Tatra Mountains Math. Publ. **5** (1995), 217–228

TATRA MOUNTAINS
Mathematical Publications

CONSTRUCTION AND AFFINE COMPLETENESS OF PRINCIPAL P–ALGEBRAS

MIROSLAV HAVIAR

Dedicated to Professor J. Jakubík on the occasion of his 70th birthday

ABSTRACT. In this paper we introduce the class of principal p-algebras which contains all quasi-modular p-algebras having a smallest dense element. We present a simple triple construction of principal p-algebras which works with pairs of elements only. We show that a principal p-algebra is locally affine complete (in the sense of [P 1972]) iff it is a Boolean algebra, and consequently, that finite Boolean algebras are the only finite affine complete principal p-algebras.

1. Introduction

The study of pseudocomplemented lattices or shortly p-algebras has a long tradition in lattice theory (see, e.g., [G 1971] or a survey paper [Ka 1980]). The best known examples of p-algebras are the Boolean and Stone algebras. In [Ka-Me 83] the class of quasi-modular p-algebras was introduced and a triple construction of all its members was presented. In this paper we introduce the class of principal p-algebras which contains all quasi-modular p-algebras having a smallest dense element, i.e., it also generalizes the Boolean algebras.

We first present a simple triple construction of any member of the class of principal p-algebras (Theorems 3.2 and 3.5), which for this class means a simplification of the general construction presented in [Ka-Me 1983]. Our construction works with pairs of elements only. In fact, it extends to a larger class the triple construction of P. Köhler [Kö 1978] for distributive p-algebras having a smallest dense element. We show that there exists a one-to-one correspondence between the principal p-algebras and so-called principal triples (Theorem 3.8, Proposition 3.9), i.e., that the principal triples uniquely represent the principal p-algebras.

In the second part of the paper we show that if a principal p-algebra L is (locally) affine complete, then its dense filter $D(L)$ is a (locally) affine complete

AMS Subject Classification (1991): 06D15, 06D30.
Key words: compatible function, (locally) affine complete algebra, (principal) p-algebra.

lattice generalizing a result from [Ha 1992] concerning distributive p-algebras having a smallest dense element (Theorem 4.3). As consequences we get that a principal p-algebra is locally affine complete (in the sense of [P 1972]) iff it is a Boolean algebra, and that finite Boolean algebras are the only finite affine complete principal p-algebras.

2. Preliminaries

A *pseudocomplemented lattice* (or *p-algebra*) is an algebra $(L; \vee, \wedge, ^*, 0, 1)$ where $(L; \vee, \wedge, 0, 1)$ is a bounded lattice and $*$ is the unary operation of pseudocomplementation, i.e., $x \leq a^*$ iff $x \wedge a = 0$. We say that a p-algebra $(L; \vee, \wedge, ^*, 0, 1)$ is *distributive* (*modular*) if the corresponding lattice $(L; \vee, \wedge, 0, 1)$ is distributive (modular).

In a p-algebra L, an element $a \in L$ is called *closed* if $a = a^{**}$. The set $B(L) = \{a \in L;\ a = a^{**}\}$ of all closed elements of L forms a Boolean algebra $\big(B(L); \triangledown, \wedge, ^*, 0, 1\big)$ where the join $\triangledown$ is defined by the rule

$$a \triangledown b = (a^* \wedge b^*)^* = (a \vee b)^{**}.$$

An element $d \in L$ is said to be *dense* if $d^* = 0$. The set $D(L) = \{d \in L;\ d^* = 0\}$ of all dense elements of L is a filter of L.

An important role in a p-algebra L is played by the *Glivenko congruence* Φ defined by $x \equiv y(\Phi)$ iff $x^* = y^*$ for all $x, y \in L$. Obviously, $B(L) \cong L/\Phi$.

We recall that a *Stone algebra* is a distributive p-algebra satisfying the *Stone identity*

$$x^* \vee x^{**} = 1\,. \tag{S}$$

In general, elements x of p-algebras satisfying (S) are called *Stone elements* and p-algebras satisfying (S) are called *S-algebras.* One can show that in an S-algebra L, $B(L)$ is a subalgebra of L, hence $(x \wedge y)^* = x^* \vee y^*$.

