm{Mon}$ such that $S = R \circ u$. Suppose $S \notin (\delta A)^*$. Then there exists $v \in \mathrm{Mon}$ such that for $w \in \mathrm{Mon}$ we have $S \circ v \circ w \notin \delta A$. But in this case $R \circ u \circ v \notin \delta A$, a contradiction with $R \in A$.

(k): First we prove that $(A^*)^* \subseteq A^*$. Let $S \in (A^*)^*$ and $u \in \mathrm{Mon}$. There exists $v_1 \in \mathrm{Mon}$ such that $S \circ u \circ v_1 \in A^*$. Therefore for $w \in \mathrm{Mon}$ there is some $v_2 \in \mathrm{Mon}$ such that $S \circ u \circ v_1 \circ w \circ v_2 \in A$. Let us take $w \in \mathrm{Mon}$ and a correspondent $v_2 \in \mathrm{Mon}$ and pose $v = v_1 \circ w \circ v_2$. Then $S \circ u \circ v \in A$ and $S \in A^*$. Now, contrariwise, suppose that $S \in A^*$ and $S \notin (A^*)^*$. There exists $u_1 \in \mathrm{Mon}$ such that $S \circ u_1 \circ v \notin A^*$ for $v \in \mathrm{Mon}$. Hence also $S \circ u_1 \notin A^*$. Then there is $u_2 \in \mathrm{Mon}$ such that $S \circ u_1 \circ u_2 \circ v \notin A$ whenever $v \in \mathrm{Mon}$. Since $S \in A^*$, for $u_1 \circ u_2$ there exists $v \in \mathrm{Mon}$ with $S \circ u_1 \circ u_2 \circ v \in A$, a contradiction.

(r): Let $S \in \langle A^*\rangle$. There are $S_1, S_2, \dots, S_n \in A^*$ and $g_1, g_2, \dots, g_n \in G$ such that $S = \sum\limits_{i=1}^{n}(g_i + S_i - g_i)$. We will show that for each $u \in \mathrm{Mon}$ there exists $v \in \mathrm{Mon}$ such that $S \circ u \circ v \in A$. Because $S_1 \in A^*$, there is $v_1 \in \mathrm{Mon}$ with $S_1 \circ u \circ v_1 \in A$. Since $S \in A^*$, there is $v_2 \in \mathrm{Mon}$ with $S_2 \circ u \circ v_1 \circ v_2 \in A$. Consequently, there are $v_3, \dots, v_n \in \mathrm{Mon}$ such that $S_3 \circ u \circ v_1 \circ v_2 \circ v_3 \in A, \dots, S_n \circ u \circ v_1 \circ \dots \circ v_n \in A$. Denote $v_1 \circ v_2 \circ \dots \circ v_n$ by v. We have $\delta A = A$ and so $S \circ u \circ v = \sum\limits_{i=1}^{n}(g_i + S_i \circ u \circ v - g_i) \in \langle A\rangle = A$. The converse inclusion is implied by (d).

(s): If $S \in [A^*]$, then there exists $T \in A^*$ such that $S \le T$. For any $u \in \mathrm{Mon}$ there exists $v \in \mathrm{Mon}$ with $T \circ u \circ v \in A$. Now, $S \circ u \circ v \le T \circ u \circ v \in A$ and thus $S \circ u \circ v \in [A] = A$. The converse inclusion is implied by (g). □

3.3. Lemma. *If $\emptyset \neq A \subseteq (G^+)^{\mathrm{N}}$ then $\left[\langle \delta A\rangle\right]^*$ is a convex G–normal subsemigroup of the semigroup $(G^+)^{\mathrm{N}}$ containing* $\mathrm{const}(0)$ *and having the properties* (α) *and* (β).

Proof. (h) and (s) of Lemma 3.2 imply the convexity and by the consecutive application of (o), (e), (m), (n), (b), and (r) we obtain that $\left[\langle \delta A\rangle\right]^*$ is a G–normal subsemigroup of the semigroup $(G^+)^{\mathrm{N}}$. From (a), (j), (l) and (i) one can derive that $\mathrm{const}(0) \in \left[\langle \delta A\rangle\right]^*$. Finally, (p) implies (α) and (k) implies (β). □

3.4. Theorem. *Let $\emptyset \neq A \subseteq (G^+)^{\mathrm{N}}$. If $\left[\langle \delta A\rangle\right]$ does not contain* $\mathrm{const}(s)$ *for any $s \neq 0$, then $\mathfrak{L} = \left\{(S;s)\colon |S-s| \in \left[\langle \delta A\rangle\right]^*\right\}$ is the smallest element of* $\mathrm{Conv}\, G$ *such that $A \subseteq \mathfrak{L}^+$. In the opposite case there exists no such convergence on G.*

