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SOME REMARKS ON THE PSEUDO–LINEAR ALGEBRA

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ABSTRACT. Recent results on pseudo-arithmetic operations and g-calculus are applied to the domain of linear algebra. As a basic notion, g-rank of a matrix is introduced. Two generators g and h are shown to be rank equivalent if and only if they differ only in a positive multiplicative constant. Some applications to the solutions of systems of pseudo-linear equations are presented.

1. Introduction

Recently, E. Pap [5] introduced and developed a so-called g-calculus generalizing the common calculus of real valued functions. The basis of g-calculus are pseudo-arithmetical operations based on a generator g. An axiomatic approach to pseudo-arithmetics can be found in Mesiar and Rybárik [4]. The theory of g-calculus was applied to some problems from differential, partially differential and difference equations, respectively, see [6]. The main idea of the above applications is in the exploiting the knowledge of the solution of some linear problem (e.g., linear differential equations) and applying it to a corresponding pseudo-linear problem.

Note that no closed theory of g-linear problems was developed till now. As a first attempt in this field, this paper is devoted to the investigation of pseudo-arithmetical operations based on a generator g. We will show that two generators g and h preserve the matrix rank if and only if they differ only in a positive multiplicative constant. Consequently, the use of normed generators is justified.

2. Pseudo-arithmetical operations

Following Mesiar and Rybárik [4], we introduce the concept of pseudo-arithmetical operations first on $[0, +\infty]$ interval and then on $[-\infty, +\infty]$ interval.

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DEFINITION 2.1. Two binary operations $\oplus$ and $\odot$ defined on $[0,+\infty]$ are called *a pseudo-addition* and *pseudo-multiplication*, respectively, if they fulfill the following axioms:

(A1) $x \oplus 0 = 0 \oplus x = x, \quad \forall x \in [0,+\infty].$

(A2) $(x \oplus y) \oplus z = x \oplus (y \oplus z), \quad \forall x,y,z \in [0,+\infty].$

(A3) If $x \leqq x'$ and $y \leqq y'$ then $x \oplus y \leqq x' \oplus y'$ for every $x,y,x',y' \in [0,+\infty]$.

(A4) If $x_n \to x$ and $y_n \to y$ then $x_n \oplus y_n \to x \oplus y$.

(A5) If $x > 0$ and $y \in [0,\infty)$ then there exists $n \in \mathbb{N}$ such that $\underbrace{x \oplus x \oplus \cdots \oplus x}_{n\text{-times}} \geqq y$.

(A6) If $x < +\infty$ and $y < +\infty$ then $x \oplus y < +\infty$.

(M1) $a \odot (x \oplus y) = (a \odot x) \oplus (a \odot y), \quad \forall a,x,y \in [0,+\infty].$

(M2) If $a \leqq b$ then $a \odot x \leqq b \odot x$, for every $x \in [0,+\infty]$.

(M3) $a \odot x = 0$ if and only if $a = 0$ or $x = 0$.

(M4) There exists a left unit, i.e., an element $e \in [0,+\infty]$ so that $e \odot x = x$ for every $x \in [0,+\infty]$.

(M5) If $a_n \to a \in (0,+\infty)$ and $x_n \to x$ then $a_n \odot x_n \to a \odot x$ and $(+\infty) \odot x = \lim\limits_{a \to +\infty} (a \odot x)$.

(M6) $x \odot y = y \odot x$ for every $x,y \in [0,+\infty]$.

THEOREM 2.1 [4]. *Two binary operations $\oplus$ and $\odot$ on $[0,+\infty]$ are pseudo-addition and pseudo-multiplication, respectively, if and only if there is a generator $\bar{g}$, $\bar{g}\colon [0,+\infty] \to [0,+\infty]$, $\bar{g}$ is an increasing bijection, so that for all $x,y \in [0,+\infty]$ it is*

$$x \oplus y = \bar{g}^{-1}\big(\bar{g}(x) + \bar{g}(y)\big) \qquad \text{and}$$
$$x \odot y = \bar{g}^{-1}\big(\bar{g}(x) \cdot \bar{g}(y)\big) \qquad \text{for} \quad \{x,y\} \neq \{0,+\infty\}.$$

Note that the unit element e of pseudo-multiplication $\odot$ is given by $e = \bar{g}^{-1}(1)$.