Besides distributive and modular p-algebras, a larger variety of *quasi-modular p-algebras* is interesting to investigate (see [Ka-Me 1983]). This subvariety of p-algebras is defined by the identity

$$\big((x \wedge y) \vee z^{**}\big) \wedge x = (x \wedge y) \vee (z^{**} \wedge x)\,.$$

It is known (see [Ka-Me 1983; 6.1]) that quasi-modular p-algebras satisfy the identity

$$x = x^{**} \wedge (x \vee x^*)\,,$$

which can be obviously weakened to the equation $x = x^{**} \wedge (x \vee d)$ in the case the filter $D(L)$ is principal and $D(L) = [d)$. In the next definition, p-algebras abstracting quasi-modular p-algebras with principal filter $D(L)$ are introduced:

2.1 Definition. A p-algebra $(L; \vee, \wedge, ^*, 0, 1)$ is called a *principal p-algebra*, if it satisfies the following conditions:

(i) the filter $D(L)$ is principal, i.e., there exists $d \in L$ such that $D(L) = [d)$;

(ii) the element d is distributive, i.e., $(x \wedge y) \vee d = (x \vee d) \wedge (y \vee d)$ for all $x, y \in L$;

(iii) $x = x^{**} \wedge (x \vee d)$ for any $x \in L$.

If L moreover satisfies the identity (S), $x^* \vee x^{**} = 1$, then it will be called a *principal S-algebra.*

2.2 Example. The following algebras are examples of principal p-algebras (S-algebras):

1. Any Boolean algebra is a principal S-algebra ($a^* := a'$).
2. Any finite distributive lattice is a principal p-algebra.
3. Heyting algebras $(H; \vee, \wedge, *, 0, 1)$ having a smallest element d such that $d * 0 = 0$ are distributive principal p-algebras.
4. A p-algebra depicted in Figure 1 is a non-distributive modular principal S-algebra.

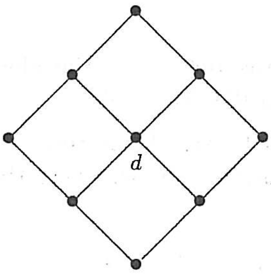

FIGURE 1

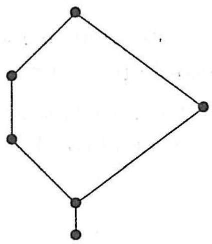

FIGURE 2

5. A non-modular principal S-algebra is depicted in Figure 2. One can easily verify that it is quasi-modular.
6. All quasi-modular p-algebras of which filter $D(L)$ is principal are principal p-algebras. We have already mentioned that these algebras satisfy (iii) above. In [Mu-En 1986; Theorem 5] it is shown that the filter $D(L)$ of a quasi-modular p-algebra L is a neutral element in the lattice $F(L)$ of all filters of L. So if $D(L) = [d)$, then for all $x, y \in L$, $([x) \vee [y)) \wedge [d) = ([x) \wedge [d)) \vee ([y) \wedge [d))$ holds in $F(L)$. Consequently, $(x \wedge y) \vee d = (x \vee d) \wedge (y \vee d)$ for all $x, y \in L$, thus (ii) of Definition 2.1 is satisfied too.

7. Take an arbitrary lattice L with 1 and add a new zero 0. Then $K = L\cup\{0\}$ is a principal S-algebra.

8. A p-algebra L in Figure 3 is a principal p-algebra which is not quasi-modular, since $\left[(a \wedge f) \vee a^*\right] \wedge a \neq (a \wedge f) \vee (a^* \wedge a)$. This algebra will be used in Example 3.6 to illustrate the triple construction of the principal p-algebras.

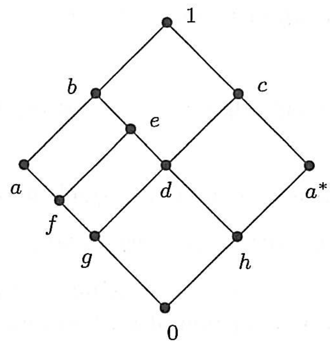

FIGURE 3.