Proof. If $\left[\langle \delta A\rangle\right]$ does not contain $\mathrm{const}(s)$ with $s \neq 0$, then by Lemma 3.3 $\left[\langle \delta A\rangle\right]^*$ is a convex G–normal subsemigroup of the semigroup $(G^+)^{\mathrm{N}}$ with (α)–(γ). By the consecutive application of the assertions of Lemma 2.2 and (c), (b), (f), (e), (i), (h), (l) and (k), $\left[\langle \delta A\rangle\right]^*$ defines the smallest element of $\mathrm{Conv}\, G$ such that each sequence of A converges to the neutral element of G. Finally, when $\mathrm{const}(s) \in \left[\langle \delta A\rangle\right]$ for some $s \neq 0$ and there is $\mathcal{K} \in \mathrm{Conv}\, G$ with $A \subseteq \mathcal{K}^+$, in the same way as before we obtain $\left[\langle \delta A\rangle\right]^* \subseteq \mathcal{K}^+$. Therefore (t) gives $\mathrm{const}(s) \in \mathcal{K}^+$ and thus $s = 0$, a contradiction. □

4. Normality and G–normality

In this section we show that the conditions of the normality and of the G–normality are not equivalent, not even if all other defining properties of a convergence are fulfilled.

The well-known condition of the normality (applied in our case: if $S \in A \subseteq (G^+)^{\mathrm{N}}$ and $\{g_n\} \in G^{\mathrm{N}}$ then $\{g_n\} + S - \{g_n\} \in A$) implies the condition of G–normality (if $S \in A \subseteq (G^+)^{\mathrm{N}}$ and $g \in G$ then also $g + S - g \in A$). Next we construct an example of an ℓ-group G which possesses a convergence $\mathfrak{L}$ such that the corresponding $\mathfrak{L}^+$ is not a normal subset of $(G^+)^{\mathrm{N}}$.

4.1. Example. Let $G = \mathbb{Z} \times \mathbb{R}^{\mathbb{Z}}$ (the set of all ordered pairs where the first member is an integer and the second one is a function from $\mathbb{Z}$ to $\mathbb{R}$). Put $\big(k; f(z)\big) \oplus \big(m; g(z)\big) = \big(k+m;\ f(z+m) + g(z)\big)$ whenever $k, m \in \mathbb{Z}$ and $f(z)$, $g(z) \in \mathbb{R}^{\mathbb{Z}}$. Then $(G, \oplus)$ is a non-abelian group with $\big(0; \mathrm{const}(0)\big)$ as a neutral element and $\ominus\big(k; f(z)\big) = \big(-k; -f(z-k)\big)$. We define an order $\trianglelefteq$ on G in the following way: $\big(k; f(z)\big) \trianglelefteq \big(m; g(z)\big)$ if $k < m$ or if $k = m$ and $f(z) \leq g(z)$ whenever $z \in \mathbb{Z}$. Then $(G, \oplus, \trianglelefteq)$ is an ℓ-group.

For a sequence $\{f_n(z)\}_{n=1}^{\infty}$ of elements of $\mathbb{R}^{\mathbb{Z}}$ we denote by $c(f_n)$ a set of all integers z for which $\{n \in \mathbb{N}\colon f_n(z) \neq 0\}$ is infinite; $c(f_n)$ will be called the *carrier* of f_n.

Let A be a subset of $G^{\mathbb{N}}$ such that $(k_n; f_n(z))_{n=1}^{\infty} \in A$ if and only if there exists $n_0 \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ the relation $n \geq n_0$ implies $k_n = 0$ and $c(f_n)$ is finite and for each $z \in c(f_n)$ the real sequence $(f_n(z))_{n=1}^{\infty}$ converges to zero in the sense of usual metric convergence on $\mathbb{R}$.

4.2. LEMMA. *We have $\delta A = \langle A \rangle = [A] = A$ and A contains only one constant sequence, namely,* $\mathrm{const}\,(0\,;\mathrm{const}\,(0))$.

The proof of the lemma is straightforward and we omit it. □

By Theorem 3.4 the set A is a positive cone of a convergence on the ℓ-group G and so A is G–normal.

Now, it suffices to take for example a sequence $\{d_n\}_{n=1}^{\infty}$, where d_n denotes the number of zeros in the decimal representation of n and functions defined by

$$f_n(z) = \begin{cases} \frac{1}{n}, & \text{if } z = 0, \\ 0, & \text{if } z \neq 0. \end{cases}$$

It is easy to see that $\{(0; f_n(z))\}_{n=1}^{\infty}$ is a sequence from A. But $\{(d_n; \mathrm{const}\,(0))\}_{n=1}^{\infty} \oplus \{(0; f_n(z))\}_{n=1}^{\infty} \ominus \{(d_n; \mathrm{const}\,(0))\}_{n=1}^{\infty} = \{(0; f_n(z - d_n))\}_{n=1}^{\infty}$. If we denote $g_n(z) = f_n(z - d_n)$, then

$$g_n = \begin{cases} \frac{1}{n}, & \text{if } z = d_n, \\ 0, & \text{if } z \neq d_n. \end{cases}$$

Therefore the carrier of the sequence g_n is infinite and thus $\{(0; g_n(z))\}_{n=1}^{\infty}$ does not belong to A.

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