An odd extension g of a given generator $\bar{g}$ from $[0,+\infty]$ to $[-\infty,+\infty]$ is called a generator on $[-\infty,+\infty]$ [4] and it allows to extend $\oplus$ and $\odot$ to the whole extended real line. Moreover, pseudo-subtraction $\ominus$ and pseudo-division $\oslash$ can be introduced.

Let g be a generator on $[-\infty,+\infty]$. For $x,y \in [-\infty,+\infty]$ we put:

$$x \oplus y = g^{-1}\big(g(x) + g(y)\big), \qquad \{x,y\} \neq \{-\infty,+\infty\};$$
$$x \odot y = g^{-1}\big(g(x) \cdot g(y)\big), \qquad \{x,y\} \neq \{0,+\infty\},\ \{x,y\} \neq \{0,-\infty\};$$
$$x \ominus y = g^{-1}\big(g(x) - g(y)\big), \qquad \text{if } x = y \text{ then } x \notin \{-\infty,+\infty\};$$
$$x \oslash y = g^{-1}\big(g(x) : g(y)\big), \qquad y \neq 0,\ \{x,y\} \not\subseteq \{-\infty,+\infty\}.$$

All the above-introduced operations are called pseudo-arithmetical operations generated by generator $\boldsymbol{g}$.

If two or more generators are taken into account, a lower index is used to distinguish the corresponding pseudo-arithmetical operations, say $\underset{g}{\oplus}$ and $\underset{h}{\oplus}$.

Note that the identity generator $\boldsymbol{g}(x) = x$ leads to the common arithmetical operations.

EXAMPLE 2.1.

(i) Let $\boldsymbol{g}(x) = \begin{cases} x^r, & \text{if } x \geqq 0, \\ -(-x)^r, & \text{if } x < 0, \end{cases}$ for some positive constant r.

Then $\boldsymbol{g}$ is a generator on $[-\infty, +\infty]$. Take e.g., $r = 3$. For corresponding pseudo-arithmetical operations we get

$$\begin{aligned} x \oplus y &= \left(x^3 + y^3\right)^{1/3}; \\ x \ominus y &= \left(x^3 - y^3\right)^{1/3}; \\ x \odot y &= \left(x^3 \cdot y^3\right)^{1/3} = x \cdot y; \\ x \oslash y &= \left(\frac{x^3}{y^3}\right)^{1/3} = \frac{x}{y}, \end{aligned}$$

i.e., the pseudo-multiplication and pseudo-division coincide with the common multiplication and division in this case (note that the last assertion holds true only for the above-introduced generators [3]).

(ii) Let $\boldsymbol{g}(x) = cx$ for some positive constant c. Then $\boldsymbol{g}$ is a generator on $[-\infty, +\infty]$. Take, e.g., $c = 3$. For the corresponding pseudo-arithmetical operations we have

$$\begin{aligned} x \oplus y &= x + y; \\ x \ominus y &= x - y; \\ x \odot y &= 3xy; \\ x \oslash y &= \frac{x}{(3y)}. \end{aligned}$$

Now, $\oplus$ and $\ominus$ coincide with the common addition and subtraction (and this is true only for $\boldsymbol{g}(x) = cx$).