Finally, we note that principal p-algebras belong to the class of so-called filter-decomposable p-algebras introduced in [Ka-Me 1983].

Now let us turn to (local) affine completeness. First recall that an n-ary function f on an algebra A is called *compatible* if it preserves the congruences of A, i.e.,

$$x_i \equiv y_i(\theta),\ (x_i, y_i \in A),\ i = 1, \dots, n$$

yields

$$f(x_1, \dots, x_n) \equiv f(y_1, \dots, y_n)(\theta)$$

for any congruence θ of A. Clearly, every polynomial function of A is compatible. An algebra A is called *affine complete* if the polynomial functions of A are the only compatible functions ([W 1971]). Further, an algebra A is said to be *locally affine complete* if any finite partial function in $A^n \to A$ (i.e., a function whose domain is a finite subset of A^n) which is compatible (where defined) can be interpolated by a polynomial of A (see, e.g., [P 1972] or [Kaa-P 1987]; note that in [Sz 1986] or [Kaa-Ma-S 1985] the notion "locally affine complete" has a different meaning).

3. Construction

The principal p-algebras, introduced in the previous section, satisfy the equation $x = x^{**} \wedge (x \vee d)$ for any element x. In general, p-algebras L having the property that for every $x \in L$ there exists $d \in D(L)$ such that

$$x = x^{**} \wedge d$$

form the largest known class of p-algebras that can be constructed by a triple construction — see [Ka-Me 1983].

The construction in [Ka-Me 1983] works with pairs of elements $(B(L) \times D(L)/a'\overline{\varphi})_{a'\overline{\varphi} \in \operatorname{Con} D(L)}$ or $B(L) \times F\big(D(L)\big)$, i.e., it uses congruence classes or filters. In this section we present for the class of principal p-algebras a simple variant of this construction using only pairs of elements of $B(L) \times D(L)$.

The triples associated with the principal p-algebras can abstractly be characterized as follows:

3.1 Definition. An (abstract) *principal triple* is (B, D, φ), where

(i) $B = (B; \vee, \wedge, ', 0, 1)$ is a Boolean algebra;

(ii) $D = (D; \vee, \wedge, 0, 1)$ is a bounded lattice;

(iii) φ is a $(0,1)$-meet-homomorphism from B into D.

The construction presented in the next theorem will be called a *principal construction.*

3.2 Theorem. *Let (B, D, φ) be a principal triple. Then*

$$L = \big\{(x, y) \in B \times D;\ y \le \varphi(x)\big\}$$

is a principal p-algebra if one defines

$$\begin{aligned}
(x_1, y_1) \vee (x_2, y_2) &= (x_1 \vee x_2, y_1 \vee y_2)\,,\\
(x_1, y_1) \wedge (x_2, y_2) &= (x_1 \wedge x_2, y_1 \wedge y_2)\,,\\
(x, y)^* &= \big(x', \varphi(x')\big)\,,\\
1_L &= (1_B, 1_D)\,,\\
0_L &= (0_B, 0_D)\,.
\end{aligned}$$

Moreover, $B(L) \cong B$ and $D(L) \cong D$.

P r o o f. Let (x_1, y_1), $(x_2, y_2) \in L$. Since φ is a $(0,1)$-meet-homomorphism, we have $y_1 \wedge y_2 \le \varphi(x_1) \wedge \varphi(x_2) = \varphi(x_1 \wedge x_2)$, $0_D = \varphi(0_B)$ and $1_D = \varphi(1_B)$. Further, for $i = 1, 2$, $y_i \le \varphi(x_i) \le \varphi(x_1 \vee x_2)$, since φ is order-preserving. Hence $y_1 \vee y_2 \le \varphi(x_1 \vee x_2)$ and L is a bounded sublattice of $B \times D$. Now assume that $(x, y) \wedge (z, w) = (0_B, 0_D)$, $(z, w) \in L$. Then $x \wedge z = 0_B$, so $z \le x'$

and $w \le \varphi(z) \le \varphi(x')$, i.e., $(z,w) \le \big(x', \varphi(x')\big)$. Obviously, $(x,y) \wedge \big(x', \varphi(x')\big) \le \big(x \wedge x', \varphi(x) \wedge \varphi(x')\big) = (0_B, 0_D)$. Hence $\big(x', \varphi(x')\big)$ is the pseudocomplement of (x,y) and L is a p-algebra.