(iii) Let $\boldsymbol{g}(x) = \begin{cases} (x+1)^{1/2} - 1, & \text{if } x \geqq 0, \\ -\left((1-x)^{1/2} - 1\right), & \text{if } x < 0. \end{cases}$

Then $\boldsymbol{g}$ is a generator on $[-\infty, +\infty]$ and for $x \geqq y < 0$ it is

$$x \oplus y = \left[(x+1)^{1/2} + (y+1)^{1/2} - 1\right]^2 - 1\,;$$
$$x \ominus y = \left[(x+1)^{1/2} - (y+1)^{1/2} - 1\right]^2 - 1\,;$$
$$x \odot y = \left[(x+1)^{1/2}(y+1)^{1/2} - (x+1)^{1/2} - (y+1)^{1/2} + 2\right]^2 - 1\,;$$
$$x \oslash y = \left[\frac{(x+1)^{1/2}}{(y+1)^{1/2}} + 1\right]^2 - 1\,.$$

The unit element $e = 3$.

3. Systems of pseudo-linear equations

In several applications [5] of $\boldsymbol{g}$-calculus based on a generator $\boldsymbol{g}$, the linear problems were replaced by corresponding pseudo-linear problems simply by means of replacing the common arithmetical operations by corresponding pseudo-arithmetical operations. We propose to generalize the basic linear problem — the solution of a system of linear equations — to the corresponding pseudo-linear problem.

DEFINITION 3.1. Let $\boldsymbol{g}$ be a given generator on $[-\infty, +\infty]$ generating $\oplus$ and $\odot$. Let $a_1, \dots, a_n$, $n \in \mathbb{N}$, and b be some given real constant and let $x_1, \dots, x_n$ be unknown real variables. Then the equation

$$a_1 \odot x_1 \oplus \cdots \oplus a_n \odot x_n = b$$

is called *a pseudo-linear equation* or equivalently *$\boldsymbol{g}$-linear equation.*

LEMMA 3.1. *Let $a_1 \odot x_1 \oplus \cdots \oplus a_n \odot x_n = b$ be a $\boldsymbol{g}$-linear equation. Then the linear equation $\sum\limits_{i=1}^{n} \boldsymbol{g}(a_i) y_i = \boldsymbol{g}(b)$ is equivalent to this $\boldsymbol{g}$-linear equation, where $y_i = \boldsymbol{g}(x_i)$, i.e., the solutions y_i of the above equation are in a one-to-one correspondence to the solutions x_i of the original $\boldsymbol{g}$-linear equation.*

P r o o f. Applying $\odot$ and $\oplus$ to the given $\boldsymbol{g}$-linear equation, one gets $\boldsymbol{g}^{-1}\big(\boldsymbol{g}(a_1) \cdot \boldsymbol{g}(x_1) + \cdots + \boldsymbol{g}(a_n) \cdot \boldsymbol{g}(x_n)\big) = b$, i.e., $\sum\limits_{i=1}^{n} \boldsymbol{g}(a_i) y_i = \boldsymbol{g}(b)$, where we put $y_i = \boldsymbol{g}(x_i)$.
On the other hand, $x_i = \boldsymbol{g}^{-1}(y_i)$. □

Let a system of $\boldsymbol{g}$-linear equations be given,

$$a_{11} \odot x_1 \oplus \cdots \oplus a_{1n} \odot x_n = b_1\,,$$
$$\dots$$
$$a_{m1} \odot x_1 \oplus \cdots \oplus a_{mn} \odot x_n = b_m\,,$$

briefly $\mathbf{A} \odot \boldsymbol{X} = \boldsymbol{B}$, where $\mathbf{A} = (a_{ij})_{\substack{1 \leqq i \leqq m \\ 1 \leqq j \leqq n}}$, $\boldsymbol{X} = (x_1, \ldots, x_n)^\top$, $\boldsymbol{B} = (b_1, \ldots, b_m)^\top$. This system is equivalent with the system of linear equations $\boldsymbol{g}(\mathbf{A}) \cdot \boldsymbol{Y} = \boldsymbol{g}(\boldsymbol{B})$, where $\boldsymbol{g}(\mathbf{A}) = \big(g(a_{ij})\big)_{\substack{1 \leqq i \leqq m \\ 1 \leqq j \leqq n}}$, $\boldsymbol{g}(\boldsymbol{B}) = \big(g(b_1), \ldots, g(b_m)\big)^\top$.

By the Frobenius theorem, both systems are solvable if and only if the rank $H\big(\boldsymbol{g}(\mathbf{A})\big)$ of matrix $\boldsymbol{g}(\mathbf{A})$ and the rank $H\big(\boldsymbol{g}(\mathbf{A}^*)\big)$ of the extended matrix $\boldsymbol{g}(\mathbf{A}^*) = \big(\boldsymbol{g}(\mathbf{A}), \boldsymbol{g}(\boldsymbol{B})\big)$ is the same, $H\big(\boldsymbol{g}(\mathbf{A})\big) = H\big(\boldsymbol{g}(\mathbf{A}^*)\big)$.

Then each solution $\boldsymbol{Y}$ of the induced linear system corresponds to the solution $\boldsymbol{X} = \boldsymbol{g}^{-1}(\boldsymbol{Y}) = \big(g^{-1}(y_1), \ldots, g^{-1}(y_n)\big)^\top$ of the original pseudo-linear system and vice versa. Hence the key role by the solution of a system $\mathbf{A} \odot \boldsymbol{X} = \boldsymbol{B}$ of g-linear equations is played by the rank of matrix $\boldsymbol{g}(\mathbf{A})$ (and $\boldsymbol{g}(\mathbf{A}^*)$).

Definition 3.2. Let $\mathbf{A}$ be a given matrix and let g be a generator on $[-\infty, +\infty]$. The rank $H\big(\boldsymbol{g}(\mathbf{A})\big)$ of the matrix $\boldsymbol{g}(\mathbf{A})$ will be called *g-rank* of $\mathbf{A}$, $g\text{–}H(\mathbf{A}) = H\big(\boldsymbol{g}(\mathbf{A})\big)$.

Example 3.1. Let $\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ and let $g(x) = x^3$. Then $H(\mathbf{A}) = 2$, but $g\text{–}H(\mathbf{A}) = 3$ because of $H\big(\boldsymbol{g}(\mathbf{A})\big) = 3$, where $\boldsymbol{g}(\mathbf{A}) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 8 \\ 1 & 8 & 27 \end{bmatrix}$

The above example shows that the g-rank depends on g and hence the solvability of a system $\mathbf{A} \odot \boldsymbol{X} = \boldsymbol{B}$ of g-linear equations varies changing the generator g.

Example 3.2. Let $g(x) = 3x$. Then for each $\mathbf{A} = (a_{ij})$ it is $\boldsymbol{g}(\mathbf{A}) = (3a_{ij})$ and thus $g\text{–}H(\mathbf{A}) = H(\mathbf{A})$. Further, the induced system to a given $\mathbf{A} \odot \boldsymbol{X} = \boldsymbol{B}$ is just $3\mathbf{A} \cdot \boldsymbol{Y} = 3\boldsymbol{B}$, i.e., $\mathbf{A} \cdot \boldsymbol{Y} = \boldsymbol{B}$. If $\boldsymbol{Y}$ is its solution, then $\boldsymbol{X} = \frac{1}{3}\boldsymbol{Y}$ is the solution of $\mathbf{A} \odot \boldsymbol{X} = \boldsymbol{B}$ (and vice versa $\boldsymbol{Y} = 3\boldsymbol{X}$). Comparing the common linear system $\mathbf{A} \cdot \boldsymbol{Y} = \boldsymbol{B}$ and the g-linear system $\mathbf{A} \odot \boldsymbol{X} = \boldsymbol{B}$, we can see their equivalency.

Note that the identity generator of a common linear system $\mathbf{A} \cdot \boldsymbol{Y} = \boldsymbol{B}$ differs from g only in a positive multiplicative constant.