To show that the filter $D(L)$ is principal, note first that

$$D(L) = \big\{(x,y) \in L;\ (x,y)^* = (0_B, 0_D)\big\} = \big\{(x,y) \in L;\ \big(x', \varphi(x')\big) = (0_B, 0_D)\big\} = \\ = \big\{(1_B, y);\ y \in D\big\} \cong D .$$

Now clearly, $d_L = (1_B, 0_D)$ is the smallest dense element of L. Further, for any (x,y), $(z,w) \in L$

$$\big((x,y) \wedge (z,w)\big) \vee (1_B, 0_D) = \big((x \wedge z) \vee 1_B\,,\ (y \wedge w) \vee 0_D\big) = (1_B, y \wedge w) = \\ = \big((x,y) \vee (1_B, 0_D)\big) \wedge \big((z,w) \vee (1_B, 0_D)\big),$$

thus d_L is a distributive element. Finally, for any $(x,y) \in L$,

$$(x,y)^{**} \wedge \big((x,y) \vee (1_B, 0_D)\big) = \big(x'', \varphi(x'')\big) \wedge (x \vee 1_B, y \vee 0_D) = \\ = \big(x, \varphi(x)\big) \wedge (1_B, y) = (x,y) .$$

Hence L satisfies also the condition (iii) of Definition 2.1, thus L is a principal p-algebra. It remains to show that $B(L) \cong B$. But

$$B(L) = \big\{(x,y) \in L;\ (x,y) = (x,y)^{**}\big\} = \big\{(x,y) \in L;\ (x,y) = \big(x'', \varphi(x'')\big)\big\} = \\ = \big\{(x,y) \in B \times D;\ y = \varphi(x)\big\} = \big\{\big(x, \varphi(x)\big);\ x \in B\big\},$$

which is evidently isomorphic to B. The proof is complete. □

3.3 Proposition ([O 1935]). *Let L be any lattice and $d \in L$. The following conditions are equivalent:*

(1) *The element d is distributive;*

(2) *The mapping*

$$\varphi : x \to x \vee d \quad (x \in L)$$

is a homomorphism of the lattice L onto the filter $[d)$;

(3) *The binary relation θ_d defined on L by the rule*

$$x \equiv y(\theta_d) \quad \textit{iff} \quad x \vee d = y \vee d$$

is a congruence of L.

3.4 Definition. Let L be a principal p-algebra with a smallest dense element d_L. By a *triple associated to* L we mean the triple $\big(B(L), D(L), \varphi(L)\big)$, where

(i) $B(L) = \big(B(L), \triangledown, \wedge, ^*, 0_L, 1_L\big)$ is the Boolean algebra of all closed elements of L;

(ii) $D(L) = \big(D(L), \vee, \wedge, d_L, 1_L\big)$ is the filter of all dense elements of L;

(iii) $\varphi(L) : B(L) \to D(L)$ is a mapping defined by the rule $\varphi(x) = x \vee d_L$.

Proposition 3.3 guarantees that the mapping $\varphi(L)$ defined above is a meet-homomorphism from $B(L)$ into $D(L)$, hence the triple associated to a principal p-algebra is principal.

Now we shall show that every principal p-algebra can be constructed from its associated triple by the principal construction.