By Example 3.2, there are some generators, say g and h, leading to the equivalent systems $\mathbf{A} \underset{g}{\odot} \boldsymbol{X} = \boldsymbol{B}$ and $\mathbf{A} \underset{h}{\odot} \boldsymbol{Y} = \boldsymbol{B}$. It is easy to see that a necessary condition for the equivalency of g-linear and h-linear systems is the rank-equivalence of their generators, i.e., $g\text{–}H(\mathbf{A}) = h\text{–}H(\mathbf{A})$ for each matrix $\mathbf{A}$.

4. Rank-equivalent generators

In this section we study the generators g and h preserving the rank.

Let $\boldsymbol{g}$ be a generator on $[-\infty,+\infty]$ and let $\boldsymbol{h} = c\cdot\boldsymbol{g}$ for some positive constant c. Then for each matrix $\mathbf{A}$ we have $\boldsymbol{g}\text{-}H(\mathbf{A}) = \boldsymbol{h}\text{-}H(\mathbf{A})$.

In the following theorem we show the necessity of $\boldsymbol{h} = c\cdot\boldsymbol{g}$ to preserve the ranks.

THEOREM 4.1. *Let $\boldsymbol{g}$ and $\boldsymbol{h}$ be two generators on $[-\infty,+\infty]$ and let for each matrix $\mathbf{A}$ it is $\boldsymbol{g}\text{-}H(\mathbf{A}) = \boldsymbol{h}\text{-}H(\mathbf{A})$. Then there is a positive constant c so that $\boldsymbol{h} = c\cdot\boldsymbol{g}$.*

Proof. Let $a,b\in\mathbb{R}-\{0\}$, $a\neq b$ and put $\mathbf{A} = \begin{bmatrix} a & b & a\\ a & b & b\\ a\underset{g}{\oplus}a & b\underset{g}{\oplus}b & a\underset{g}{\oplus}b\end{bmatrix}$.

Then $\boldsymbol{g}(\mathbf{A}) = \begin{bmatrix} \boldsymbol{g}(a) & \boldsymbol{g}(b) & \boldsymbol{g}(a)\\ \boldsymbol{g}(a) & \boldsymbol{g}(b) & \boldsymbol{g}(b)\\ 2\boldsymbol{g}(a) & 2\boldsymbol{g}(b) & \boldsymbol{g}(a)+\boldsymbol{g}(b)\end{bmatrix}$. It is easy to see that the rank $H\big(\boldsymbol{g}(\mathbf{A})\big) = 2$, and hence $\boldsymbol{g}\text{-}H(\mathbf{A}) = 2$. But then also $\boldsymbol{h}\text{-}H(\mathbf{A}) = H\big(\boldsymbol{h}(\mathbf{A})\big) = 2$.

We have $\boldsymbol{h}(\mathbf{A}) = \begin{bmatrix} \boldsymbol{h}(a) & \boldsymbol{h}(b) & \boldsymbol{h}(a)\\ \boldsymbol{h}(a) & \boldsymbol{h}(b) & \boldsymbol{h}(b)\\ \boldsymbol{h}(a\underset{g}{\oplus}a) & \boldsymbol{h}(b\underset{g}{\oplus}b) & \boldsymbol{h}(a\underset{g}{\oplus}b)\end{bmatrix}$ and hence

$$S_3 = c_1S_1 + c_2S_2 \tag{1}$$

or

$$S_1 = c_3S_2\,, \tag{2}$$

where S_i are columns of the matrix $\boldsymbol{h}(\mathbf{A})$ and c_i are some real constants. If (1) is true, then $\boldsymbol{h}(a) = c_1\boldsymbol{h}(a) + c_2\boldsymbol{h}(b)$ and $\boldsymbol{h}(b) = c_1\boldsymbol{h}(a) + c_2\boldsymbol{h}(b)$ and thus $\boldsymbol{h}(a) = \boldsymbol{h}(b)$. But this means that $a = b$, a contradiction.

It follows that (2) should be true and hence $\boldsymbol{h}(a) = c_3\boldsymbol{h}(b)$ and $\boldsymbol{h}(a\underset{g}{\odot}a) = c_3\boldsymbol{h}(b\underset{g}{\odot}b)$. Then $c_3 = \frac{\boldsymbol{h}(a)}{\boldsymbol{h}(b)} = \frac{\boldsymbol{h}(a\underset{g}{\odot}a)}{\boldsymbol{h}(b\underset{g}{\odot}b)}$ and consequently

$$\frac{\boldsymbol{h}(a)}{\boldsymbol{h}(b)} = \frac{\boldsymbol{h}\big(\boldsymbol{g}^{-1}(2\boldsymbol{g}(a))\big)}{\boldsymbol{h}\big(\boldsymbol{g}^{-1}(2\boldsymbol{g}(b))\big)}\,. \tag{3}$$

Put $x = \boldsymbol{h}(a)$, $y = \boldsymbol{h}(b)$ and $\boldsymbol{f} = \boldsymbol{g}\circ\boldsymbol{h}^{-1}$. Then $\boldsymbol{f}^{-1} = \boldsymbol{h}\circ\boldsymbol{g}^{-1}$ and (3) turns to

$$\frac{x}{y} = \frac{\boldsymbol{f}^{-1}\big(2\boldsymbol{f}(x)\big)}{\boldsymbol{f}^{-1}\big(2\boldsymbol{f}(y)\big)}\,, \tag{4}$$

because of $\boldsymbol{g}(a) = \boldsymbol{g}\big(\boldsymbol{h}^{-1}(\boldsymbol{h}(a))\big) = \boldsymbol{f}(x)$, and similarly $\boldsymbol{g}(b) = \boldsymbol{f}(y)$. Simple rewriting of (4) leads to

$$\frac{x}{\boldsymbol{f}^{-1}\big(2\boldsymbol{f}(x)\big)} = \frac{y}{\boldsymbol{f}^{-1}\big(2\boldsymbol{f}(y)\big)}\,, \tag{5}$$

for all $x, y \in \mathbb{R} - \{0\}$, $x \neq y$.
The last equality is true for all couples of $x \neq y$ and thus, fixing $y = 1$, one gets

$$\frac{x}{f^{-1}(2f(x))} = \frac{1}{f^{-1}(2f(1))} = \frac{1}{K}, \tag{6}$$

where $K = f^{-1}(2f(1))$ is some positive real constant greater than 1. It follows $K \cdot x = f^{-1}(2f(x))$ and thus

$$f(K \cdot x) = 2f(x). \tag{7}$$

Recall that $f = g \circ h^{-1}$ is continuos strictly increasing odd bijection on $[-\infty, +\infty]$. By Aczel [1], a general solution of functional equation (7) is

$$f(x) = \begin{cases} d \cdot x^p, & \text{if } x \geqq 0, \\ -d \cdot (-x)^p, & \text{if } x < 0, \end{cases}$$

where d is some positive real constant (in fact, $d = f(1)$), and $p = \log_K 2$, i.e., p is a positive real constant. We will show that $p = 1$.

Put $B = \begin{bmatrix} 0 & h^{-1}(1) & h^{-1}(1) \\ h^{-1}(1) & 0 & h^{-1}(1) \\ h^{-1}(1) & h^{-1}(1) & h^{-1}(2) \end{bmatrix}$, then $h(B) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ and hence $H(h(B)) = h\text{–}H(B) = 2$. Then also $H(g(B)) = g\text{–}H(B) = 2$. We have

$$g(b) = \begin{bmatrix} 0 & gh^{-1}(1) & gh^{-1}(1) \\ gh^{-1}(1) & 0 & gh^{-1}(1) \\ gh^{-1}(1) & gh^{-1}(1) & gh^{-1}(2) \end{bmatrix} = \begin{bmatrix} 0 & f(1) & f(1) \\ f(1) & 0 & f(1) \\ f(1) & f(1) & f(2) \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & d & d \\ d & 0 & d \\ d & d & d2^p \end{bmatrix}.$$