3.5 THEOREM. *Let L be a principal p-algebra. Let $\big(B(L), D(L), \varphi(L)\big)$ be its associated triple and let L' be the principal p-algebra constructed from $\big(B(L), D(L), \varphi(L)\big)$ by the principal construction. Then L and L' are isomorphic.*

Proof. We shall show that the mapping $f : L \to L'$ defined by

$$f(a) = (a^{**},\ a \vee d_L)$$

is the desired isomorphism. Obviously, $f(a) \in L'$ because $a \vee d_L \leq \varphi(a^{**}) = a^{**} \vee d_L$. Further, since d_L is a distributive element

$$\begin{aligned} f(a \wedge b) &= \big((a \wedge b)^{**}, (a \wedge b) \vee d_L\big) = \big(a^{**} \wedge b^{**}, (a \vee d_L) \wedge (b \vee d_L)\big) = \\ &= (a^{**}, a \vee d_L) \wedge (b^{**}, b \vee d_L) = f(a) \wedge f(b)\,, \end{aligned}$$

thus f is a meet-homomorphism. By the definition of the join $\triangledown$ in $B(L)$ we have

$$f(a \vee b) = \big((a \vee b)^{**}, a \vee b \vee d_L\big) = (a^{**} \triangledown b^{**}, a \vee b \vee d_L) = f(a) \vee f(b)\,.$$

Finally,

$$\begin{aligned} &f(a^*) = (a^{***}, a^* \vee d_L) = \big(a^*, \varphi(a^*)\big) = f(a)^*, \\ &f(0_L) = (0_L, d_L) = 0_{L'} \qquad \text{and} \qquad f(1_L) = (1_L, 1_L) = 1_{L'}. \end{aligned}$$

Hence f is a $(0,1)$-lattice homomorphism from L into L'. To prove the injectivity of f, suppose that $f(a_1) = f(a_2)$. Then $a_1^{**} = a_2^{**}$ and $a_1 \vee d_L = a_2 \vee d_L$, thus $a_1^{**} \wedge (a_1 \vee d_L) = a_2^{**} \wedge (a_2 \vee d_L)$. Hence by the (iii) of Definition 1, $a_1 = a_2$. It remains to show that f is an onto map. Let $(u,v) \in L'$, i.e., $u \in B(L)$, $v \in D(L)$, $v \leq \varphi(u)$. Put $a = u \wedge v$. Then again by the distributivity of d_L and standard rules of computation we get

$$\begin{aligned} f(a) &= \big((u \wedge v)^{**}, (u \wedge v) \vee d_L\big) = \big(u^{**} \wedge v^{**}, (u \vee d_L) \wedge (v \vee d_L)\big) = \\ &= \big(u \wedge 1_L, (u \vee d_L) \wedge v\big) = \big(u, \varphi(u) \wedge v\big) = (u,v)\,, \end{aligned}$$

and the proof is finished. □

3.6 EXAMPLE. Take the non-quasi-modular principal p-algebra L in Figure 3. Obviously, the triple associated to L is $\big(B(L), D(L), \varphi(L)\big)$, where

$$\begin{aligned} B(L) &= \{0, a, a^*, 1\}\,, \\ D(L) &= [d) \end{aligned}$$

and

$$\varphi(L) = \{(0,d),\ (a,b),\ (a^*,c),\ (1,1)\}.$$

By the principal construction we get the principal p-algebra

$$L = \{(0,d),\ (a^*,d),\ (a^*,c),\ (a,d),\ (a,e),\ (a,b),\ (1,d),\ (1,c),\ (1,e),\ (1,b),\ (1,1)\}$$

which is isomorphic to L by Theorem 3.5.

Finally, we shall show that the principal p-algebras are represented by the principal triples uniquely.

3.7 Definition. By an *isomorphism* of the principal triples (B_1, D_1, φ_1) and (B_2, D_2, φ_2) we shall call a pair of maps (f, g) such that f is an isomorphism of B_1 and B_2, g is an isomorphism of D_1 and D_2 and the diagram

$$\begin{array}{ccc} B_1 & \xrightarrow{\varphi_1} & D_1 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B_2 & \xrightarrow{\varphi_2} & D_2 \end{array}$$

is commutative.

3.8 Theorem. *Two principal p-algebras are isomorphic if and only if their associated principal triples are isomorphic.*

Proof. If $L_1 \cong L_2$ under a p-algebra-isomorphism h, then obviously the restrictions $h \restriction B(L_1)$ and $h \restriction D(L_1)$ form the required pair of isomorphisms. Conversely, let (B_1, D_1, φ_1) and (B_2, D_2, φ_2) be the triples associated to principal p-algebras L_1 and L_2, and let these triples be isomorphic under a pair (f, g). Let L_1' and L_2', respectively, denote the principal p-algebras constructed from these triples by the principal construction. Define a map $h\colon L_1' \to L_2'$ by the rule $h\big((x,y)\big) = \big(f(x), g(y)\big)$. Obviously, h is a bijection as well as a lattice homomorphism because f and g are so. Finally,

$$\begin{aligned} h\big((x,y)^*\big) &= h\big((x^*, \varphi_1(x^*))\big) = \big(f(x^*), g(\varphi_1(x^*))\big) = \big(f(x^*), \varphi_2(f(x^*))\big) = \\ &= \big(f(x)^*, \varphi_2(f(x)^*)\big) = \big(f(x), g(y)\big)^* = h\big((x,y)\big)^*. \end{aligned}$$

The rest of the proof follows from Theorem 3.5. □

It is important to answer a question whether or not a principal p-algebra can be constructed by the présented construction from two non-isomorphic triples as well. We shall show that this is not the case. Thus a one-to-one correspondence between principal p-algebras and principal triples can be stated.

3.9 Proposition. *Let (B, D, φ) be a principal triple and let L be a principal p-algebra constructed from (B, D, φ) by the principal construction. Then*

$$\big(B(L), D(L), \varphi(L)\big) \cong (B, D, \varphi) .$$

Proof. In Theorem 3.2 we have shown that the maps $f\colon B(L) \to B$, $f\big((x, y)\big) = x$ and $g\colon D(L) \to D$, $g\big((x, y)\big) = y$ are isomorphisms. So it remains to show that the diagram

$$\begin{array}{ccc} B(L) & \xrightarrow{\varphi(L)} & D(L) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow{\varphi} & D \end{array}$$

is commutative. Obviously, any $(x, y) \in B(L)$ is of the form $\big(x, \varphi(x)\big)$ for some $x \in B$. So $g\Big(\varphi(L)\big((x, \varphi(x))\big)\Big) = g\big((x \vee 1_B, \varphi(x) \vee 0_D)\big) = g\big((1_B, \varphi(x))\big) = \varphi(x) = \varphi\Big(f\big((x, \varphi(x))\big)\Big)$. The proof is complete. □

4. On (local) affine completeness

A basic result of G. Grätzer says that every Boolean algebra is affine complete ([G 1962]). In ([P 1979]) it was shown that a variety is arithmetical iff it is locally affine complete. This means that every Boolean algebra is also locally affine complete.

The following propositions can be considered as a part of folklore:

4.1 Proposition. *A lattice L is locally affine complete iff $|L| = 1$.*

Proof. Let L be locally affine complete and let $a, b \in L$, $a < b$. The function $f = \big\{(a, b), (b, a)\big\}$ is a finite partial compatible function on L, thus by hypothesis it can be interpolated on $\{a, b\}$ by a polynomial of L which obviously is an isotone function. But we have $f(a) = b$, $f(b) = a$, a contradiction. □

4.2 Proposition. *A finite lattice L is affine complete iff $|L| = 1$.*

Proof. If $[a, b]$ is a (two-element) Boolean interval in a finite lattice L then the function $f(x) = \big[(x \vee a) \wedge b\big]'$ is compatible, but not isotone. □

In [Ha 1992] it was shown that if a distributive p-algebra L with a smallest dense element is affine complete, then its dense filter $D(L)$ is an affine complete distributive lattice, i.e., no proper interval of $D(L)$ is a Boolean algebra. The method of the proof was based on a special canonical form of the polynomials of a distributive p-algebra found by the author. Using a different approach, we now generalize this result to the class of principal p-algebras:

4.3 Theorem. *Let L be a principal p-algebra. If L is (locally) affine complete then $D(L)$ is a (locally) affine complete lattice.*