It follows $d + d = d \cdot 2^p$, i.e., $p = 1$. But then $f(x) = d \cdot x = g \circ h^{-1}(x)$ which leads to $g^{-1}(d \cdot x) = h^{-1}(x)$ and consequently $h(x) = \frac{1}{d} \cdot g(x) = c \cdot g(x)$, where $c = \frac{1}{d}$ is some positive real constant. □

COROLLARY 4.1. *Two generators g and h are rank-equivalent, i.e., $g\text{–}H(\mathbf{A}) = h\text{–}H(\mathbf{A})$ for each matrix $\mathbf{A}$, if and only if $h = c \cdot g$ for some positive constant c.*

Remark 4.1. The rank of matrices with respect to special binary operations $\oplus$ and $\odot$ (covering e.g., $\max = \vee$ and $\min = \wedge$) is discussed in [7], see also [2], where two different ranks, namely the column rank $c(\mathbf{A})$ and the semiring rank $r(\mathbf{A})$, are introduced. In our case, both types of ranks coincide, $r(\mathbf{A}) = c(\mathbf{A}) = g\text{–}H(\mathbf{A})$.

THEOREM 4.2. *Let $h = c \cdot g$. Then and only then the systems $\mathbf{A} \underset{g}{\odot} \mathbf{X} = \mathbf{B}$ and $\mathbf{A} \underset{h}{\odot} \mathbf{Y} = \mathbf{B}$ are equivalent, i.e., the knowledge of $\mathbf{X}$ implies the knowledge of $\mathbf{Y}$ and vice versa, and $g(\mathbf{X}) = h(\mathbf{Y})$.*

Proof. The system $\mathbf{A} \underset{g}{\odot} \mathbf{X} = \mathbf{B}$ is equivalent to the linear system $g(\mathbf{A}) \cdot Z = g(\mathbf{B})$, where $Z = g(\mathbf{X})$. The system $\mathbf{A} \underset{h}{\odot} \mathbf{Y} = \mathbf{B}$ is equivalent to the linear system $h(\mathbf{A}) \cdot Q = c \cdot g(\mathbf{A}) \cdot Q = h(\mathbf{B}) = c \cdot g(\mathbf{B})$, i.e., $g(\mathbf{A}) \cdot Q = g(\mathbf{B})$, where $Q = h(\mathbf{Y})$. It is easy to see the equivalency of both linear systems and if they are solvable, then $Z = g(\mathbf{X}) = h(\mathbf{Y}) = Q$. □

Remark 4.2. If $h = c \cdot g$, $c > 0$, and $g(\mathbf{X}) = h(\mathbf{Y})$, then $\mathbf{Y} = h^{-1}(g(\mathbf{X})) = g^{-1}\left(\frac{g(\mathbf{X})}{c}\right) = g^{-1}\left(\frac{g(\mathbf{X})}{g(g^{-1}(c))}\right) = \mathbf{X} \underset{g}{\oslash} g^{-1}(c)$.

Further, the unity e_h of pseudo-multiplication $\underset{h}{\odot}$ is given by $e_h = h^{-1}(1) = g^{-1}(\frac{1}{c})$. If we put $g(1) = 1$, i.e., g is normed generator, then $e_h = 1 \underset{g}{\oslash} g^{-1}(c)$ and consequently $\mathbf{Y} = \mathbf{X} \underset{g}{\odot} e_h = e_h \underset{g}{\odot} \mathbf{X}$. For the further investigation we propose to use the normed generators g on $[-\infty, +\infty]$ only, as long as the general casc can be easily obtained as remarked above.

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