Proof. Let L be affine complete and $f'\colon D(L)^n \to D(L)$ be an n-ary compatible function of the lattice $D(L)$ $(n \geq 1)$. Define a function $f\colon L^n \to L$ by the rule

$$f(x_1,\dots,x_n) = f'(x_1 \vee d,\dots,x_n \vee d).$$

Obviously, $f \restriction D(L)^n = f'$ and f is a compatible function of the algebra L. Indeed, if θ is a congruence of L and $x_i \equiv y_i\,(\theta)$, $x_i, y_i \in L$, $i = 1,\dots,n$, then $x_i \vee d \equiv y_i \vee d\,(\theta \restriction D(L))$, so $f'(x_1 \vee d,\dots,x_n \vee d) \equiv f'(y_1 \vee d,\dots,y_n \vee d)\,(\theta \restriction D(L))$ since f' is compatible on $D(L)$. Hence, $f(x_1,\dots,x_n) \equiv f(y_1,\dots,y_n)\,(\theta)$ and f is compatible. Thus by the assumption, there exists a polynomial $p(x_1,\dots,x_n)$ of L representing f on L. Using the formulas $(x \wedge y)^* = x^* \bigtriangledown y^*$, $(x \vee y)^* = (x^{**} \bigtriangledown y^{**})^* = x^* \wedge y^*$, we can rewrite the polynomial $p(x_1,\dots,x_n)$ as a polynomial $p_1(x_1,\dots,x_n, x_1^*,\dots,x_n^*, x_1^{**},\dots,x_n^{**})$ of the partial algebra $(L;\vee,\wedge,\bigtriangledown,{}^*,0,1)$ where $p_1(x_1,\dots,x_{3n})$ is a polynomial of its reduct $L_1 = (L;\vee,\wedge,\bigtriangledown,0,1)$ only. Here, of course, the partial operation $\bigtriangledown$ is defined for closed elements only. Now, if $a_1,\dots,a_m$ denote all constant symbols appearing in p_1, then p_1 can be expressed as a term $t_1(x_1,\dots,x_n, x_1^*,\dots, x_n^*, x_1^{**},\dots, x_n^{**}, a_1,\dots, a_m)$ of the partial algebra $(L;\vee,\wedge,\bigtriangledown,{}^*,0,1,a_1,\dots,a_m)$ where $t_1(x_1,\dots, x_{3n+m})$ is a term of the partial algebra $L_1 = (L;\vee,\wedge,\bigtriangledown,0,1)$. Since for $x_1,\dots,x_n \in D(L)$ we have $f'(x_1,\dots,x_n) = f(x_1,\dots,x_n)$ and $x_i^* = 0$, $x_i^{**} = 1$, $i = 1,\dots,n$, the function f' can be represented on $D(L)$ by a term $t_1(x_1,\dots,x_n, 0,\dots,0, 1,\dots,1, a_1,\dots,a_m)$ of the algebra $(L;\vee,\wedge,\bigtriangledown,0,1,a_1,\dots,a_m)$. Since in $t_1(x_1,\dots,x_n, 0,\dots,0, 1,\dots,1, a_1,\dots, a_m)$ the operation symbols $\bigtriangledown$ join constant symbols only, they can be omitted by replacing each subterm of the form $a_i \bigtriangledown a_j$, $(a_i, a_j \in \{a_1,\dots,a_m\}\cup\{0,1\})$ by a new constant symbol b denoting the element $a_i \bigtriangledown a_j \in B(L)$. Thus f' can be represented by a term $t_2(x_1,\dots,x_n, b_1,\dots,b_k)$ of an algebra $(L;\vee,\wedge,b_1,\dots,b_k)$, $(b_1,\dots,b_k \in L)$ where $t_2(x_1,\dots,x_{n+k})$ is a term of the lattice $L_2 = (L;\vee,\wedge)$ only. Hence for any $x_1,\dots,x_n \in D(L)$ we have $f'(x_1,\dots,x_n) = t_2(x_1,\dots,x_n, b_1,\dots,b_k) = t_2(x_1,\dots,x_n, b_1,\dots,b_k) \vee d$ as $f'(x_1,\dots,x_n) \in D(L)$. Since d is a distributive element, the mapping $\varphi\colon L \to D(L)$, $\varphi(x) = x \vee d$ is a lattice homomorphism by Proposition 3.3. Thus for $x_1,\dots,x_n \in D(L)$ we get $f'(x_1,\dots,x_n) = \varphi\big(t_2(x_1,\dots,x_n, b_1,\dots,b_k)\big) = t_2\big(\varphi(x_1),\dots,\varphi(x_n), \varphi(b_1),\dots,\varphi(b_k)\big) = t_2(x_1 \vee d,\dots,x_n \vee d, b_1 \vee d,\dots,b_k \vee d) = p_2(x_1,\dots,x_n)$ where p_2 is a polynomial of the lattice $(D(L),\vee,\wedge)$. Hence the function f' is a polynomial function of the lattice $D(L)$, which was to be proved.

Note that if L is locally affine complete and f' is a finite partial compatible function of $D(L)$, then $f \equiv f'$ is a finite partial compatible function of the algebra L, too. Hence by hypothesis, there is a polynomial $p(x_1,\dots,x_n)$ of L interpolating f on its finite domain. By the same procedure as above one can show that the restriction of $p(x_1,\dots,x_n)$ to the domain of $f \equiv f'$ is a polynomial of the lattice $D(L)$. □

4.4 COROLLARY. *A principal p-algebra is locally affine complete if and only if it is a Boolean algebra.*

Proof. From the construction presented in Section 3 it follows that L is a sublattice of $B(L) \times D(L)$. If L is locally affine complete, then $D(L)$ is a locally affine complete lattice by 4.3, hence $|D(L)| = 1$ by 4.1. Consequently, L is a Boolean algebra. □

4.5 COROLLARY. *Finite Boolean algebras are the only finite affine complete principal p-algebras.*

REFERENCES

[B 1982] BEAZER, R.: *Affine complete Stone algebras*, Acta. Math. Acad. Sci. Hungar. **39** (1982), 169–174.

[G 1962] GRÄTZER, G.: *On Boolean functions (notes on Lattice theory II)*, Revue de Math. Pures et Appliquées **7** (1962), 693–697.

[G 1971] GRÄTZER, G.: *Lattice theory. First concepts and distributive lattices*, W. H. Freeman and Co., San Francisco, Calif., 1971.

[Ha 1992] HAVIAR, M.: *On affine completeness of distributive p-algebras*, Glasgow Math. J. **34** (1992), 365–368.

[Kaa-Ma-S 1985] KAARLI, K.—MÁRKI, L.—SCHMIDT, E. T.: *Affine complete semilattices*, Mh. Math. **99** (1985), 297–309.

[Kaa-P 1987] KAARLI, K.—PIXLEY, A. F.: *Affine complete varieties*, Algebra Universalis **24** (1987), 74–90.

[Ka 1980] KATRIŇÁK, T.: *p-algebras*, Colloq. Math. Soc. János Bolyai (1980), 549–573.

[Ka-Me 1983] KATRIŇÁK, T.—MEDERLY, P.: *Construction of p-algebras*, Algebra Universalis **17** (1983), 288–316.

[Kö 1978] KÖHLER, P.: *The triple method and free distributive pseudo-complemented lattices*, Algebra Universalis **8** (1978), 139–150.

[Mu-En 1986] MURTY, P. V. R.—ENGELBERT, Sr. T.: *On "constructions of p-algebras"*, Algebra Universalis **22** (1986), 215–228.

[O 1935] ORE, O.: *On the foundation of abstract algebra, I*, Ann. of Math. **36** (1935), 406–437.

[P 1972] PIXLEY, A. F.: *Completeness in arithmetical algebras*, Algebra Universalis **2** (1972), 179–196.

[P 1979] PIXLEY, A. F.: *Characterizations of arithmetical varieties*, Algebra Universalis **9** (1979), 87–98.

[Sz 1986] SZENDREI, A.: *Clones in Universal Algebra*, Les Presses de L'Université de Montréal, Montreal, 1986.

[W 1971] WERNER, H.: *Produkte von Kongruenzklassengeometrien universeller Algebren*, Math. Z. **121** (1971), 111–140.

Received October 14, 1994

Department of Mathematics
Matej Bel University
Zvolenská cesta 6
SK–974 01 Banská Bystrica
SLOVAKIA

E-mail: haviar@bb.sanet.sk